Horikawa surfaces $H(2n)$, $H'(n)$.

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1 Definition of $X(m,n)$, $U(m,n)$, $H(n)$ and $H'(n)$.

This note is a part of Section 7.3, 7.4, 8.4, 8.5 in [1].

Let Y be a complex surface and B a curve, and the map

$$X \to Y \supset B$$

the branched covering X of Y along B. Then, the characteristic classes are computed as follows:

$$c_1^2(X) = 2\left(c_1(Y) - \frac{1}{2}[B]\right)^2 = 2(c_1^2(Y) - \chi(B)) - \frac{3}{2}[B]^2, \ \ c_2(X) = 2c_2(Y) - \chi(B)$$

$$\chi_h(X) = \frac{c_1^2(X) + c_2(X)}{12} = 2\chi_h(Y) - \frac{\chi(B)}{4} - \frac{1}{8}[B]^2.$$

Thus, one can get what the $c_1^2(X)$ and $\chi_h(X)$ do not depend only on the homology class of $B \subset Y$.

Definition 1. *$X(m,n)$ is the desingularization of the double branched cover of $\mathbb{F}_0$ branched along $B_{m,n}$, where $\mathbb{F}_p$ is the Hirzebruch surface with the Euler number p and*

$$B_{m,n} = \cup_{i=1}^{2m}(\mathbb{C}P^1 \times \{p_i\}) \cup \cup_{j=1}^{2n}(\{q_j\} \times \mathbb{C}P^1).$$

$X(m+1,n+1)$ is also constructed by resolving the singularities of the quotient

$$\mathrm{pr}_2 : \Sigma_m \times \Sigma_n/(\sigma_m \times \sigma_n) = (\Sigma_m \times \Sigma_n)/\mathbb{Z}_2 \to \Sigma_n/\sigma_n = S^2.$$

This is a singular fibration with the fiber genus m.

$X(m,n)$ is the double branched cover of $\mathbb{F}_0$ branched $\tilde{B}_{m,n}$. $\tilde{B}_{m,n}$ is a smooth curve by slight deformation of $B_{m,n}$. The characteristic classes are

$$c_1^2(X(m,n)) = 4(n-2)(m-2), \ \ \chi_h(X(m,n)) = (n-1)(m-1) + 1.$$

Proposition 1. *$X(m,n)$ are all simply-connected.*

Proposition 2. *$X(2,n)$ is diffeomorphic to $E(n)$.*

In the Hirzebruch surface $\mathbb{F}_{2n}$, we consider the following class:

$$[B'_{m,n}] = 2m[S_{2n}] - 2n[F_{2n}],$$

where S_{2n} and F_{2n} are the section and the fiber of $\mathbb{F}_{2n}$ respectively. Thus, the homology class is the same as $[B'_{m,n}] = [B_{m,(m-1)n}] \in H_2(\mathbb{F}_0, \mathbb{Z})$. Via the diffeomophism $\mathbb{F}_0 \cong \mathbb{F}_{2n}$, the homology classes of the fiber and the section are moved as follows:

$$[F_0] \mapsto [F_{2n}]$$

$$[S_0] \mapsto [S_{2n}] - n[F_{2n}].$$

The class of the infinite section S_∞ of $\mathbb{F}_{2n}$ is represented by $[S_\infty] = [S_{2n}] - 2n[F_{2n}]$ and $[S_\infty]^2 = -2n$ holds. Thus, the class of $B'_{m,n}$ is equivalent to $[B'_{m,n}] = (2m-1)[S_{2n}] + [S_\infty]$.

Definition 2. *We define $U(m,n)$ to be the desingularization of the double branched cover of $\mathbb{F}_{2n}$ branched along $2m-1$ affine sections S_{2n} and the infinite section S_∞.*

$$U(m,n) \to \mathbb{F}_{2n} \supset 2m-1 \ S_{2n}\text{'s and } S_\infty$$

Since we have

$$[B'_{m,n}] = [B_{m,(m-1)n}].$$

Hence, we have the following:

Proposition 3. *$U(m,n)$ and $X(m,(m-1)n)$ have the same characteristic number.*

Proposition 4. *$U(2,n)$ is diffeomorphic to $E(n) = X(2,n)$.*

We consider the case of $m = 3$. We call $X(3,n)$ and $U(3,n)$

$$H(n) := X(3,n),\ H'(n) = U(3,n) \text{ respectively.}$$

Proposition 5. *$H(2n)$ and $H'(n)$ have the same characteristic number.*

Proposition 6. *If n is even, then $H(2n)$ and $H'(n)$ are not homotopy equivalent.*

Proposition 7. *$H(2n)$ and $H'(n)$ are homoemorphic iff n is odd.*

Proposition 8 (Horikawa[2])**.** *$H(2n)$ and $H'(n)$ are not deformation equivalent.*

Proposition 9 (The $n = 1$ case)**.** *$H(2) = X(3, 2) = X(2, 3) = E(3)$ is homeomorphic but non-diffeomorphic to $H'(1) = U(3, 1)$.*

$H'(1) = U(3, 1)$ contains a -1-sphere coming from the infinite section, but $H(2) = E(3)$ is minimal.

Problem 1. *For any odd number n, $H(2n)$ is non-diffeomorphic to $H'(n)$.*

2 Horikawa surface.

$H(n)$ admits a genus 2 fibration and a generalized fiber-sum of three copies of $X(1, n) = \mathbb{C}P^2 \#^{4n+1} \overline{\mathbb{C}P^2}$. Also, since

$$c_1^2(H(n)) = 4(n-2), \chi_h(H(n)) = 2n - 1,$$

we have $c_1^2(H(n)) = 2\chi_h(H(n)) - 6$ holds, thus these manifolds lie on the Noether line $y = 2x - 6$ on the (χ_h, c_1^2)-coodinate.

Definition 3. *If a complex surface X satisfies $c_1^2(X) = 2\chi_h(X) - 6$, then X is called Horikawa surface.*

Thus $H(2n)$ and $H'(n)$ are Horikawa surfaces and satisfy

$$c_1^2(H'(n)) = c_1^2(H(2n)) = 8(n-1), \ \chi_h(H'(n)) = \chi_h(H(2n)) = 4n - 1.$$

When $n > 1$, $H(2n)$ and $H'(n)$ are both minimal surfaces of general type. Thus they have the same Seiberg-Witten invariant

$$SW(\pm c_1) = \pm 1.$$

We put on the next page the pictures of $U(m, n)$ and $X(m, n)$ in [1].

References

[1] R. Gompf and A. Stipsicz, *4-manifolds and Kirby calculus*, Graduate Studies in Mathematics, 20. American Mathematical Society.

[2] E. Horikawa, *Local deformation of pencil of curves of genus two*, Proc. Japan Acad. 64(series A), 1988 241-244.

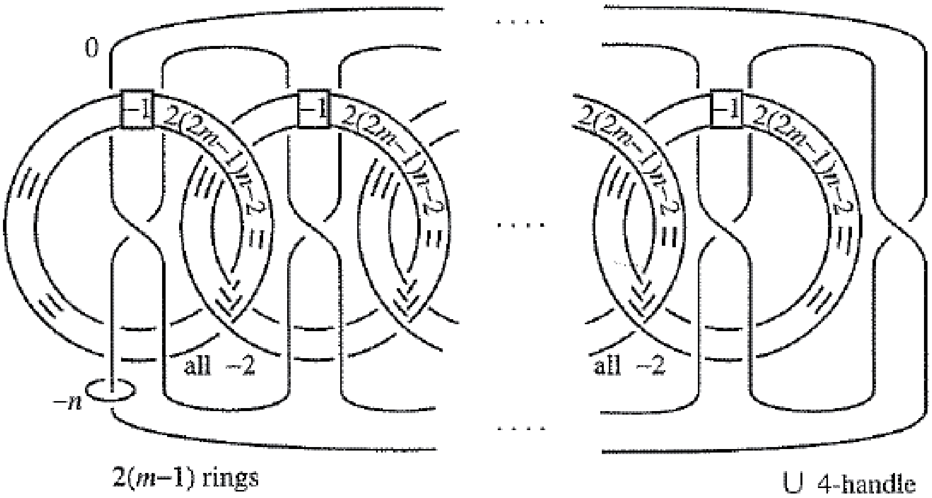

Figure 8.32. Complex surface $U(m,n)$ — general type for $m \geq 3$.

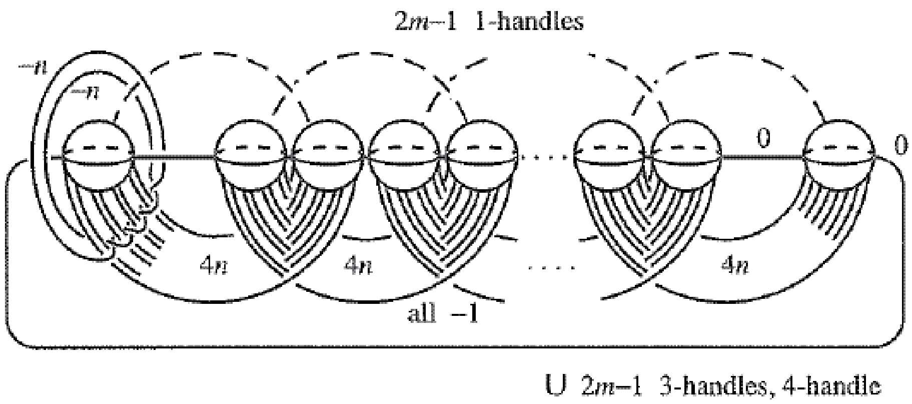

Figure 8.33. Lefschetz fibration $X(m,n) \to S^2$.