# Homework

May 2024

## 1 Manifolds

**Exercise 1.1.** *Construct an canonical isomorphism* $H^2(X,\mathbb{Z}) = [X, K(\mathbb{Z},2)]$*, where* $K(\mathbb{Z},2)$ *is the Eilenberg-MacLane space.*

Here $[X,Y]$ is the homotopy set of continuous maps $f : X \to Y$

**Exercise 1.2.** *For a group* $G$ *prove that any principal* $G$*-bundle* $P \to X$ *is isomorphic to the pull-back bundle* $f^*\pi_G$ *of* $\pi_G$ *by* $f : X \to BG$*. Here* $\pi_G : EG \to BG$ *is the universal* $G$*-bundle.*

**Exercise 1.3.** *Prove* $BU(1) = K(\mathbb{Z},2)$ *and it is homotopy equivalent to infinite dimension complex projective space.*

**Exercise 1.4.** *Construct for any group* $G$ *the classifying space* $BG$ *and the universal principal* $G$*-bundle* $\pi_G : EG \to BG$*.*

**Exercise 1.5.** *Let* $X$ *be a closed 4-manifold. Prove for* $\alpha, \beta \in H_2(X,\mathbb{Z})$*,* $\alpha \cdot \beta = Q(PD(\alpha), PD(\beta)) = \int_X \eta_A \wedge \eta_B$*. Here* $\eta_A$ *and* $\eta_B$ *are closed 2-forms corresponding to the integral classes for the natural isomorphism* $H_2(X,\mathbb{R}) \cong H^2_{dR}(X,\mathbb{R})$

**Exercise 1.6.** *For any element* $\alpha \in H_2(X^4)$ *is equivalent to the homology class of the zero set of a generic section* $s : X \to \mathcal{L}$ *of a line bundle* $\mathcal{L}$*.*

## 2 Knot theory

**Exercise 2.1.** *Prove any knot in* $S^3$ *has a Seifert surface. Use the Seifert Algorigthm.*

**Exercise 2.2.** *Prove that Hopf link has a Seifert surface homeomorphic to annulus.*

# 3 Quantum invariant

## 3.1 The trace function

Let $\{e_i\}_{i=1}^n$ be a basis in $V$. Then since we have $\mathrm{trace}(A) = \sum_{j=1}^m \langle e_j, Ae_j\rangle$. Here we identify $e_i$ with the standard basis in $\mathbb{C}^n$ by the isomorphism $V \cong \mathbb{C}^n$ by the basis. We can have its dual basis $\{e_i^*\}_{i=1}^n$ in its dual basis in $V^*$. Naturally we obtain an inner product $\langle\cdot,\cdot\rangle$ over $V$ by inducing the standard inner product on $\mathbb{C}^n$. In general $\mathrm{End}(V)$ and $M(n,\mathbb{C})$ is identified as following

$$\mathrm{End}(V) \ni f \mapsto (e_i^*(f(e_j))) \in M(n,\mathbb{C}).$$

Thus the values the trace function for $f \in \mathrm{End}(V)$ is as follows:

$$\begin{aligned}\mathrm{trace}(f) &= \mathrm{trace}\left((e_j^*(f(e_i)))\right) = \sum_{k=1}^n \langle e_k, ((e_i^*(f(e_j))))\,(e_k)\rangle \\ &= \sum_{k=1}^n \langle e_k, f(e_k)\rangle = \sum_{k=1}^n e_k^*(f(e_k))\end{aligned}$$

The identification $\mathrm{End}(V) = V^* \otimes V$ can be seen that for $f \in \mathrm{End}(V)$ we have

$$\mathrm{End}(V) \ni f \mapsto \sum_{i,j=1}^n e_j^*(f(e_i))e_j^* \otimes e_i \in V^* \otimes V.$$

Therefore the trace function on $V^* \otimes V$

$$\mathrm{trace} : V^* \otimes V \to \mathbb{C}$$

means

$$\mathrm{trace}\left(\sum_{i,j=1}^n e_j^*(f(e_i))e_j^* \otimes e_j\right) = \sum_{k=1}^n e_k^*(f(e_k)).$$

This is the linear expansion of $e_j^* \otimes e_i \mapsto e_j^*(e_i) = \delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker's delta. In particular for a linear function $g \in V^*$ and $x \in V$, we have $\mathrm{trace}(g \otimes x) = g(x)$. This is also called the *contraction* in general. Note that the trace function is independent of the choice of the basis on $V$.

## 3.2 The trace$_2$ function

Let $V$ be a $k$-dimensional vector space with the basis $\{e_l\}_{l=1}^k$. We put $e_{i_1,\cdots,i_n} = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}$. The the trace function on $\mathrm{End}(V^{\otimes n})$ is

$$\mathrm{trace} : f \mapsto \sum_{i_1,\cdots,i_n,j_1,\cdots,j_n=1}^k e_{i_1,\cdots,i_n}^*(f(e_{j_1\cdots j_n}))$$

Then the trace$_2$ is a function:

$$\mathrm{trace}_2 : \mathrm{End}(V^{\otimes(n+1)}) = \mathrm{End}(V^{\otimes n} \otimes V) \to \mathrm{End}(V^{\otimes n})$$

is the linear expansion of $\mathrm{trace}_2(g_1 \otimes \cdots \otimes g_{n+1} \otimes x_1 \otimes \cdots \otimes x_{n+1}) = g_{n+1}(x_{n+1}) g_1 \otimes \cdots \otimes g_n \otimes x_1 \otimes \cdots \otimes x_n$, where $g_i \in V^*$ and $x_j \in V$.

This is equivalent to the map that for $f \in \mathrm{End}(V^{\otimes n+1})$ we move it to the linear expansion $\mathrm{trace}_2(f)$ of

$$e_{i_1,\cdots,i_n} \mapsto \sum_{l=1}^{k} (id^{\otimes n} \otimes e_l^*)(f(e_{i_1 \cdots e_{i_n}} \otimes e_l)).$$

**Exercise 3.1.** *Show that trace* $\circ$ *trace*$_2$ = *trace.*

*Proof.* $\mathrm{trace}(\mathrm{trace}_2(f_1 \otimes \cdots \otimes f_n \otimes x_1 \otimes \cdots \otimes x_n)) = f_n(x_n)\mathrm{trace}(f_1 \otimes \cdots \otimes f_{n-1} \otimes x_1 \otimes \cdots \otimes x_{n-1}) = f_n(x_n) f_1(x_1) \cdots f_{n-1}(x_{n-1}) = \mathrm{trace}(f_1 \otimes \cdots \otimes f_n \otimes x_1 \otimes \cdots \otimes x_n)$. □

We redecompose the map $h^{\otimes n+1} \cdot \psi_{n+1}(b\sigma_n)$ as follows:

$$\begin{aligned}
& h^{\otimes n+1} \cdot \psi_{n+1}(b\sigma_n) \\
= \;& h^{\otimes n+1} \cdot \psi_{n+1}(b) \cdot \psi_{n+1}(\sigma_n) \\
= \;& h^{\otimes n+1} \cdot \psi_{n+1}(b) \cdot (id^{\otimes n-1} \otimes R) \\
= \;& (h^{\otimes n} \otimes h) \cdot \psi_{n+1}(b) \cdot (id^{\otimes n-1} \otimes R) \\
= \;& (h^{\otimes n} \otimes id) \cdot \psi_{n+1}(b) \cdot (id^{\otimes n} \otimes h) \cdot (id^{\otimes n-1} \otimes R) \\
= \;& ((h^{\otimes n} \otimes id) \cdot (\psi_n(b) \otimes id)) \cdot (id^{\otimes n} \otimes h) \cdot (id^{\otimes n-1} \otimes R) \\
= \;& ((h^{\otimes n} \cdot \psi_n(b)) \otimes id) \cdot (id^{\otimes n} \otimes h) \cdot (id^{\otimes n-1} \otimes R) \\
= \;& ((h^{\otimes n} \cdot \psi_n(b)) \otimes id) \cdot (id^{\otimes n-1} \otimes ((id \otimes h) \cdot R))
\end{aligned}$$

$\mathrm{trace}_2$ is as follows for $f = h^{\otimes n+1} \cdot \psi_{n+1}(b\sigma_n))$:

$$\begin{aligned}
e_{i_1,\cdots,i_n} \mapsto \;& \sum_{l=1}^{k} (id^{\otimes n} \otimes e_l^*)(f(e_{i_1,\cdots,i_n} \otimes e_l)) \\
= \;& \sum_{l=1}^{k} (id^{\otimes n} \otimes e_l^*)((h^{\otimes n} \cdot \psi_n(b)) \otimes id)(e_{i_1,\cdots,i_{n-1}} \otimes ((id \otimes h) \cdot R)(e_{i_n,i_l})) \\
= \;& \sum_{l=1}^{k} (h^{\otimes n} \cdot \psi_n(b)) \otimes e_l^*)(e_{i_1,\cdots,i_{n-1}} \otimes ((id \otimes h) \cdot R)(e_{i_n,i_l})) \\
= \;& \sum_{l=1}^{k} (h^{\otimes n} \cdot \psi_n(b))(id^{\otimes n} \otimes e_l^*)(e_{i_1,\cdots,i_{n-1}} \otimes ((id \otimes h) \cdot R)(e_{i_n,i_l})) \\
= \;& \sum_{l=1}^{k} (h^{\otimes n} \cdot \psi_n(b))(e_{i_1,\cdots,i_{n-1}} \otimes ((id \otimes e_l^*)((id \otimes h) \cdot R)(e_{i_n,i_l}))) \\
= \;& (h^{\otimes n} \cdot \psi_n(b))(e_{i_1,\cdots,i_{n-1}} \otimes \left( \sum_{l=1}^{k} (id \otimes e_l^*)((id \otimes h) \cdot R)(e_{i_n,i_l}) \right)) \\
= \;& (h^{\otimes n} \cdot \psi_n(b))(e_{i_1,\cdots,i_{n-1}} \otimes (\mathrm{trace}_2((id \otimes h) \cdot R)(e_{i_n}))) \\
= \;& (h^{\otimes n} \cdot \psi_n(b))(e_{i_1,\cdots,i_{n-1}} \otimes e_{i_n}) = (h^{\otimes n} \cdot \psi_n(b))(e_{i_1,\cdots,i_n})
\end{aligned}$$

Thus $\mathrm{trace}_2(h^{\otimes n+1} \cdot \psi_{n+1}(b\sigma_n)) = h^{\otimes n}\psi_n(b)$. By applying Exercise 3.1, we obtain

$$\mathrm{trace}(h^{\otimes n+1} \cdot \psi_{n+1}(b\sigma_n))) = \mathrm{trace}(\mathrm{trace}_2(h^{\otimes n+1} \cdot \psi_{n+1}(b\sigma_n))) = \mathrm{trace}(h^{\otimes n} \cdot \psi_n(b)).$$