Some plug twists with infinite order and Seiberg Witten invariant

Def (Cork).

(C, φ). is a cork with order $p \in \mathbb{N} \cup \{\infty\}$

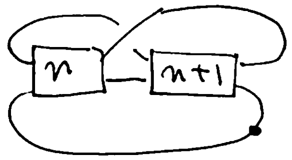
- C : contractible compact Stein
- $\varphi : \partial C \to \partial C$ diffeo.
- the order of φ is p.
- φ can not extend to $C \to C$ as a diffeo

Def (Plug)

(P, φ) is a plug with order $p \in \mathbb{N} \cup \{\infty\}$.

- P : compact Stein
- $\varphi : \partial P \to \partial P$ diffeo.
- The order of φ is p.
- φ cannot extend to $P \to P$ as a homeo.
- $\exists X, X'$ exotic. pair
  $(X - P) \cup_\varphi P = X'$

examples :

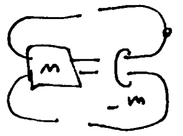
W_n : 

$\circlearrowleft \varphi$

C

$E(n) \# \overline{\mathbb{C}P^2}$, $(2n-1)\mathbb{C}P^2 \# 10n\,\overline{\mathbb{C}P^2}$

$\left.\begin{array}{l} W_{2n-1} \\ W_{2n} \end{array}\right\} \to E(2n) \# \overline{\mathbb{C}P^2}$

$W_{m,n}$: 

$\circlearrowleft \varphi$

P

$E(2n) \# m\,\overline{\mathbb{C}P^2}$

$W_{m,2n}$, $W_{m,2n+1}$

$E(n)_{p,q} \# \overline{\mathbb{C}P^2}$

$W_{1,3}$

$E(n)_K \# \overline{\mathbb{C}P^2}$

$W_{1,3}$

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Thm (Matveyev. Akbulut)

X. X', exotic simply conn. closed. 4-mfd.

$\exists\,(C, \varphi)$ a cork with order 2.

$X' = (X - C) \cup_\varphi C$ (possibly not Stein)

Question. 1 $K3 = X$

X_K: knot surgery of $K3$

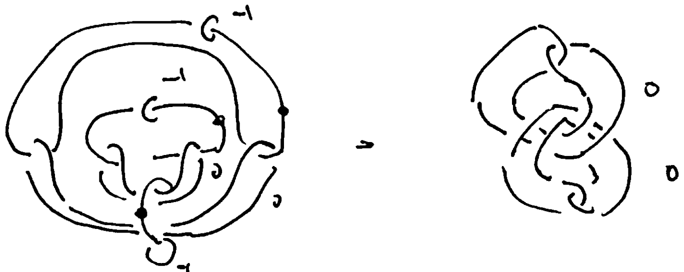
X. X_K exotic

Find. a cork for $\{X, X_K\}$.
a plug

2. X, X' spin exotic 4-mfd.

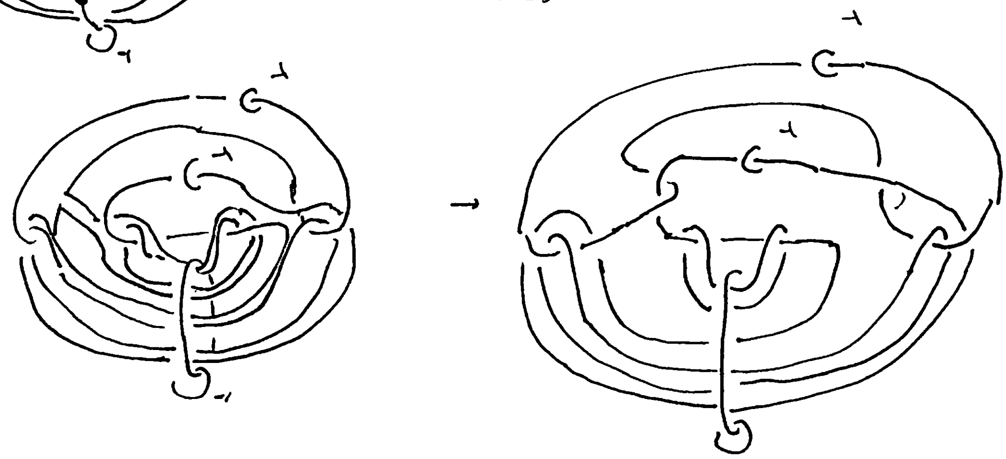
Find a cork (C, φ). for $(X X')$.
(a plug)

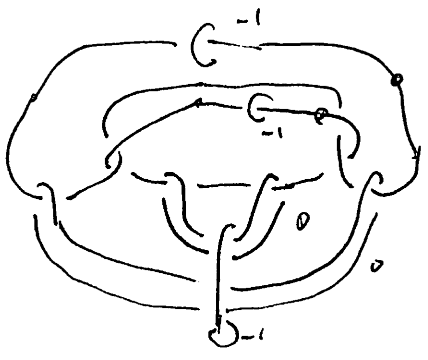
Thm (T.)



$= P$

φ

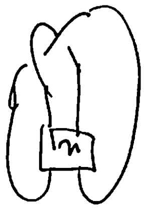




P : Stein surface. $\pi_1 = e$. $H_*(P) = H_*(S^2 \vee S^2)$

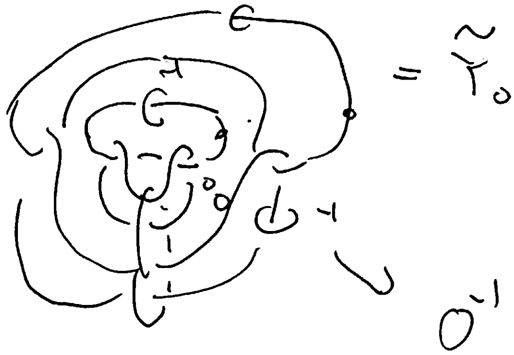
$H_*(\partial P) = H_2(S^2\times S^1 \# S^2\times S^1)$

$$\left(E(2)_{K_n} \setminus P\right) \cup_{\varphi} P = E(2)_{K_{n+1}}$$



K_n.

$Y_0 =$

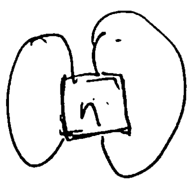


$= \tilde{Y}_0 \# \overline{\mathbb{CP}^2}$

$$Y_n := (Y_0 \setminus P) \cup_{\varphi^n} P$$

$\rightsquigarrow$

L_n.



<u>Fact</u>. $Y_0 \hookrightarrow E(1)_{L_0} = 3\mathbb{CP}^2 \# 19\overline{\mathbb{CP}^2}$ $\quad \Delta = 0$.

$Y_1 \hookrightarrow E(1)_{L_1} = K3$ $\quad \Delta = 1$

L_2 $\quad \Delta = (t_1t_2)^{1/2} + (t_1t_2)^{-1/2}$

L_3 $\quad \Delta = t_1t_2 + 1 + (t_1t_2)^{-1}$

$\vdots$

L_4 $\quad \Delta = (t_1t_2)^{3/2} + (t_1t_2)^{1/2} + (t_1t_2)^{-1/2} + (t_1t_2)^{-3/2}$

<u>Thm</u>. $Y_n \simeq Y_m$ (homeo)

$\iff n \equiv m \ (2)$

$\because$ (P, φ^n) extend to $P \to P$ as a homeo. $n \equiv 0 \ (2)$

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$(Y_n \setminus P) \cup_{\varphi_m} P = Y_{n+m}$

& $Q_{Y_n} = \begin{cases} \langle 0 \rangle \oplus H & n \text{ odd.} \\ \langle 0 \rangle \oplus \langle 1 \rangle \oplus \langle -1 \rangle & n \text{ even.} \end{cases}$ //

Question $\{Y_{2n}\}$, $\{Y_{2n+1}\}$ are exotic mfds?

$\boxed{\text{Thm } Y_0 \not\cong Y_{2n} \ \forall n.}$

$\tilde{Y}_0 = Y_0 \# \overline{\mathbb{C}P^2}$

If $Y_0 \cong Y_{2n}$. $\Rightarrow$ $\underbrace{E(1)_{L_{2n}}}_{\text{sym.}} = \underbrace{E}_{\text{sym}} \# \overline{\mathbb{C}P^2}$.

$\therefore$ $\mathcal{B}_X$: SW-basic classes of X.

$\mathcal{B}_{E(1)_{L_{2n}}} = \{\pm(T_1+T_2), \pm 3(T_1+T_2), \cdots, \pm(2n-1)(T_1+T_2)\}$

$(\pm k(T_1+T_2))^2 = 0 \quad \forall k \in \mathbb{Z}$

self int #

On the other hands. ; $H_2 = H^2$ identify

$\mathcal{B}_{E \# \overline{\mathbb{C}P^2}} = \{k \pm [C] \mid k \in \mathcal{B}_E\}$

$k + [C]$, $k - [C]$ basic class.

$k + [C] - (k - [C]) = 2[C]$

$(2[C])^2 = -4$

$\therefore$ contradiction.

$\therefore$ $Y_0 \not\cong Y_{2n}$ $\forall n$

in particular Y_n is minimal symplectic mfd.
$n \neq 0$

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Thm $m>0$

$\{Y_{2m}\}$ has inf. exotic strs. Assume. $Y_{2n} \cong Y_{2m}$.

$Q = \langle 0\rangle \oplus \langle -1\rangle \oplus \langle 1\rangle$. とする.

$(\mathbb{Z}^3, Q) \xrightarrow{\varphi} (\mathbb{Z}^3, Q)$ isomorphism

$$\Rightarrow \quad \varphi = \begin{pmatrix} \varepsilon_1 & a & b \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}$$

$\because$ $e_1\ e_2\ e_3$ $\quad e_1^2 = 0.\ e_2^2 = 1.\ e_3^2 = -1.$

$$\varphi(e_i) = \sum_{j=1}^{3} \alpha_{ij} e_j$$

inte. #. $\rightsquigarrow \begin{pmatrix} \pm 1 & a & b \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}$

$$\begin{array}{ccc} (H_2(Y_{2m}), Q_{Y_{2m}}) & \xrightarrow{f} & (H_2(Y_{2n}), Q_{Y_{2n}}) \\ \downarrow \varphi_m & & \downarrow \varphi_n \\ (\mathbb{Z}^3, Q) & \xrightarrow{\varphi} & (\mathbb{Z}^3, Q) \end{array}$$

$(H_2(Y_{2m}), Q_{Y_{2m}}) \to (\mathbb{Z}^3, \langle 0\rangle \oplus \langle -1\rangle \oplus \langle 1\rangle)$

$\{T_1\ T_2\ S\}$
$$Q_{Y_{2m}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -2m^2-m-1 \end{pmatrix}$$

$$(T_1\ T_2\ S)\begin{pmatrix} 1 & 0 & 0 \\ 0 & -m^2-\frac{m}{2} & m^2+\frac{m}{2}+1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\left(\left(-m^2-\frac{m}{2}\right)T_2 - S\right)^2 = -2\left(-m^2-\frac{m}{2}\right) + (-2m^2-m-1)$$
$$= 2m^2+m + (-2m^2-m-1) = -1$$

$$\left(\left(m^2+\frac{m}{2}+1\right)T_2 + S\right)^2 = 2\left(m^2+\frac{m}{2}+1\right) + (-2m^2-m-1) = 1$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -m^2-\frac{m}{2} & m^2+\frac{m}{2}+1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} \varepsilon_1 & a & b \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -n^2-\frac{n}{2}-1 \\ 0 & 1 & -n^2-\frac{n}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & (\varepsilon_3-\varepsilon_2)\left(m^2-\frac{m}{2}\right)+\varepsilon_3 & (\varepsilon_3-\varepsilon_2)\left(m^2-\frac{m}{2}\right)\left(n^2-\frac{n}{2}\right) - \varepsilon_3\left(n^2+\frac{n}{2}\right) + \varepsilon_2\left(m^2-\frac{m}{2}\right. \\ 0 & \varepsilon_3-\varepsilon_2 & -(\varepsilon_3-\varepsilon_2)\left(n^2+\frac{n}{2}\right)+\varepsilon_2 \end{pmatrix}$$

$n \equiv m \equiv 0 \ (2)$

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$$f(S) = \varepsilon_1\left((a\varepsilon_2 + b\varepsilon_3)(m^2+\tfrac{m}{2}) + a\varepsilon_2\right)T_1'$$
$$+\left\{(\varepsilon_2-\varepsilon_3)(m^2+\tfrac{m}{2})(n^2+\tfrac{n}{2}) - \varepsilon_3(m^2+\tfrac{m}{2}) + \varepsilon_2(n^2+\tfrac{n}{2})\right\}T_2'$$
$$+\left\{(\varepsilon_2-\varepsilon_3)(m^2+\tfrac{m}{2}) + \varepsilon_2\right\}S'$$

$$T_1', T_2', S' \subset Y_{2n}$$

$$f(S)\cdot(T_1+T_2) = (\varepsilon_2-\varepsilon_3)(m^2+\tfrac{m}{2}) + \varepsilon_2.$$

$$k = PD\left(\varepsilon_2(n-1)(T_1+T_2)'\right)$$

$$k(S) = (n-1)\left((2m^2+m)\eta+1\right) \qquad \eta = \frac{1-\varepsilon_2\varepsilon_3}{2} = \begin{cases}0\\1\end{cases}$$

$$\chi(S) - S^2 - k(S) = 2 - 2m(m-1) + (2m^2+m+1)$$
$$-(n-1)\left((2m^2+m)\eta+1\right)$$
$$= 3m - n + 4 - (n-1)(2m^2+m)\eta$$

$$\cancel{k\cdot f(S) = (n-1)\left((\varepsilon_3\varepsilon_2-1)(4n^2+1\right)}$$

$$\cancel{(aT_1+bT_2+cS)^2 = 2bc\,T_2S + c^2S^2}$$

$$\cancel{f(S) = \qquad S' \subset Y_{2n}.}$$

$$\cancel{(T_1\ T_2\ S)\,P \qquad S \to}$$

$$\cancel{f(S') \quad \left((m^2+\tfrac{m}{2}+1)T_1 + S\right)^2}$$

$$\cancel{= 2(m^2+\tfrac{m}{2}+1) + (-2m^2-2n-1) = 1}$$

$$\cancel{S \to (T_1\ T_2\ S)\,Q\begin{pmatrix}T_1\\T_2\\S\end{pmatrix}}$$

$$\cancel{Q_{Y_{2m}} = \begin{pmatrix}(T_1\ T_1) & \\ & (S\ S)\end{pmatrix}}$$

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If $3m+4<n$, then. $\chi(S)-S^2-k(S)=2l<0$
Then $k+2PD(S)$ is also basic class.

[Thm (0.S) X: smooth 4-mfd
$\Sigma\subset X$: a smooth embedded closed 2-dim. submfd.

K: a basic class of X.
$$\chi(\Sigma)-\Sigma^2-K(\Sigma)=2n<0.$$
$$\varepsilon=\operatorname{sgn}(K(\Sigma))$$
Then, $K+2\varepsilon PD(\Sigma)$ is also a basic class.]

Namely. $2f(S)=Z(*)T_1'+2(**)T_2'+2\underbrace{\left((\varepsilon_2-\varepsilon_3)(m^2+\frac{m}{2})+\varepsilon_2\right)}_{\overset{\shortparallel}{0}}S$

$$\therefore (\varepsilon_2-\varepsilon_3)(2m^2+m)+\varepsilon_2=0$$
Since $m\equiv 0\ (2)$ $(\varepsilon_2-\varepsilon_3)(2m^2+m)+\varepsilon_2\equiv 1\ (2)$
contradiction

$m=n=1\ (2)$ case
$$\begin{pmatrix}1&0&0\\0&1&m^2+\frac{m+1}{2}\\0&0&1\end{pmatrix}\begin{pmatrix}\varepsilon_1&a&b\\0&\varepsilon_2&0\\0&0&\varepsilon_2\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&-n^2-\frac{n+1}{2}\\0&0&1\end{pmatrix}$$
$$\wr$$
$$\begin{pmatrix}1&0&0\\0&1&m^2+\frac{m+1}{2}\\0&0&1\end{pmatrix}\begin{pmatrix}\varepsilon_1&a&b\\0&0&\varepsilon_2\\0&\varepsilon_2&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&-n^2-\frac{n+1}{2}\\0&0&1\end{pmatrix}$$

$$\varphi(S)=\varepsilon_1\varepsilon_2\left(a(m^2+\frac{m+1}{2})-b\right)T_1'+\varepsilon_2(n-m)\left((n+m)+\frac{1}{2}\right)T_2'+S'$$
$$\wr$$
$$\varphi(S)=\varepsilon_1\varepsilon_2\left(b(m^2+\frac{m+1}{2})-a\right)T_1'+\varepsilon_2\left(1-(m^2+\frac{m+1}{2})(n^2+\frac{n+1}{2})\right)T_2'+\varepsilon_2\left(m^2+\frac{m+1}{2}\right)S'$$

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$3m+3 < n-1$

$\varphi(S)\cdot(T_1'+T_2') = 1$

or

$\varphi(S)\cdot(T_1'+T_2') = \varepsilon_2(m^2+\frac{m+1}{2})$

$k = (n-1)(T_1'+T_2')$ or $\varepsilon_2(n-1)(T_1'+T_2')$

$\chi(S) - S^2 - k(S) = 2-2m(m-1) + 2m^2+m+1 - n-1.$

$= 3m+2-n.$

or

$\chi(S) - S^2 - k(S) = 2-2m(m-1) + 2m^2+m+1 - (n-1)(m^2+\frac{m+1}{2})$

$= 3m+3 - (n-1)(m^2+\frac{m+1}{2}).$

$=$

If $3m+4 < n$

$3m+3 - (n-1)(m^2+\frac{m+1}{2}) < n-1-(n-1)(m^2+\frac{m+1}{2})$

$< -(n-1)(m^2+\frac{m-1}{2})$

$< -(n-1)(1^2+\frac{1-1}{2}) = -n+1 < 0.$

$\therefore$ $k+2S$ is also a basic class.

But the coefficients of S' in S is not zero..

$\Rightarrow$ contradiction

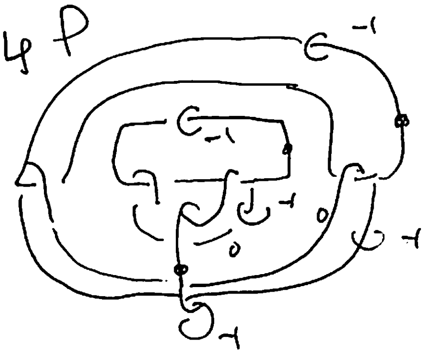
$\therefore$ $Y_m \neq Y_{2n}.$

$\therefore$ $\{Y_{2n}\}$

$\{Y_{2n+1}\}$ has both infinitely many exotic structures.

$Z_n = (Z_0 \setminus P) \cup_\varphi P$

$Z_0 =$



$Z_n \hookrightarrow E(1)_{L_n}$.

$b_2(Z_n) = 4$.

$Z_0 = \tilde{Z}_0 \# 2\overline{\mathbb{CP}}^2$

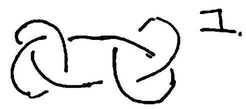
$Q_{Z_0} \cong \oplus^2\langle 1\rangle \oplus^2 \langle -1\rangle$

Thm. (1) Z_n : simply conn.

(2) $Q_{Z_{2n}} = \oplus^2\langle 1\rangle \oplus^2\langle -1\rangle$

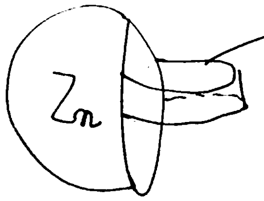
$Q_{Z_{2n+1}} = \oplus^2 H$

(3) $\partial Z_{2n} = S^3_1(\text{granny})$.



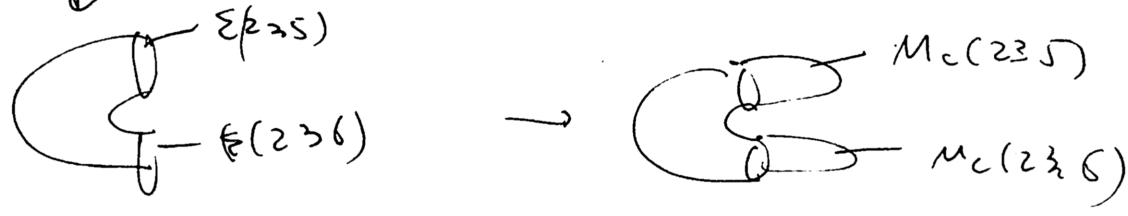
(4) Z_n : irreducible

$Z_n \hookrightarrow E(1)_{L_n}$



along essential torus in S^3_1(granny)

this torus is unique.



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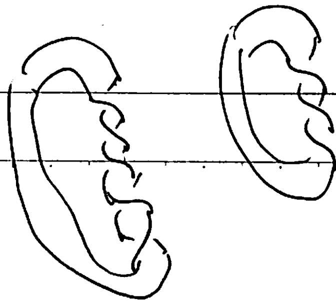
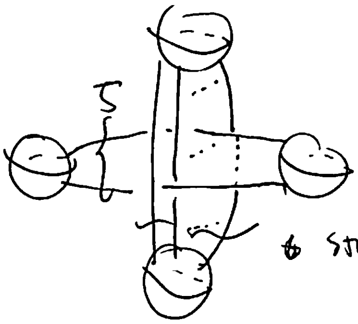


Way to attach these Milnor fibers are unique.

Lemma any diffeo of $\Sigma(2,3,5)$ or $\widetilde{\Sigma}(2,3,6)$ extends to $M_c(2,3,5)$, $\widetilde{M}_c(2,3,5)$



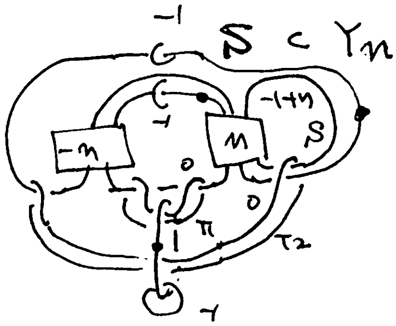
$\Rightarrow Z_{2n} \cup (\text{round-}h) \cup \widetilde{M}_c(235) \cup M_c(235)$

$= E(1)_{L_{2n}}$

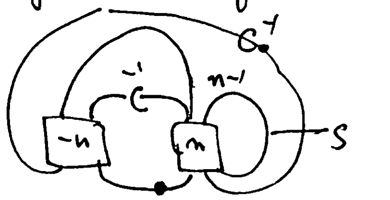
$Z_{2n} \cong Z_{2m}$

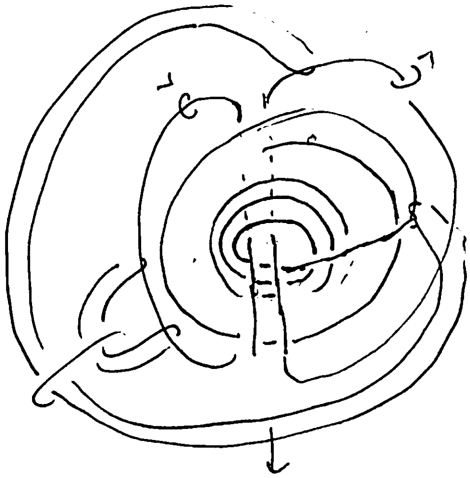
$\Rightarrow E(1)_{L_{2n}} \cong E(1)_{L_{2m}}$

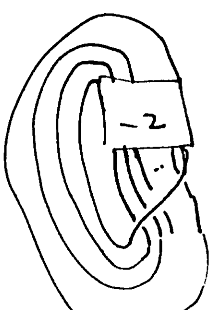
$S \subset Y_n$



genus. self int. #







$g(S) = \frac{(n-1)(2n-2)}{2}$

$= (n-1)^2$

$(n, 2n-1)$ torus knot