B.-Y. Chen’s conjecture on hypersurfaces of Euclidean spaces

1. Introduction

Definition 1.1. (a) A map \( f : (M, g) \to (N, g_N) \) is harmonic iff \( f \) is a solution of the variational problem defined by \( \int_M |df|^2 v_g \). Its Euler-Lagrange equation is \( \tau \equiv 0 \), where \( \tau \) is the tension field of \( f \). Roughly speaking, it means that \( f \) is “close to a constant map”.
(b) A map \( f : (M, g) \to (N, g_N) \) is bi-harmonic iff \( f \) is a solution of the variational problem defined by \( \int_M |\tau|^2 v_g \). Roughly speaking, it means that \( f \) is “close to a harmonic map”.
(c) A submanifold \( M \subset (\overline{M}, \overline{g}) \) is a bi-harmonic submanifold iff the inclusion map \( \iota \) is bi-harmonic map w.r.t. \( g = \iota^* \overline{g} \). Its Euler-Lagrange equation becomes

\[
\begin{cases}
\quad (\bot): \Delta \tau + \alpha^2(\tau) - g^{ij}(\overline{R}(\tau, \partial_i) \partial_j)^{\bot} = 0, \\
\quad (\top): -2\overline{g}(\delta \alpha)(\partial_i) + 2\overline{g}(\alpha(\partial_j, \partial_i), \nabla^i \tau) - \frac{1}{2} \nabla_i |\tau|^2 = 0,
\end{cases}
\]

where \( \alpha \) is the second fundamental equation. The equation (\( \bot \)) is a 4th order elliptic equation.

Note that the conditions of bi-harmonic submanifold: “the inclusion map \( \iota \) is bi-harmonic map” and “the source metric \( g \) is the induced metric \( \iota^* \overline{g} \)” are independent, and their combination becomes an over-determined PDE. Therefore, its solutions rarely exist, in general.

However, every minimal submanifold is bi-harmonic, and we have a lot of bi-harmonic submanifolds. B.-Y. Chen conjectured as follows.

B.-Y. Chen’s conjecture: There are no non-minimal bi-harmonic submanifolds in \( E^m \).

We consider this conjecture for local submanifolds \( M^n \subset E^m \). Note that bi-harmonic submanifolds in \( E^m \) automatically become \( C^\infty \) submanifolds.

Known results: Chen’s conjecture is true for
(1) Curves in \( E^m \) (I. Dimitric, 1992),
(2) Surfaces in \( E^3 \) (B.-Y. Chen, 1991; G. Y. Jiang, 1986),
(3) Hypersurfaces in \( E^4 \) (T. Hasanis & T. Vlachos, 1995),
(4) Hypersurfaces in $E^{n+1}$ with the number of principal curvatures $\#pc \leq 2$
(I. Dimitric, 1989).

Our main results are as follows.

**Theorem 1.2 (N).** There are no non-minimal hypersurfaces in $E^{n+1}$ with $\#pc \leq 3$.

**Theorem 1.3 (I).** There are no non-minimal bi-harmonic hypersurfaces $M^n$ in $E^{n+1}$ with the following properties:

1. Each principal curvature $\lambda_i$ of $M$ is simple at some point in $M$.
2. $g(\nabla v_i,v_j,v_k) \neq 0$ for all distinct unit principal curvature vector fields $v_i, v_j, v_k \in \text{Ker } d\tau$ at some point in $M$.

2. Proof of the theorem

Let $M$ be a non-minimal bi-harmonic hypersurface of $R^{n+1}$. Since principal curvatures are simple, unit principal curvature vector fields $\{v_i\}$ forms an orthonormal frame field on $M$, $\alpha(v_i, v_j) = \delta_{ij}\lambda_i$. (1.1) becomes

\begin{align}
(A) & \quad \Delta \tau + |\alpha|^2 \tau = 0, \\
(B) & \quad (\tau + 2\lambda_i)v_i[\tau] = 0 \quad (1 \leq i \leq n),
\end{align}

where $v_i[*]$ is differentiation of function *. From (A), we see that if $|\tau|^2$ takes local maximum at some point, then $M$ is minimal. In particular, $\tau$ is not constant. From (B), if there are no $\lambda_i$ with $\tau + 2\lambda_i = 0$, then $\tau$ is constant. Hence there exists $\lambda_i$ with $\tau + 2\lambda_i = 0$. We may assume $\tau + 2\lambda_n = 0$.

Since $\tau$ is not constant, equation: $\tau = c$ defines a hypersurface $F$ in $M$ at generic points. We call $F$ a characteristic hypersurface of $M$. Put $n_1 = n-1$. The vectors $\{v_i\}_{1 \leq i \leq n_1}$ consist an orthonormal tangent frame field on $F$. Put $\mu_i := g(\nabla v_i,v_i,v_n)$, which turns out to be principal curvatures of $F$ in $M$. From (A), (B) and Gauss, Codazzi equation, we can derive the following ODE.

**Proposition 2.1.** $\lambda_i$ and $\mu_i$ satisfy the over-determined ODE:

\begin{align}
(D) & \quad (\lambda_i)' = (\frac{1}{2} \tau + \lambda_i)\mu_i, \quad (\mu_i)' = (\mu_i)^2 - \frac{1}{2} \tau \lambda_i, \\
(#) & \quad -\tau'' + \tau'\sum_{i<n}\mu_i + \tau(\frac{1}{4}\tau^2 + \sum_{i<n}(\lambda_i)^2) = 0,
\end{align}

where $\tau' = v_n[*]$ and $\tau := (2/3)\sum_{i<n}\lambda_i$.

**Remark 2.2.** This proposition holds even if principal curvatures are not simple.

Since ODE (##) is algebraic, we get the following

**Proposition 2.3.** Solutions $(\lambda_i,\mu_i) \in R^{2n_1}$ to (##) runs in the zero-set of a homogeneous polynomial $P_3$ of degree 3. Put $P_{k+1} := (P_k)'$. The set $S$ of initial data of solutions to (##) becomes an algebraic manifold $\cap_{k=3}^{\infty}(P_k)^{-1}(0)$. 

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Conjecture 2.4. $S = \cap_{k \geq 3}(P_k)^{-1}(0) \subset \tau^{-1}(0)$, and so Chen’s conjecture is true.

3. Proof of Theorem N

Based on ODE $(\#)$, we prove theorem N. First, we prepare the following

Lemma 3.1. $\lambda_n = -\tau/2$ is simple. If $\lambda_i \equiv \lambda_j$, then $\mu_i \equiv \mu_j$.

Therefore, solutions to $(\#)$ are considered as curves in $\mathbb{C}^4(\lambda_1, \lambda_2, \mu_1, \mu_2)$. Let $m_i$ be the multiplicity of $\lambda_i$. We denote by $\pi$ the projection $\mathbb{C}^4 \to \mathbb{C}^2(\lambda_1, \lambda_2)$, and by $p$ the projection $\mathbb{C}^2 \setminus \{0\} \to \mathbb{P}^1(\mathbb{C})$. The set $S$ becomes an algebraic manifold in $\mathbb{C}^4$.

Therefore, $p(\pi(S))$ is whole $\mathbb{P}^1(\mathbb{C})$ or a finite point set.

On the other hand we can show, may be using a computer, that

Step 1. $(m_2 + 3, -m_1) \notin \pi(S)$.

Thus, $p(\pi(S))$ is a finite point set, and the ratio $\lambda_2/\lambda_1$ is constant along each solution to $(\#)$.

Step 2. Any solution to $(\#)$ with constant ratio $\lambda_2/\lambda_1$ is in $\tau^{-1}(0)$.

Q.E.D.

4. Proof of Theorem I

To prove Theorem I, we have to analyze the characteristic submanifold $F$.

Definition 4.1. Put $J = \{\{i, j\} \mid 1 \leq i, j \leq n_1, i \neq j\}$. If a distinct triplet $\{i, j, k\}$ satisfies $g(\nabla v_i, v_j, v_k) \neq 0$, then we define $\{i, j\} \sim \{j, k\} \sim \{i, k\}$. Let $\sim_J$ be the equivalence relation on $J$ generated by $\sim$. If all $\{i, j\} \in J$ are equivalent under $\sim_J$, the frame field $\{v_i\}$ is irreducible.

Remark. It is weaker than the assumption of Theorem 1.3.

Definition 4.2. If there exist functions $\varphi, \psi$ on $M$ s.t. $\mu_i = \varphi \lambda_i + \psi$ for $\forall i \leq n_1$, then $\{\lambda_i\}$ and $\{\mu_i\}$ are linearly related.

Lemma 4.3. We assume that $n_1 \geq 3$. (1) If the frame field $\{v_i\}$ is irreducible, then $\{\lambda_i\}$ and $\{\mu_i\}$ are linearly related, and $\lambda_i, \mu_i, \varphi, \psi$ are constant on each characteristic hypersurface $F$. (2) If $\varphi$ or $\psi$ is constant in $t$, then $\tau \equiv 0$.

Theorem 4.4 (I). There are no non-minimal bi-harmonic hypersurfaces $M^n$ in $E^{n+1}$ with following properties:

(1) $\{\lambda_i\}$ are simple . (2) $\{v_i\}_{i \leq n_1}$ is irreducible.
To prove Theorem 4.4, we need simple, but length calculation. Last equation to prove Theorem I is
\[
12n_1(n_1 - 1)\psi^2 \phi^3(1 + \psi^2)^2
\]
x{-105(12 + n_1 + 3(n_1)^2) + (-5901 + 875n_1 + 1026(n_1)^2)\psi^2}
- 2(-351 + 1264n_1 + 567(n_1)^2)\psi^4 + 8(-159 + 34n_1 + 45(n_1)^2)\phi^6
\]
x{-7(33 - 17n_1)^2(27 + 3n_1 - 13(n_1)^2 - 6(n_1)^3 + 5(n_1)^4)
+ (-2755134 + 421064n_1 - 839475(n_1)^2 - 1289439(n_1)^3 + 362329(n_1)^4 + 269159(n_1)^5 - 109964(n_1)^6)\psi^2}
+ 3(-3211164 + 5957928n_1 - 3766311(n_1)^2 + 651168(n_1)^3 + 904142(n_1)^4 - 789504(n_1)^5 + 167725(n_1)^6)\phi^4
- (268288 + 1109001n_1 - 2174458(n_1)^2 + 1018458(n_1)^3 + 11702488(n_1)^4 - 534593(n_1)^5 + 825166(n_1)^6)\phi^6
+ (-20731545 + 44245224n_1 - 22777452(n_1)^2 - 9103320(n_1)^3 + 13627127(n_1)^4 - 5693096(n_1)^5 + 576422(n_1)^6)\phi^8
- 12(-683640 + 1720305n_1 - 1262460(n_1)^2 - 367722(n_1)^3 + 758200(n_1)^4 - 199503(n_1)^5 + 13322(n_1)^6)\phi^{10}
- 4(843453 - 2056212n_1 + 731808(n_1)^2 + 834336(n_1)^3 - 446779(n_1)^4 + 56460(n_1)^5 + 1094(n_1)^6)\phi^{12}
+ 16(9 - 7n_1)(-9 + 2(n_1)^2)(-43 - 26n_1 + 5(n_1)^2)\psi^{14}.
\]

References