Limit points of fractional parts of geometric sequences *

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Abstract

Let $\alpha > 1$ be an algebraic number and ξ a nonzero real number. In this paper, we compute the range of the fractional parts $\{\xi\alpha^n\}$ $(n = 0, 1, \ldots)$. In particular, we estimate the maximal and minimal limit points. Our results show, for example, that if $\theta(= 24.97\ldots)$ is the unique zero of the polynomial $2X^2 - 50X + 1$ with $X > 1$, then there exists a nonzero ξ^* satisfying $\limsup_{n\to\infty}\{\xi^*\theta^n\} \le 0.02127\ldots$. On the other hand, we also prove for any nonzero ξ that $\limsup_{n\to\infty}\{\xi\theta^n\} \ge 0.02003\ldots$.

1 Introduction

Koksma [11] proved for nonzero ξ that the geometric progressions $\xi\alpha^n$ $(n \ge 0)$ are uniformly distributed modulo 1 for almost all $\alpha > 1$. He also showed for $\alpha > 1$ that $\xi\alpha^n$ $(n \ge 0)$ are uniformly distributed modulo 1 for almost all real ξ. There is, however, no criterion of uniform distribution for the series $\xi\alpha^n$ $(n \ge 0)$ with given $\alpha > 1$ and $\xi \ne 0$.

Let μ be the Haar measure of the torus $\mathbb{R}/\mathbb{Z}$ with $\mu(\mathbb{R}/\mathbb{Z}) = 1$. We write the canonical map from $\mathbb{R}$ onto $\mathbb{R}/\mathbb{Z}$ by τ. For any interval $I \subset \mathbb{R}$, we call $J = \tau(I)$ an interval in $\mathbb{R}/\mathbb{Z}$.

We take $\alpha > 1$ and $\xi \ne 0$. Let $J(\alpha, \xi)$ be one of the shortest interval in $\mathbb{R}/\mathbb{Z}$ containing all limit points of $\xi\alpha^n \bmod \mathbb{Z}$ $(n \ge 0)$. Note that $J(\alpha, \xi)$ is uniquely determined unless the set of limit points of $\xi\alpha^n \bmod \mathbb{Z}$ $(n \ge 0)$ consists of two elements. We now recall the definition of Pisot and Salem numbers. Pisot numbers are algebraic integers greater than 1 whose conjugates different from themselves have absolute values strictly less than 1. Salem numbers are algebraic integers greater than 1 which have at least one conjugate with modulus 1 and exactly one conjugate outside the unit circle. Pisot [14] proved for an algebraic $\alpha > 1$ and a nonzero ξ that if the sequence $\xi\alpha^n \bmod \mathbb{Z}$ $(n \ge 0)$ has only finitely many limit points, then α is a Pisot number and $\xi \in \mathbb{Q}(\alpha)$. For further details of powers of Pisot and Salem numbers we refer the reader to [2].

We put

$$\mu(\alpha, \xi) = \mu(J(\alpha, \xi)).$$

For example, $J(\alpha, 1) = \{0 \bmod \mathbb{Z}\}$ and $\mu(\alpha, 1) = 0$, where α is a Pisot number, because the trace of α^n is a rational integer. Tijdeman [15] proved for every

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*2000 Mathematics Subject Classification : 11J71, 11J54, 11J25, 11A63 half integer $\alpha = N/2 > 2$ that there exists a nonzero $\xi = \xi(\alpha)$ such that

$$\mu(\alpha, \xi) \leq \frac{1}{2(\alpha - 1)}.$$

Flatto [8] pointed out that, for each rational $\alpha = a/b > 1$, there is a nonzero $\xi = \xi(\alpha)$ with

$$\mu(\alpha, \xi) \leq \frac{b-1}{b(\alpha - 1)} = \frac{b-1}{a-b}. \tag{1.1}$$

He proved the inequality above using Tijdeman's method.

Koksma's Theorem implies that if $\alpha > 1$ is given, then, for almost all ξ, the set

$$\{\xi\alpha^n \bmod \mathbb{Z} | n = 0, 1, \ldots\}$$

is dense in $\mathbb{R}/\mathbb{Z}$. In particular, $\mu(\alpha, \xi) = 1$. On the other hand, Tijdeman [15] showed that if $\alpha > 2$ is given, then there exists a nonzero $\xi = \xi(\alpha)$ with

$$\{\xi\alpha^n\} \leq \frac{1}{\alpha - 1} \ (n = 0, 1, \ldots), \tag{1.2}$$

where $\{\xi\alpha^n\}$ denotes the fractional part of $\xi\alpha^n$. In particular, such α and ξ satisfy

$$\mu(\alpha, \xi) \leq \frac{1}{\alpha - 1}. \tag{1.3}$$

The author [10] proved the following:
Let ξ be a nonzero real number. Take arbitrary positive numbers δ and M. Then there exists an α satisfying $\alpha > M$ and

$$\mu(\xi, \alpha) \leq \frac{1 + \delta}{\alpha}.$$

Let $\iota(= 2.025\ldots)$ be the unique solution of $34X^3 - 102X^2 + 75X - 16 = 0$ with $X > 2$. Dubickas [7] verified for $1 < \alpha < \iota$ that there is a nonzero $\xi = \xi(\alpha)$ such that

$$\mu(\alpha, \xi) \leq 1 - \frac{2(\alpha - 1)^2}{9(2\alpha - 1)^2}. \tag{1.4}$$

It is easy to check that if $2 < \alpha < \iota$ is given, then

$$1 - \frac{2(\alpha - 1)^2}{9(2\alpha - 1)^2} < \frac{1}{\alpha - 1}.$$

Thus (1.4) is stronger than (1.3) for $2 < \alpha < \iota$. We now review Dubickas's estimation of maximal and minimal limit points of the sequence $\{\xi\alpha^n\}$ $(n = 0, 1, \ldots)$. Let us define notation about polynomials and algebraic numbers. Let $B(X) = b_m X^m + \cdots + b_0$ be an arbitrary polynomial with real coefficients. We denote the length of $B(X)$ by

$$L(B) = |b_m| + \cdots + |b_0|.$$

Let $\alpha > 1$ be an algebraic number with minimal polynomial $P_\alpha(X) = a_d X^d + \cdots + a_0 \in \mathbb{Z}[X]$, where $a_d > 0$ and $\gcd(a_d, \ldots, a_0)$=1. Define the length of α by

$$L(\alpha) = L(P_\alpha(X)).$$

Put furthermore

$$L_+(\alpha) = \sum_{i=0}^{d} \max\{0, a_i\},\ L_-(\alpha) = \sum_{i=0}^{d} \max\{0, -a_i\}.$$

Next, let $l(\alpha)$ be the reduced length of α defined by

$$l(\alpha) = \min\{l'(\alpha), l'(\alpha^{-1})\},$$

where

$$l'(\alpha) = \inf_{B(X)\in\mathbb{R}[X]} \left\{L(B(X)P_\alpha(X)) \middle| B(X) \text{ is monic}\right\}.$$

Formulae about $l(\alpha)$ and $l'(\alpha)$ were studied by Dubickas [5]. Take a nonzero real ξ. If α is a Pisot or Salem number, then assume $\xi \notin \mathbb{Q}(\alpha)$. We write the integral part of a real number x by $[x]$. Dubickas [6] showed that the sequence

$$\left(\sum_{i=0}^{d} a_{d-i}[\xi\alpha^{n-i}]\right)\ (n = 0, 1, \ldots)$$

is not ultimately periodic. In particular,

$$\left|\sum_{i=0}^{d} a_{d-i}[\xi\alpha^{n-i}]\right| \geq 1$$

for infinitely many $n \geq 0$ because nonzero integers occur infinitely many times in this sequence. Since

$$0 = \sum_{i=0}^{d} a_{d-i}\xi\alpha^{n-i} = \sum_{i=0}^{d} a_{d-i}([\xi\alpha^{n-i}] + \{\xi\alpha^{n-i}\}),$$

we have

$$\left|\sum_{i=0}^{d} a_{d-i}\{\xi\alpha^{n-i}\}\right| = \left|\sum_{i=0}^{d} a_{d-i}[\xi\alpha^{n-i}]\right| \geq 1$$

for infinitely many n. Thus we get

$$\limsup_{n\to\infty}\{\xi\alpha^n\} \geq \min\left\{\frac{1}{L_+(\alpha)}, \frac{1}{L_-(\alpha)}\right\}. \tag{1.5}$$

Moreover, Dubickas [6] proved

$$\limsup_{n\to\infty}\{\xi\alpha^n\} - \liminf_{n\to\infty}\{\xi\alpha^n\} \geq \frac{1}{l(\alpha)}. \tag{1.6}$$

In this paper, we calculate the range of the sequence $\{\xi\alpha^n\}$ $(n = 0, 1, \ldots)$ in the case where $\alpha > 1$ is an algebraic number. The main results are stated in Section 2 and proved in Section 6. First, we construct a nonzero $\xi = \xi(\alpha)$ and improve (1.3), (1.4) by giving an interval in $\mathbb{R}/\mathbb{Z}$ which includes all limit points of the sequence $\xi\alpha^n \bmod \mathbb{Z}$ $(n \geq 0)$. Next, we give new estimation of the maximal and minimal limit points of the sequence $\{\xi\alpha^n\}$ $(n = 0, 1, \ldots)$. The auxiliary results are given in Sections 3,4, and 5. Moreover, in Section 7 we introduce Mahler's Z-numbers (cf. [3, 8, 9, 12]) and discuss their generalization.

2 Main results

At first, we sharpen the inequality (1.3) in the case where $\alpha > 1$ is an algebraic number whose conjugates different from itself have absolute values less than 1. For $t, m \geq 1$, put

$$\rho_m(X_1, \ldots, X_t) = \begin{cases} 1 & t = m = 0 \\ 0 & t = 0,\ m \geq 1 \\ \sum_{\substack{i_1, \ldots, i_t \geq 0 \\ i_1 + \cdots + i_t = m}} X_1^{i_1} \cdots X_t^{i_t} & t \geq 1 \end{cases} \tag{2.1}$$

THEOREM 2.1. *Let $\alpha > 1$ be an algebraic number of degree d and let $a_d (> 0)$ be the leading coefficient of the minimal polynomial of α. We denote the conjugates of α by $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$. Assume that $|\alpha_j| < 1$ for $2 \leq j \leq d$. Let*

$$\nu = \sum_{h=0}^{\infty} |\rho_h(\alpha_2, \ldots, \alpha_d)|. \tag{2.2}$$

Then there exists a nonzero $\xi = \xi(\alpha)$ such that

$$\mu(\alpha, \xi) \leq \frac{(a_d - 1)\nu}{a_d(\alpha - 1)}. \tag{2.3}$$

Note that if α is a rational number, then (2.3) coincides with (1.1). Next, we consider the case where α is a quadratic irrational number. We give an interval in $\mathbb{R}/\mathbb{Z}$ which includes $J(\alpha, \xi)$.

COROLLARY 2.2. *Let $\alpha > 1$ be an quadratic irrational number and $P_\alpha(X)$ be its minimal polynomial. We denote the leading coefficient of $P_\alpha(X)$ by $a_2 (> 0)$. Assume that the conjugate α_2 of α has the absolute value less than 1 and that $a_2 \geq 2$.*
(1) *If $0 < \alpha_2 < 1$, then there exists a nonzero $\xi = \xi(\alpha)$ such that, for any $n \geq 0$*

$$\{\xi \alpha^n\} < \frac{a_2 - 1}{|P_\alpha(1)|}.$$

In particular,

$$J(\xi, \alpha) \subset \tau\left(\left[0, \frac{a_2 - 1}{|P_\alpha(1)|}\right]\right).$$

(2) *If $-1 < \alpha_2 < 0$, then there exists a nonzero $\xi = \xi(\alpha)$ such that*

$$J(\xi, \alpha) \subset \tau\left(\left[\frac{(a_2 - 1)\alpha_2}{a_2(\alpha - 1)(1 - \alpha_2^2)}, \frac{a_2 - 1}{a_2(\alpha - 1)(1 - \alpha_2^2)}\right]\right).$$

Example 2.1. Let $\theta_1 (= 24.97\ldots)$ be the unique zero of the polynomial $2X^2 - 50X + 1$ with $X > 1$. Then by Tijdeman's result (1.2) there exists a nonzero $\xi = \xi(\theta_1)$ with

$$\{\xi \theta_1^n\} \leq \frac{1}{\theta_1 - 1} = 0.04170\ldots$$

for each $n \geq 0$. Since the conjugate of θ_1 is on the interval $(0, 1)$, by Corollary 2.2 there exists a nonzero $\xi = \xi(\theta_1)$ such that for each $n \geq 0$

$$\{\xi \theta_1^n\} < \frac{1}{47} = 0.02127\ldots.$$

We now compare these estimations with the Dubickas's lower bound (1.5) of the maximal limit point. For any nonzero ξ we have

$$\limsup_{n\to\infty} \{\xi \theta_1^n\} \geq \min\left\{\frac{1}{L_+(\theta_1)}, \frac{1}{L_-(\theta_1)}\right\} = \frac{1}{50} = 0.02.$$

Note that the first statement of Corollary 2.2 gives an upper bound of the maximal limit point of the sequence $\{\xi\alpha^n\}$ $(n = 0, 1, \ldots)$. We generalize this estimation in the case where $\alpha > 1$ is an algebraic number with arbitrary degree whose conjugates different from itself are on the interval $(0, 1)$. Next, we give also an upper bound of the difference between the maximal and minimal limit points in the case where the absolute values of the conjugates of α different from itself are sufficiently small.

THEOREM 2.3. *Let $\alpha > 1$ be an algebraic number of degree d. We denote the conjugates of α by $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$. Moreover, let $P_\alpha(X)$ be the minimal polynomial of α and $a_d(> 0)$ its leading coefficient. Suppose that $a_d \geq 2$.*
(1) *Assume that*

$$0 < \alpha_j < 1 \ (2 \leq j \leq d).$$

Then there exists a nonzero $\xi = \xi(\alpha)$ satisfying

$$\{\xi\alpha^n\} < \frac{a_d - 1}{|P_\alpha(1)|}$$

for all $n \geq 0$.
(2) *Let ν be defined by (2.2). Assume that, for any j with $2 \leq j \leq d$,*

$$|\alpha_j| < 1$$

and that

$$\frac{a_d - 1}{a_d(\alpha - 1)}\nu < \frac{1}{2}, \ |P_\alpha(1)| \geq 2.$$

Then there is a nonzero $\xi = \xi(\alpha)$ satisfying

$$\limsup_{n\to\infty}\{\xi\alpha^n\} - \liminf_{n\to\infty}\{\xi\alpha^n\} \leq \frac{a_d - 1}{a_d(\alpha - 1)}\nu.$$

REMARK 2.1. If the absolute values of the conjugates of α different from itself are sufficiently small, then the assumptions of the second statement of Theorem 2.3 follow. In fact, they are rewritten by

$$\nu = \sum_{h=0}^{\infty} |\rho_h(\alpha_2, \ldots, \alpha_d)| < \frac{a_d(\alpha - 1)}{2(a_d - 1)}$$

and

$$\prod_{i=2}^{d} |1-\alpha_i| \geq \frac{2}{a_d(\alpha-1)}.$$

Example 2.2. We give an example of the first statement. Let $\theta_2(= 24.69\ldots)$ be the unique solution of $2X^3 - 50X^2 + 15X - 1 = 0$ with $X > 1$. Then Tijdeman's result (1.2) implies that there exists a nonzero $\xi = \xi(\theta_2)$ with

$$\{\xi\theta_2^n\} \leq \frac{1}{\theta_2 - 1} = 0.04219\ldots$$

for all $n \geq 0$. Since θ_2 is an algebraic number of degree 3 whose conjugates different from itself are on the interval $(0,1)$, the first statement of Theorem 2.3 means that there is a nonzero $\xi = \xi(\theta_2)$ satisfying

$$\{\xi\theta_2^n\} < \frac{1}{34} = 0.02941\ldots$$

for any n.

On the other hand, Dubickas's lower bound (1.5) implies that if $\xi \neq 0$, then

$$\limsup_{n\to\infty}\{\xi\theta_2^n\} \geq \min\left\{\frac{1}{L_+(\theta_2)}, \frac{1}{L_-(\theta_2)}\right\} = \frac{1}{51} = 0.01960\ldots.$$

Example 2.3. We introduce an example of the second statement of Theorem 2.3. Let $\theta_3(= 25.01\ldots)$ be the unique positive zero of the polynomial $2X^2 - 50X - 1$. Then, by Tijdeman's result (1.2) there exists a nonzero $\xi = \xi(\theta_3)$ fulfilling

$$\limsup_{n\to\infty}\{\xi\theta_3^n\} - \liminf_{n\to\infty}\{\xi\theta_3^n\} \leq \frac{1}{\theta_3 - 1} = 0.04163\ldots.$$

Theorem 2.3 means there is a nonzero $\xi = \xi(\theta_3)$ with

$$\limsup_{n\to\infty}\{\xi\theta_3^n\} - \liminf_{n\to\infty}\{\xi\theta_3^n\} \leq \frac{a_d - 1}{a_d(\theta_3 - 1)}\nu = 0.02124\ldots.$$

We next compare with Dubickas's lower bound (1.6). Dubickas [5] verified that if $\alpha > 1$ is a quadratic irrational number whose conjugate has absolute value less than 1, then

$$l(\alpha) = a_2\alpha + \min\{a_2, |a_0|\}.$$

Therefore, for any nonzero ξ

$$\limsup_{n\to\infty}\{\xi\theta_3^n\} - \liminf_{n\to\infty}\{\xi\theta_3^n\} \geq \frac{1}{l(\theta_3)} = 0.01959\ldots.$$

Finally, we improve Dubickas's lower bound (1.5) of the maximal limit point $\limsup_{n\to\infty}\{\xi\alpha^n\}$ in the case where $\alpha > 1$ whose conjugates are all positive.

THEOREM 2.4. *Let ξ be a nonzero real number and $\alpha > 1$ an algebraic number of degree d. We denote the leading coefficient of the minimal polynomial of α by $a_d(> 0)$. Suppose that the conjugates of α are all positive. If α is a Pisot number, then assume further $\xi \notin \mathbb{Q}(\alpha)$. We denote the conjugates of α by $\alpha_1 = \alpha, \ldots, \alpha_p, \alpha_{1+p}, \ldots, \alpha_d$, where $\alpha_i > 1$ $(1 \le i \le p)$ and $0 < \alpha_j < 1$ $(1+p \le j \le d)$. Put*

$$\eta_l = \sum_{\substack{i,j \ge 0 \\ j-i=l}} \rho_i(\alpha_1^{-1}, \ldots, \alpha_p^{-1}) \rho_j(\alpha_{1+p}, \ldots, \alpha_d).$$

Let

$$\delta_1 = \max\left\{ \frac{1}{L_+(\alpha)}, \frac{1}{L_-(\alpha)} \right\}$$

and

$$\delta_2 = \frac{1}{a_d \alpha_1 \cdots \alpha_p} \sup_{l \in \mathbb{Z}} \eta_l,$$

respectively. Then

$$\limsup_{n \to \infty} \{\xi \alpha^n\} \ge \min\{\delta_1, \delta_2\}. \tag{2.4}$$

Example 2.4. We consider the case of $\alpha = \theta_1, \theta_2$ which are defined in Examples 2.1 and 2.2, respectively. Tijdeman's result (1.2) and Dubickas's lower bound (1.5) imply

$$0.02 \le \inf_{\xi \ne 0} \limsup_{n \to \infty} \{\xi \theta_1^n\} \le 0.04170\ldots$$

and

$$0.01960\ldots \le \inf_{\xi \ne 0} \limsup_{n \to \infty} \{\xi \theta_2^n\} \le 0.04219\ldots.$$

By using Theorems 2.3 and 2.4, we obtain

$$0.02003\ldots \le \inf_{\xi \ne 0} \limsup_{n \to \infty} \{\xi \theta_1^n\} \le 0.02127\ldots$$

and

$$0.02049\ldots \le \inf_{\xi \ne 0} \limsup_{n \to \infty} \{\xi \theta_2^n\} \le 0.02941\ldots,$$

respectively. In particular, Theorem 2.4 gives improvements of (1.5) in these cases.

In the case of $\alpha = \theta_2$, we calculate δ_2 in the following way. If $l \le 0$, then

$$\eta_l = \frac{\alpha^l}{(1 - \alpha^{-1}\alpha_2)(1 - \alpha^{-1}\alpha_3)};$$

otherwise,

$$\begin{aligned}\eta_l &\le \sum_{i=0}^{\infty} \alpha^{-i}\rho_i(\alpha_2,\alpha_3)\rho_l(\alpha_2,\alpha_3) \\ &= \frac{\rho_l(\alpha_2,\alpha_3)}{(1-\alpha^{-1}\alpha_2)(1-\alpha^{-1}\alpha_3)}.\end{aligned}$$

Thus, we obtain

$$\delta_2 = \frac{1}{2\alpha(1-\alpha^{-1}\alpha_2)(1-\alpha^{-1}\alpha_3)}.$$

Let us show that Theorem 2.4 gives the best result in the case where α is a Pisot number satisfying $\delta_1 \ge \delta_2$.

THEOREM 2.5. *Let α be a Pisot number. We denote the conjugates of α by $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_d$. Suppose that all α_j are positive. Let δ_1, δ_2 be defined as in Theorem 2.4. Assume further $\delta_1 \ge \delta_2$. Then*

$$\inf_{\xi \notin \mathbb{Q}(\alpha)} \limsup_{n\to\infty} \{\xi\alpha^n\} = \delta_2.$$

Moreover, the infimum is attained by the transcendental number

$$\xi_0(\alpha) = \frac{1}{\alpha\prod_{j=2}^{d}(1-\alpha^{-1}\alpha_j)} \sum_{m=1}^{\infty} \alpha^{-m!}.$$

By applying Theorem 2.5 in the case where α is a quadratic Pisot number, we obtain the following:

COROLLARY 2.6. *Let α be a quadratic Pisot number with the conjugate α_2. Assume that $0 < \alpha_2 < 2-\sqrt{2}(= 0.5857\dots)$. Then*

$$\inf_{\xi \notin \mathbb{Q}(\alpha)} \limsup_{n\to\infty} \{\xi\alpha^n\} = \frac{1}{\alpha-\alpha_2}.$$

Moreover, the infimum is attained by the transcendental number

$$\xi_0(\alpha) = \frac{1}{\alpha-\alpha_2} \sum_{m=1}^{\infty} \alpha^{-m!}.$$

Example 2.5. Let $\theta_4 = 2+\sqrt{3}(= 3.732\cdots)$. Then the conjugate θ_4' satisfies $0 < \theta_4' < 2-\sqrt{2}$. Thus Corollary 2.6 implies

$$\inf_{\xi \notin \mathbb{Q}(\theta_4)} \limsup_{n\to\infty} \{\xi\theta_4^n\} = \frac{1}{2\sqrt{3}} = 0.2886\dots.$$

3 Symmetric homogeneous polynomials

Let us introduce basic results of the symmetric polynomials $\rho_m(X_1, \ldots, X_t)$ with $t, m \geq 0$ defined by (2.1). In this section we fix $t \geq 1$. The generating function of these polynomials is given by

$$\begin{aligned}\sum_{m=0}^{\infty} \rho_m(X_1, \ldots, X_t) Y^m &= \sum_{m=0}^{\infty} \sum_{\substack{i_1, i_2, \ldots, i_t \geq 0 \\ i_1+i_2+\cdots+i_t=m}} (X_1 Y)^{i_1} (X_2 Y)^{i_2} \cdots (X_t Y)^{i_t} \\ &= \sum_{i_1, i_2, \ldots, i_t \geq 0} (X_1 Y)^{i_1} (X_2 Y)^{i_2} \cdots (X_t Y)^{i_t} \\ &= \frac{1}{\prod_{i=1}^{t} (1 - X_i Y)}. \end{aligned} \tag{3.1}$$

Therefore

$$\left(\sum_{m=0}^{\infty} \rho_m(X_1, \ldots, X_t) Y^m \right) \prod_{i=1}^{t} (1 - X_i Y) = 1,$$

and so, for $m \geq 1$,

$$\sum_{h=0}^{\min\{m,t\}} (-1)^h \rho_{m-h}(X_1, \ldots, X_t) e_h(X_1, \ldots, X_t) = 0, \tag{3.2}$$

where $e_h(X_1, \ldots, X_t)$ is the elementary symmetric polynomial of degree h, namely,

$$e_h(X_1, \ldots, X_t) = \begin{cases} 1 & (h = 0), \\ \displaystyle\sum_{1 \leq i_1 < i_2 < \cdots < i_h \leq t} X_{i_1} X_{i_2} \cdots X_{i_h} & (h \geq 1). \end{cases} \tag{3.3}$$

The following result is Lemma 3.1 of [10]:

LEMMA 3.1. *If $t \geq 1$, then*

$$\rho_m(X_1, \ldots, X_t) = \sum_{i=1}^{t} \left(\prod_{\substack{1 \leq j \leq t \\ j \neq i}} \frac{1}{X_i - X_j} \right) X_i^{m+t-1} \tag{3.4}$$

for any $m \geq 0$.

Let us define $\rho_m(X_1, \ldots, X_t)$ also for a negative integer m by (3.4). Then we have the following:

LEMMA 3.2. *If $t \geq 1$ and if $-t + 1 \leq l \leq -1$, then*

$$\rho_l(X_1, \ldots, X_t) = 0.$$

Proof. Put

$$g_m(X_1,\ldots,X_t)=\sum_{h=0}^{t}(-1)^h\rho_{m-h}(X_1,\ldots,X_t)e_h(X_1,\ldots,X_t)$$

for $m\in\mathbb{Z}$. Then, by Lemma 3.1, there exist rational functions $b_i(X_1,\ldots,X_t)\in\mathbb{Q}(X_1,\ldots,X_t)$ with $1\le i\le t$ such that

$$g_m(X_1,\ldots,X_t)=\sum_{i=1}^{t}b_i(X_1,\ldots,X_t)X_i^m.$$

If $m\ge t$, then $g_m(X_1,\ldots,X_t)=0$ by (3.2). Thus $b_i(X_1,\ldots,X_t)=0$ for any i with $1\le i\le t$ and so

$$g_m(X_1,\ldots,X_t)=0 \tag{3.5}$$

for every $m\in\mathbb{Z}$.

In the case of $1\le m\le t-1$, by combining (3.2) and (3.5), we get

$$\begin{aligned}0&=\sum_{h=m+1}^{t}(-1)^h\rho_{m-h}(X_1,\ldots,X_t)e_h(X_1,\ldots,X_t)\\&=\sum_{h=m-t}^{-1}(-1)^{m-h}e_{m-h}(X_1,\ldots,X_t)\rho_h(X_1,\ldots,X_t)\end{aligned} \tag{3.6}$$

We now show Lemma 3.2 by induction on l. In the case of $l=-1$, we can deduce $\rho_{-1}(X_1,\ldots,X_t)=0$ by substituting $m=t-1$ into (3.6). Next, assume for l with $-t+1\le l\le -2$ that

$$\rho_{-1}(X_1,\ldots,X_t)=\cdots=\rho_{l+1}(X_1,\ldots,X_t)=0.$$

Then, by substituting $m=t+l$ into (3.6), we obtain

$$\rho_l(X_1,\ldots,X_t)=0.$$

$\square$

4 Representation of fractional parts

Let us recall the relation of the decimal expansion of a real number ξ to the fractional parts of the geometric sequence $\xi 10^n$ $(n=0,1,\ldots)$. For simplicity, assume $0<\xi<1$. We now write the decimal expansion of ξ by $\sum_{i=1}^{\infty}s_{-i}(10;\xi)10^{-i}$ with $0\le s_{-i}(10;\xi)\le 9$. Then

$$\{\xi 10^n\}=\sum_{i=1}^{\infty}s_{-i-n}(10;\xi)10^{-i}\ (n=0,1,\ldots), \tag{4.1}$$

Note that the right-hand side of (4.1) is expressed by the iteration of the shift operator to the sequence $(s_{-i}(10;\xi))_{i=1}^{\infty}$.

In this section, we give an analogue of the decimal numeral system to calculate powers of algebraic numbers; we represent the integral and fractional parts by using the symmetric polynomials ρ_m defined in the previous section. Let $\alpha > 1$ be an algebraic number with minimal polynomial $a_d X^d + \cdots + a_0 \in \mathbb{Z}[X]$ $(a_d > 0)$. In what follows, we assume that α has no conjugate with absolute value 1. Let p be the number of the conjugates of α whose absolute values are greater than 1. Moreover, we write the conjugates of α by $\alpha_1 = \alpha, \ldots, \alpha_p, \alpha_{1+p}, \ldots, \alpha_d$, where

$$|\alpha_i| > 1 \ (i = 1, \ldots, p)$$

and

$$|\alpha_j| < 1 \ (j = 1 + p, \ldots, d).$$

We define the m-th digit of a real number ξ by

$$s_m(\alpha; \xi) = a_d[\xi\alpha^{-m}] + a_{d-1}[\xi\alpha^{-m-1}] + \cdots + a_0[\xi\alpha^{-m-d}].$$

For instance, if $\alpha = 10$ and if $\xi \geq 0$, then the m-th digit is

$$s_m(10; \xi) = [\xi 10^{-m}] - 10[\xi 10^{-m-1}],$$

which coincides with the usual decimal digit. Let us call $(s_m(\alpha;\xi))_{m=-\infty}^{\infty}$ the digital sequence of ξ. We now introduce some easy consequences from the definition.

LEMMA 4.1. (1) *If $\xi \geq 0$, then $s_m(\alpha; \xi) = 0$ for sufficiently large m.*

(2) *For any integer m,*

$$-L_+(\alpha) < s_m(\alpha; \xi) < L_-(\alpha).$$

Proof. The first statement is obvious. Note that $a_d > 0$ and $\min\{a_d, \ldots, a_0\} < 0$. The second statement is obtained by

$$s_m(\alpha; \xi) + \sum_{i=0}^{d} a_{d-i}\{\xi\alpha^{-m-i}\} = \xi\alpha^{-m-d}\sum_{i=0}^{d} a_{d-i}\alpha^{d-i} = 0$$

and $0 \leq \{\xi\alpha^{-m-i}\} < 1$ for any i with $0 \leq i \leq d$. □

PROPOSITION 4.1. (1) *If $\xi \geq 0$, then the integral part $[\xi\alpha^n]$ and fractional part $\{\xi\alpha^n\}$ are given by*

$$[\xi\alpha^n] = \frac{1}{a_d}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\rho_i(\alpha_1, \ldots, \alpha_p)\rho_j(\alpha_{1+p}, \ldots, \alpha_d)s_{i+j-n}(\alpha; \xi) \tag{4.2}$$

and

$$\{\xi\alpha^n\} = \frac{1}{a_d}\sum_{i=-\infty}^{-1}\sum_{j=0}^{\infty}\rho_i(\alpha_1, \ldots, \alpha_p)\rho_j(\alpha_{1+p}, \ldots, \alpha_d)s_{i+j-n}(\alpha; \xi), \tag{4.3}$$

respectively. In particular,

$$\xi\alpha^n = \frac{1}{a_d}\sum_{i=-\infty}^{\infty}\sum_{j=0}^{\infty}\rho_i(\alpha_1, \ldots, \alpha_p)\rho_j(\alpha_{1+p}, \ldots, \alpha_d)s_{i+j-n}(\alpha; \xi). \tag{4.4}$$

(2) *If $\xi < 0$, then the representation of fractional part (4.3) holds.*

REMARK 4.1. Let $\xi \geq 0$. Then, by the first statement of Lemma 4.1, the right-hand side of (4.2) is a finite sum.

Now note that the sequence $(s_m(\alpha;\xi))_{m=-\infty}^{\infty}$ is bounded by the second statement of Lemma 4.1 and that the series

$$\sum_{i=-\infty}^{-1} \sum_{j=0}^{\infty} \alpha_h^i \alpha_l^j$$

converges for every h, l with $1 \leq h \leq p$, $1+p \leq l \leq d$. Thus, by using Lemma 3.1, we conclude that the right-hand side of (4.3) converges.

REMARK 4.2. Let $M(\alpha) = a_d|\alpha_1 \cdots \alpha_p|$ be the Mahler measure of α and put

$$\sigma(\alpha) = (-1)^{p-1} \frac{a_d \alpha_1 \cdots \alpha_p}{M(\alpha)} \in \{1, -1\}.$$

Then by Lemma 3.2 and

$$\begin{aligned} \rho_i(\alpha_1, \ldots, \alpha_p) &= \sum_{l=1}^{p} \left(\prod_{\substack{1 \leq h \leq p \\ h \neq l}} \frac{-\alpha_l^{-1}\alpha_h^{-1}}{\alpha_l^{-1} - \alpha_h^{-1}} \right) \alpha_l^{i+p-1} \\ &= (-1)^{p-1} \left(\prod_{h=1}^{p} \alpha_h^{-1} \right) \rho_{-i-p}(\alpha_1^{-1}, \ldots, \alpha_p^{-1}), \end{aligned}$$

the representation (4.3) is rewritten by

$$\{\xi\alpha^n\} = \frac{\sigma(\alpha)}{M(\alpha)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \rho_i(\alpha_1^{-1}, \ldots, \alpha_p^{-1}) \rho_j(\alpha_{1+p}, \ldots, \alpha_d) s_{j-i-n-p}(\alpha;\xi) \quad (4.5)$$

Moreover, if $\xi \geq 0$, then

$$[\xi\alpha^n] = \frac{1}{a_d} \sum_{i=0}^{\infty} \rho_i(\alpha_1, \ldots, \alpha_d) s_{i-n}(\alpha;\xi) \quad (4.6)$$

by using (2.1).

Proof of Proposition 4.1. It suffices to check (4.5) and (4.6). We put

$$q = d - p, \ \boldsymbol{a} = (\alpha_1, \ldots, \alpha_p), \ \text{and } \boldsymbol{b} = (\alpha_{1+p}, \ldots, \alpha_d).$$

Moreover, write

$$\begin{aligned} \boldsymbol{a} \cdot \boldsymbol{b} &= (\alpha_1, \ldots, \alpha_p, \alpha_{1+p}, \ldots, \alpha_d), \\ \boldsymbol{a}^{-1} &= (\alpha_1^{-1}, \ldots, \alpha_p^{-1}). \end{aligned}$$

For $h \geq 0$ and $t \geq 1$, let $e_h(X_1, \ldots, X_t)$ be defined by (3.3). By relations between coefficients and roots of a polynomial, we get

$$\frac{1}{a_d} s_m(\alpha;\xi) = \sum_{h=0}^{d} (-1)^h e_h(\boldsymbol{a} \cdot \boldsymbol{b}) [\xi\alpha^{-m-h}],$$

Thus, if $\xi \geq 0$, then

$$\begin{aligned}\frac{1}{a_d}\sum_{i=0}^{\infty}\rho_i(\boldsymbol{a}\cdot\boldsymbol{b})s_{i-n}(\alpha;\xi) &= \sum_{i=0}^{\infty}\sum_{h=0}^{d}(-1)^h e_h(\boldsymbol{a}\cdot\boldsymbol{b})\rho_i(\boldsymbol{a}\cdot\boldsymbol{b})[\xi\alpha^{n-i-h}] \\ &= \sum_{l=0}^{\infty}[\xi\alpha^{n-l}]\sum_{h=0}^{\min\{l,d\}}(-1)^h\rho_{l-h}(\boldsymbol{a}\cdot\boldsymbol{b})e_h(\boldsymbol{a}\cdot\boldsymbol{b}) \\ &= [\xi\alpha^n],\end{aligned}$$

where the last equality follows from (3.2).

Similarly, by

$$s_m(\alpha;\xi) = -\sum_{i=0}^{d} a_{d-i}\{\xi\alpha^{-m-i}\}$$

and

$$e_m(\boldsymbol{a}) = \alpha_1\cdots\alpha_p e_{p-m}(\boldsymbol{a}^{-1})\ (0 \leq m \leq p),$$

we get

$$\frac{1}{a_d}s_m(\alpha;\xi) = -\sum_{h=0}^{d}(-1)^h e_h(\boldsymbol{a}\cdot\boldsymbol{b})\{\xi\alpha^{-m-h}\}.$$

If $q = 0$, then $p = d$. Thus

$$\frac{1}{a_d}s_m(\alpha;\xi) = (-1)^{d-1}\alpha_1\alpha_2\cdots\alpha_d\sum_{h=0}^{d}(-1)^h e_h(\boldsymbol{a}^{-1})\{\xi\alpha^{h-d-m}\},$$

and so by (3.2)

$$\begin{aligned}&\frac{\sigma(\alpha)}{M(\alpha)}\sum_{i=0}^{\infty}\rho_i(\boldsymbol{a}^{-1})s_{-i-n-d}(\alpha;\xi) \\ &\quad = \sum_{i=0}^{\infty}\sum_{h=0}^{d}(-1)^h e_h(\boldsymbol{a}^{-1})\rho_i(\boldsymbol{a}^{-1})\{\xi\alpha^{h+i+n}\} = \{\xi\alpha^n\},\end{aligned}$$

which implies (4.5).

In the case of $q \geq 1$, we have

$$\begin{aligned}\frac{1}{a_d}s_m(\alpha;\xi) &= -\sum_{h=0}^{p}\sum_{l=0}^{q}(-1)^{h+l}e_h(\boldsymbol{a})e_l(\boldsymbol{b})\{\xi\alpha^{-m-h-l}\} \\ &= (-1)^{p-1}\alpha_1\cdots\alpha_p\sum_{h=0}^{p}\sum_{l=0}^{q}(-1)^{h+l}e_h(\boldsymbol{a}^{-1})e_l(\boldsymbol{b})\{\xi\alpha^{h-p-l-m}\}.\end{aligned}$$

Thus by using (3.2) we obtain

$$\begin{aligned}
&\frac{\sigma(\alpha)}{M(\alpha)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\rho_i(\boldsymbol{a}^{-1})\rho_j(\boldsymbol{b})s_{j-i-n-p}(\alpha;\xi)\\
&= \sum_{i=0}^{\infty}\sum_{h=0}^{p}\sum_{j=0}^{\infty}\sum_{l=0}^{q}(-1)^h e_h(\boldsymbol{a}^{-1})\rho_i(\boldsymbol{a}^{-1})(-1)^l e_l(\boldsymbol{b})\rho_j(\boldsymbol{b})\{\xi\alpha^{h+i+n-j-l}\}\\
&= \sum_{i=0}^{\infty}\sum_{h=0}^{p}(-1)^h e_h(\boldsymbol{a}^{-1})\rho_i(\boldsymbol{a}^{-1})\{\xi\alpha^{h+i+n}\} = \{\xi\alpha^n\}.
\end{aligned}$$

□

Example 4.1. Let α be a rational number a/b, where $a > b > 0$ and $\gcd(a,b) = 1$. Then Proposition 4.1 implies

$$\begin{aligned}
\left[\xi\left(\frac{a}{b}\right)^n\right] &= \frac{1}{b}\sum_{i=0}^{\infty}\left(\frac{a}{b}\right)^i s_{i-n}\left(\frac{a}{b};\xi\right),\\
\left\{\xi\left(\frac{a}{b}\right)^n\right\} &= \frac{1}{b}\sum_{i=-\infty}^{-1}\left(\frac{a}{b}\right)^i s_{i-n}\left(\frac{a}{b};\xi\right)
\end{aligned}$$

for $\xi \geq 0$. This is the companion representation of ξ, which is written in [1].

Example 4.2. Let $\alpha > 1$ be a quadratic irrational number. We assume $p = 1$. Then by Proposition 4.1

$$\begin{aligned}
[\xi\alpha^n] &= \frac{1}{a_2}\sum_{i=0}^{\infty}\rho_i(\alpha,\alpha_2)s_{i-n}(\alpha;\xi),\\
\{\xi\alpha^n\} &= \frac{1}{a_2\alpha}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\alpha^{-i}\alpha_2^j s_{j-i-n-1}(\alpha;\xi)\\
&= \frac{1}{a_2(\alpha-\alpha_2)}\sum_{h=-\infty}^{\infty}\alpha^{\min\{0,h\}}\alpha_2^{\max\{0,h\}}s_{h-n-1}(\alpha;\xi).
\end{aligned}$$

5 Digital sequences

Let $\alpha > 1$ be an algebraic number with no conjugate whose absolute value is 1. We use the same notation as in the previous section. We observed for a nonnegative ξ that the integral part $[\xi\alpha^n]$ and the fractional part $\{\xi\alpha^n\}$ are written by the digital sequence $(s_m(\alpha;\xi))_{m=-\infty}^{\infty}$. We now characterize this sequence by considering the generating function of $[\xi\alpha^n]$ and $\{\xi\alpha^n\}$ $(n = 0, 1, \ldots)$. Recall that if $\xi \geq 0$, then $s_m(\alpha;\xi) = 0$ for any sufficiently large m.

PROPOSITION 5.1. *Let ξ be a nonnegative number.*
(1) *For any integer n, the finite sum*

$$\frac{1}{a_d}\sum_{i=0}^{\infty}\rho_i(\alpha_1,\ldots,\alpha_d)s_{i-n}(\alpha;\xi)$$

is a rational integer.
(2) *If $2 \leq k \leq p$, then*

$$\sum_{i=-\infty}^{\infty} \alpha_k^i s_i(\alpha;\xi) = 0.$$

Proof. The first statement is obvious by Proposition 4.1. Now we prove the second one. Since $s_m(\alpha;\alpha^{-1}\xi) = s_{m+1}(\alpha;\xi)$, we may assume $[\xi\alpha^m] = 0$ for any $m < 0$. Put

$$f(z) = \sum_{n=0}^{\infty}[\xi\alpha^n]z^n,\ g(z) = \sum_{n=0}^{\infty}\{\xi\alpha^n\}z^n.$$

Then we have

$$\frac{\xi}{1-\alpha z} - g(z) = f(z).$$

Let $P_\alpha^*(z) = a_0z^d + a_1z^{d-1} + \cdots + a_d$. Thus we get

$$\begin{aligned}
\left(\frac{\xi}{1-\alpha z} - g(z)\right)P_\alpha^*(z) &= f(z)P_\alpha^*(z) \\
&= \sum_{h=0}^{\infty}\sum_{\substack{i,j\geq 0\\ i+j=h}}[\xi\alpha^i]a_{d-j}z^h \\
&= \sum_{h=0}^{\infty}\sum_{i=h-d}^{h}[\xi\alpha^i]a_{d-h+i}z^h = \sum_{h=0}^{\infty}s_{-h}(\alpha;\xi)z^h.
\end{aligned}$$

Consider the region of $z \in \mathbb{C}$ satisfying

$$\left(\frac{\xi}{1-\alpha z} - g(z)\right)P_\alpha^*(z) = \sum_{h=0}^{\infty}s_{-h}(\alpha;\xi)z^h. \tag{5.1}$$

Since $0 \leq \{\xi\alpha^n\} < 1$ for any n, the left-hand side of (5.1) is a meromorphic function on $\{z \in \mathbb{C}||z| < 1\}$. Moreover, because the sequence $s_{-h}(\alpha;\xi)$ ($h = 0,1,\ldots$) is bounded, the right-hand side of (5.1) converges for $|z| < 1$. Hence (5.1) holds for $|z| < 1$. In particular, since the left-hand side of (5.1) has a zero at $z = \alpha_k^{-1}$ with $2 \leq k \leq p$, we obtain

$$0 = \sum_{i=0}^{\infty}\alpha_k^{-i}s_{-i}(\alpha;\xi) = \sum_{i=-\infty}^{\infty}\alpha_k^i s_i(\alpha;\xi).$$

□

The decimal numeral system gives the correspondence between nonnegative numbers and sequences of digits $0,1,\ldots,9$. In what follows, we show that sequences satisfying the assumptions of Proposition 5.1 represents the fractional parts of certain geometric progressions.

PROPOSITION 5.2. *Let $\boldsymbol{x} = (x_m)_{m=-\infty}^{\infty}$ be a bounded sequence of integers. Assume that $x_m = 0$ for all sufficiently large m. Suppose further that*

$$\sum_{i=-\infty}^{\infty} \alpha_k^i x_i = 0 \tag{5.2}$$

for any k with $2 \le k \le p$ and that the finite sum

$$\frac{1}{a_d} \sum_{i=0}^{\infty} \rho_i(\alpha_1, \ldots, \alpha_d) x_{i-n} \tag{5.3}$$

is a rational integer for any n. Let

$$\xi = \xi(\boldsymbol{x}) = \frac{1}{a_d} \sum_{i=-\infty}^{\infty} \sum_{j=0}^{\infty} \rho_i(\alpha_1, \ldots, \alpha_p) \rho_j(\alpha_{1+p}, \ldots, \alpha_d) x_{i+j}. \tag{5.4}$$

Then for any n

$$\xi \alpha^n = \frac{1}{a_d} \sum_{i=-\infty}^{\infty} \sum_{j=0}^{\infty} \rho_i(\alpha_1, \ldots, \alpha_p) \rho_j(\alpha_{1+p}, \ldots, \alpha_d) x_{i+j-n}. \tag{5.5}$$

In particular,

$$\xi \alpha^n \equiv \frac{1}{a_d} \sum_{i=-\infty}^{-1} \sum_{j=0}^{\infty} \rho_i(\alpha_1, \ldots, \alpha_p) \rho_j(\alpha_{1+p}, \ldots, \alpha_d) x_{i+j-n} \mod \mathbb{Z}$$

REMARK 5.1. Let n be an integer. Then, since $x_m = 0$ for all sufficiently large m, the series

$$\begin{aligned} &\frac{1}{a_d} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \rho_i(\alpha_1, \ldots, \alpha_p) \rho_j(\alpha_{1+p}, \ldots, \alpha_d) x_{i+j-n} \\ &= \frac{1}{a_d} \sum_{i=0}^{\infty} \rho_i(\alpha_1, \ldots, \alpha_d) x_{i-n} \end{aligned}$$

is a finite sum. By using Lemma 3.1, we also deduce that the series

$$\frac{1}{a_d} \sum_{i=-\infty}^{-1} \sum_{j=0}^{\infty} \rho_i(\alpha_1, \ldots, \alpha_p) \rho_j(\alpha_{1+p}, \ldots, \alpha_d) x_{i+j-n}$$

converges.

Proof. Since (5.3) is a rational integer, it suffices to check (5.5). By using (3.4) and (5.2), we get

$$\begin{aligned} \xi &= \frac{1}{a_d} \sum_{j=0}^{\infty} \rho_j(\alpha_{1+p}, \ldots, \alpha_d) \sum_{h=-\infty}^{\infty} \rho_{h-j}(\alpha_1, \ldots, \alpha_p) x_h \\ &= \frac{1}{a_d} \sum_{j=0}^{\infty} \rho_j(\alpha_{1+p}, \ldots, \alpha_d) \sum_{h=-\infty}^{\infty} \left(\prod_{l=2}^{p} \frac{1}{\alpha - \alpha_l} \right) \alpha^{h-j+p-1} x_h. \end{aligned}$$

Thus we get

$$\begin{aligned}
\xi\alpha^n &= \frac{1}{a_d}\sum_{j=0}^{\infty}\rho_j(\alpha_{1+p},\ldots,\alpha_d)\sum_{h=-\infty}^{\infty}\left(\prod_{l=2}^{p}\frac{1}{\alpha-\alpha_l}\right)\alpha^{n+h-j+p-1}x_h \\
&= \frac{1}{a_d}\sum_{j=0}^{\infty}\rho_j(\alpha_{1+p},\ldots,\alpha_d)\sum_{h=-\infty}^{\infty}\rho_{h-j}(\alpha_1,\ldots,\alpha_p)x_{h-n} \\
&= \frac{1}{a_d}\sum_{i=-\infty}^{\infty}\sum_{j=0}^{\infty}\rho_i(\alpha_1,\ldots,\alpha_p)\rho_j(\alpha_{1+p},\ldots,\alpha_d)x_{i+j-n}.
\end{aligned}$$

□

REMARK 5.2. $\xi(\boldsymbol{x})$ defined by Proposition 5.2 is not necessarily a nonnegative number.

In the end of this section, we introduce a lemma which we use to prove Corollary 2.2 and the first statement of Theorem 2.3.

LEMMA 5.1. *Let $(u_m)_{m=-d}^{\infty}$ and $(y_m)_{m=0}^{\infty}$ be sequences of integers. Assume that $(u_m)_{m=-d}^{\infty}$ is not ultimately periodic and that $(y_m)_{m=0}^{\infty}$ is ultimately periodic. Suppose further*

$$y_m = a_d u_m + a_{d-1}u_{m-1} + \cdots + a_0 u_{m-d}$$

for any $m \geq 0$. Then $a_d = 1$, namely, α is an algebraic integer.

For the proof of Lemma 5.1, we begin with Lemma 1 of [5] which is rewritten from [4]:

LEMMA 5.2. *If $P(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0 = a_d(x-\alpha_1)\cdots(x-\alpha_d) \in \mathbb{C}[x]$ has distinct roots and*

$$X_1\alpha_1^j + \cdots + X_d\alpha_d^j = Z_j,\ j = 0, 1, \ldots, d-1,$$

then, for $j = 1, 2, \ldots, d$,

$$X_j = \frac{1}{P'(\alpha_j)}\sum_{k=0}^{d-1}\beta_{j,k}Z_k,$$

where

$$\beta_{j,k} = \sum_{l=k+1}^{d} a_l\alpha_j^{l-k-1}.$$

Proof of Lemma 5.1. Assume that $a_d \geq 2$. Write the period of the sequence $(y_m)_{m=0}^{\infty}$ by T. Put $w_n = u_{n+T} - u_n$. If n is sufficiently large, then $y_{n+T} = y_n$ and so

$$a_d w_n + a_{d-1}w_{n-1} + \cdots + a_0 w_{n-d} = 0.$$

Hence, there are a natural number n_0 and complex numbers $\xi_1, \ldots, \xi_d$ such that, for any $n \geq n_0$,

$$w_n = \xi_1 \alpha_1^n + \cdots + \xi_d \alpha_d^n. \tag{5.6}$$

Let $m \geq n_0$. Apply Lemma 5.2 to the linear system

$$X_1 \alpha_1^{n-m} + \cdots + X_d \alpha_d^{n-m} = w_n,\ n = m, m+1, \ldots, m+d-1$$

with variables $X_j = \xi_j \alpha_j^m$, $j = 1, 2, \ldots, d$. Thus we get

$$P'_\alpha(\alpha_j)\xi_j \alpha_j^m = G_m(\alpha_j) \tag{5.7}$$

for each $j = 1, 2, \ldots, d$, where G_m is an integer polynomial of degree at most $d-1$.

Now suppose that $\xi_1 = 0$. By (5.7), $\xi_1, \ldots, \xi_d$ are algebraic numbers and conjugate over $\mathbb{Q}$. Therefore, $\xi_1 = \cdots = \xi_d = 0$. By (5.6) we have $w_n = u_{n+T} - u_n = 0$ for $n \geq n_0$. This is impossible since $(u_m)_{m=-d}^\infty$ is not ultimately periodic. Finally we obtain $\xi_1 \neq 0$.

Take a nonzero integer R for which

$$\frac{R}{P'_\alpha(\alpha)\xi_1}, \frac{R\alpha}{P'_\alpha(\alpha)\xi_1}, \ldots, \frac{R\alpha^{d-1}}{P'_\alpha(\alpha)\xi_1}$$

are algebraic integers. Then $R\alpha^m = (RG_m(\alpha))/(P'_\alpha(\alpha)\xi_1)$ is an algebraic integer for every sufficiently large m. However, by considering the factorization of $R\alpha^m$ into prime ideals, we see that this is impossible since α is not an algebraic integer. □

6 Proof of the main results

Proof of Theorem 2.1. Let $a_d X^d + \cdots + a_0 \in \mathbb{Z}[X]$ be the minimal polynomial of α. Define the sequences $(u_m)_{m=-d}^\infty$ and $(y_m)_{m=0}^\infty$ by

$$\begin{aligned} u_{-d} &= u_{-d+1} = \cdots = u_{-1} = 0, \\ u_0 &= 1,\ y_0 = a_d \end{aligned}$$

and, for $m \geq 1$,

$$\begin{aligned} u_m &= -\left[\frac{a_{d-1}u_{m-1} + \cdots + a_0 u_{m-d}}{a_d}\right], \\ y_m &= a_d \left\{\frac{a_{d-1}u_{m-1} + \cdots + a_0 u_{m-d}}{a_d}\right\}. \end{aligned}$$

Then we have

$$y_m = a_d u_m + a_{d-1}u_{m-1} + \cdots + a_0 u_{m-d}$$

for any $m \geq 0$. Moreover, $y_m \in \{0, 1, \ldots, a_d - 1\}$ for $m \geq 1$.

Put

$$f(z) = \sum_{n=0}^\infty y_n z^n,\ g(z) = \sum_{n=0}^\infty u_n z^n,$$

and so

$$\begin{aligned} f(z) &= (a_d + a_{d-1}z + \cdots + a_0 z^d)g(z) \\ &= a_d(1-\alpha z)\prod_{i=2}^{d}(1-\alpha_i z)g(z). \end{aligned}$$

Therefore, by using (3.1) we get

$$\begin{aligned} g(z) &= \frac{1}{a_d}\sum_{i=0}^{\infty} y_i z^i \sum_{j=0}^{\infty} \rho_j(\alpha, \alpha_2, \ldots, \alpha_d) z^j \\ &= \frac{1}{a_d}\sum_{n=0}^{\infty} \sum_{\substack{i+j\geq 0 \\ i+j=n}} y_i \rho_j(\alpha, \alpha_2, \ldots, \alpha_d) z^n. \end{aligned}$$

We now define the two-sided sequence $\boldsymbol{x} = (x_m)_{m=-\infty}^{\infty}$ as follows:

$$x_m = \begin{cases} 0 & (m > 0), \\ y_{-m} & (m \leq 0). \end{cases}$$

Then $\boldsymbol{x}$ satisfies the assumptions of Proposition 5.2. In fact, if $n < 0$, then (5.3) is zero. In the case where $n \geq 0$,

$$\frac{1}{a_d}\sum_{i=0}^{\infty} \rho_i(\alpha, \alpha_2, \ldots, \alpha_d) x_{i-n} = u_n$$

is a rational integer. Moreover, (5.2) clearly holds since $p = 1$. Put

$$v_n = \frac{1}{a_d}\sum_{i=-\infty}^{-1}\sum_{j=0}^{\infty} \alpha^i \rho_j(\alpha_2, \ldots, \alpha_d) x_{i+j-n}$$

for integer n. Then Proposition 5.2 implies

$$\xi(\boldsymbol{x})\alpha^n = u_n + v_n \tag{6.1}$$

and

$$\xi(\boldsymbol{x})\alpha^n \equiv v_n \bmod \mathbb{Z}, \tag{6.2}$$

where

$$\begin{aligned} \xi(\boldsymbol{x}) &= \frac{1}{a_d}\sum_{i=-\infty}^{\infty}\sum_{j=0}^{\infty} \alpha^i \rho_j(\alpha_2, \ldots, \alpha_d) x_{i+j} \\ &= \frac{1}{a_d}\sum_{j=0}^{\infty} \rho_j(\alpha_2, \ldots, \alpha_d)\alpha^{-j} \sum_{h=-\infty}^{\infty} x_h \alpha^h \\ &= \frac{1}{a_d}\sum_{j=0}^{\infty} \rho_j(\alpha^{-1}\alpha_2, \ldots, \alpha^{-1}\alpha_d) \sum_{h=-\infty}^{0} x_h \alpha^h \\ &= \frac{1}{a_d \prod_{i=2}^{d}(1-\alpha^{-1}\alpha_i)} \sum_{h=-\infty}^{0} x_h \alpha^h. \end{aligned}$$

Thus $\xi(\boldsymbol{x}) \neq 0$ since $x_0 = a_d$ and $x_m \geq 0$ for $m \leq -1$. Since $0 \leq x_m \leq a_d - 1$ for $m \leq -1$ and since

$$\lim_{n\to\infty} \frac{1}{a_d} \sum_{\substack{i<0, j\geq 0\\ i+j=n}} \alpha^i \rho_j(\alpha_2, \ldots, \alpha_d) x_0 = 0,$$

every limit point of the sequence $(v_m)_{m=0}^{\infty}$ is denoted by

$$v' = \frac{1}{a_d} \sum_{i=-\infty}^{-1} \sum_{j=0}^{\infty} \alpha^i \rho_j(\alpha_2, \ldots, \alpha_d) \theta_{i,j},$$

where $\theta_{i,j} \in \{0, 1, \ldots, a_d - 1\}$. Putting

$$\nu_+ = \sum_{j=0}^{\infty} \max\{0, \rho_j(\alpha_2, \ldots, \alpha_d)\} \tag{6.3}$$

and

$$\nu_- = \sum_{j=0}^{\infty} \max\{0, -\rho_j(\alpha_2, \ldots, \alpha_d)\}, \tag{6.4}$$

we obtain

$$-\frac{a_d - 1}{a_d(\alpha - 1)} \nu_- \leq v' \leq \frac{a_d - 1}{a_d(\alpha - 1)} \nu_+. \tag{6.5}$$

By (6.2), (6.5) and $\nu = \nu_+ + \nu_-$, we verified the theorem. □

Proof of Corollary 2.2. We use the same notation as the proof of Theorem 2.1. In the case of $-1 < \alpha_2 < 0$, the corollary follows from (6.5) and

$$\nu_+ = \frac{1}{1 - \alpha_2^2}, \ \nu_- = -\frac{\alpha_2}{1 - \alpha_2^2}.$$

We now assume $0 < \alpha_2 < 1$. Then v_n is rewritten by

$$v_n = \frac{1}{a_2(\alpha - \alpha_2)} \sum_{h=-\infty}^{n+1} \alpha^{\min\{0,h\}} \alpha_2^{\max\{0,h\}} x_{h-n-1}.$$

(6.5) implies that the sequence $(v_m)_{m=0}^{\infty}$ is bounded. On the other hand, by using (6.1) and $\xi(\boldsymbol{x}) \neq 0$, we deduce that the sequence $(u_m)_{m=-d}^{\infty}$ is not ultimately periodic. Thus, since $a_2 \geq 2$, Lemma 5.1 means that the sequence $(x_{-m})_{m=0}^{\infty}$ is not ultimately periodic. In particular, by $x_m \in \{0, 1, \ldots, a_2 - 1\}$ $(m \leq -1)$, there exists an $M > 0$ with

$$x_{-M} \leq a_2 - 2. \tag{6.6}$$

By (6.6) and $x_0 = a_2$, if $n \geq M$, then

$$\begin{aligned} v_n &\leq \frac{1}{a_2(\alpha - \alpha_2)} \Bigg(\sum_{\substack{h=-\infty \\ h \neq n+1, n+1-M}}^{\infty} \alpha^{\min\{0,h\}} \alpha_2^{\max\{0,h\}} (a_2 - 1) \\ &\qquad\qquad + \alpha_2^{n+1} a_2 + \alpha_2^{n+1-M}(a_2 - 2) \Bigg) \\ &= \frac{1}{a_2(\alpha - \alpha_2)} \left((a_2 - 1) \sum_{h=-\infty}^{\infty} \alpha^{\min\{0,h\}} \alpha_2^{\max\{0,h\}} + \alpha_2^{n+1} - \alpha_2^{n+1-M} \right) \\ &< \frac{a_2 - 1}{a_2(\alpha - 1)(1 - \alpha_2)} = \frac{a_2 - 1}{|P_\alpha(1)|} \end{aligned}$$

for $n \geq M$. By putting

$$\xi' = \xi(\boldsymbol{x})\alpha^M,$$

we obtain

$$\{\xi' \alpha^n\} < \frac{a_2 - 1}{|P_\alpha(1)|}$$

for any $n \geq 0$. $\square$

Proof of Theorem 2.3. For the proof of the first statement, we use the same notation as the proof of Theorem 2.1. If $d \geq 2$, then we may assume that $1 > \alpha_2 > \ldots > \alpha_d > 0$. Then by using Lemma 3.1 we get

$$\lim_{m \to \infty} \rho_m(\alpha_2, \ldots, \alpha_d) \alpha_2^{-m} = \prod_{j=3}^{d} \frac{\alpha_2}{\alpha_2 - \alpha_j}.$$

Hence, there is an $M > 0$ such that, for any $m_1, m_2 \geq 0$ with $m_1 \geq m_2 + M$,

$$\rho_{m_1}(\alpha_2, \ldots, \alpha_d) < \rho_{m_2}(\alpha_2, \ldots, \alpha_d).$$

On the other hand, we can deduce that the sequence $(x_{-m})_{m=0}^{\infty}$ is not ultimately periodic in the same way as the proof of Corollary 2.2. Therefore, there exists an $\widetilde{M} > 0$ satisfying $\widetilde{M} > M$ and $x_{-\widetilde{M}} \leq a_d - 2$. Thus by using $x_0 = a_d$ we get, for $n \geq \widetilde{M}$,

$$\begin{aligned} 0 \leq v_n &\leq \frac{1}{a_d} \sum_{\substack{i<0, j\geq 0 \\ i+j \neq n, n-\widetilde{M}}} \alpha^i \rho_j(\alpha_2, \ldots, \alpha_d)(a_d - 1) \\ &\quad + \frac{1}{a_d} \sum_{\substack{i<0, j\geq 0 \\ i+j = n}} \alpha^i \rho_j(\alpha_2, \ldots, \alpha_d) a_d + \frac{1}{a_d} \sum_{\substack{i<0, j\geq 0 \\ i+j = n-\widetilde{M}}} \alpha^i \rho_j(\alpha_2, \ldots, \alpha_d)(a_d - 2) \\ &= \frac{1}{a_d} \sum_{i<0, j\geq 0} \alpha^i \rho_j(\alpha_2, \ldots, \alpha_d)(a_d - 1) \\ &\quad + \frac{1}{a_d} \sum_{\substack{i<0, j\geq 0 \\ i+j = n}} \alpha^i \rho_j(\alpha_2, \ldots, \alpha_d) - \frac{1}{a_d} \sum_{\substack{i<0, j\geq 0 \\ i+j = n-\widetilde{M}}} \alpha^i \rho_j(\alpha_2, \ldots, \alpha_d) \end{aligned}$$

Since $\widetilde{M} > M$, we obtain

$$\frac{1}{a_d} \sum_{\substack{i<0,j\geq 0\\ i+j=n}} \alpha^i \rho_j(\alpha_2,\ldots,\alpha_d) - \frac{1}{a_d} \sum_{\substack{i<0,j\geq 0\\ i+j=n-\widetilde{M}}} \alpha^i \rho_j(\alpha_2,\ldots,\alpha_d)$$
$$= \frac{1}{a_d} \sum_{i=-\infty}^{-1} \alpha^i \Big(\rho_{n-i}(\alpha_2,\ldots,\alpha_d) - \rho_{n-i-\widetilde{M}}(\alpha_2,\ldots,\alpha_d)\Big) < 0,$$

so

$$0 \leq v_n < \frac{a_d-1}{a_d} \sum_{i=-\infty}^{-1} \alpha^i \sum_{j=0}^{\infty} \rho_j(\alpha_2,\ldots,\alpha_d) = \frac{a_d-1}{|P_\alpha(1)|}.$$

By combining this inequality with (6.2), we proved the first statement.

We now verify the second statement. We define ν_+ and ν_- by (6.3) and (6.4), respectively. Let us choose a positive integer A with

$$\left\{ -\frac{a_d-1}{a_d(\alpha-1)}\nu_- + \frac{A}{|P_\alpha(1)|} \right\} \in \left(0, \frac{1}{|P_\alpha(1)|}\right]. \tag{6.7}$$

Write the left-hand side of (6.7) by η. Put $P_\alpha(X) = a_d X^d + \cdots + a_0$. We define the sequences $(u'_m)_{m=-d}^{\infty}$ and $(y'_m)_{m=0}^{\infty}$ by

$$u'_{-d} = u'_{-d+1} = \cdots = u'_{-1} = 0$$

and, for $m \geq 0$,

$$\begin{aligned} u'_m &= -\left\lfloor \frac{-A + a_{d-1}u'_{m-1} + \cdots + a_0 u'_{m-d}}{a_d} \right\rfloor, \\ y'_m &= A + a_d \left\{ \frac{-A + a_{d-1}u'_{m-1} + \cdots + a_0 u'_{m-d}}{a_d} \right\}. \end{aligned}$$

Thus we get, for any $m \geq 0$,

$$y'_m = a_d u'_m + a_{d-1}u'_{m-1} + \cdots + a_0 u'_{m-d}$$

and

$$y'_m \in \{A, A+1, \ldots, A + a_d - 1\}.$$

Since the rest of proof is same as that of Theorem 2.1 we give only its sketch. Define $\boldsymbol{x}' = (x'_m)_{m=-\infty}^{\infty}$ and $\xi(\boldsymbol{x}')$ by

$$x'_m = \begin{cases} 0 & (m > 0), \\ y'_{-m} & (m \leq 0) \end{cases}$$

and by (5.4), respectively. Then, because $x'_m > 0$ for $m \leq 0$, we get $\xi(\boldsymbol{x}') \neq 0$. Moreover, every limit point of the sequence $\xi\alpha^n \bmod \mathbb{Z}$ $(n = 0, 1, \ldots)$ is written by

$$\frac{1}{a_d} \sum_{i=-\infty}^{-1} \sum_{j=0}^{\infty} \alpha^i \rho_j(\alpha_2,\ldots,\alpha_d)\theta'_{i,j} \bmod \mathbb{Z},$$

where $\theta_{i,j} \in \{A, A+1, \ldots, A+a_d-1\}$. By putting

$$w' = \frac{1}{a_d} \sum_{i=-\infty}^{-1} \sum_{j=0}^{\infty} \alpha^i \rho_j(\alpha_2, \ldots, \alpha_d)\theta'_{i,j},$$

we get

$$-\frac{a_d-1}{a_d(\alpha-1)}\nu_- + \frac{A}{|P_\alpha(1)|} \leq w' \leq \frac{a_d-1}{a_d(\alpha-1)}\nu_+ + \frac{A}{|P_\alpha(1)|}.$$

Therefore,

$$\begin{aligned} 0 < \eta &\leq w' - \left[-\frac{a_d-1}{a_d(\alpha-1)}\nu_- + \frac{A}{|P_\alpha(1)|}\right] \\ &\leq \eta + \frac{a_d-1}{a_d(\alpha-1)}\nu < \frac{1}{|P_\alpha(1)|} + \frac{1}{2} \leq 1. \end{aligned}$$

Consequently, we get

$$w' \bmod \mathbb{Z} \in \tau\left(\left[\eta, \eta + \frac{a_d-1}{a_d(\alpha-1)}\nu\right]\right).$$

Since

$$\left[\eta, \eta + \frac{a_d-1}{a_d(\alpha-1)}\nu\right] \subset (0,1),$$

we obtain

$$\eta \leq \liminf_{n\to\infty}\{\xi\alpha^n\} \leq \limsup_{n\to\infty}\{\xi\alpha^n\} \leq \eta + \frac{a_d-1}{a_d(\alpha-1)}\nu$$

□

Proof of Theorem 2.4. Let $a_dX^d + \cdots + a_0 \in \mathbb{Z}[X]$ be the minimal polynomial of α. It suffices to prove the theorem in the case of

$$\limsup_{n\to\infty}\{\xi\alpha^n\} < \delta_1.$$

Moreover, we may assume

$$\{\xi\alpha^n\} < \delta_1$$

for any $n \geq -d$. Let $\sigma(\alpha)$ be defined as in Remark 4.2. We verify that if $m \leq 0$, then $\sigma(\alpha)s_m(\alpha;\xi)$ is a nonnegative integer. Suppose $\sigma(\alpha) = (-1)^{p-1} = 1$. Then, since $P_\alpha(X)$ has exactly p zeros on the interval $(1,\infty)$,

$$0 > P_\alpha(1) = L_+(\alpha) - L_-(\alpha),$$

namely,

$$\delta_1 = \frac{1}{L_+(\alpha)}.$$

Thus we get

$$s_m(\alpha;\xi) = -\sum_{i=0}^{d} a_{d-i}\{\xi\alpha^{-m-i}\} > -L_+(\alpha)\delta_1 = -1.$$

In the case of $\sigma(\alpha) = -1$, we have

$$0 < P_\alpha(1) = L_+(\alpha) - L_-(\alpha),$$

namely,

$$\delta_1 = \frac{1}{L_-(\alpha)}.$$

Hence,

$$s_m(\alpha;\xi) = -\sum_{i=0}^{d} a_{d-i}\{\xi\alpha^{-m-i}\} < L_-(\alpha)\delta_1 = 1.$$

Since $\lim_{|l|\to\infty} \eta_l = 0$, there exists an $N \in \mathbb{Z}$ such that $\eta_N = \sup_{l\in\mathbb{Z}} \eta_l$. By (4.5) we get

$$\{\xi\alpha^n\} = \frac{1}{M(\alpha)} \sum_{l=-\infty}^{\infty} \eta_l \sigma(\alpha) s_{l-n-p}(\alpha;\xi).$$

Lemma 1 of [6] implies that $\sigma(\alpha)s_m(\alpha;\xi) \geq 1$ for infinitely many $m \leq 0$. Thus, since $\eta_l \geq 0$ for any integer l and

$$\lim_{n\to\infty} \frac{1}{M(\alpha)} \sum_{l=n+p+1}^{\infty} \eta_l \sigma(\alpha) s_{l-n-p}(\alpha;\xi) = 0,$$

we obtain

$$\limsup_{n\to\infty}\{\xi\alpha^n\} \quad\geq\quad \frac{1}{M(\alpha)}\eta_N = \delta_2.$$

□

Proof of Theorem 2.5. Theorem 2.4 means

$$\inf_{\xi\notin\mathbb{Q}(\alpha)} \limsup_{n\to\infty}\{\xi\alpha^n\} \geq \delta_2.$$

It suffices to show that there exists a $\xi \notin \mathbb{Q}(\alpha)$ with

$$\limsup_{n\to\infty}\{\xi\alpha^n\} = \delta_2.$$

Let the sequence $\boldsymbol{x} = (x_m)_{m=-\infty}^{\infty}$ be defined as follows:

$$x_m = \begin{cases} 1 & (n = -m! \text{ for some } m \geq 1), \\ 0 & \text{(otherwise)}. \end{cases}$$

Then $\boldsymbol{x}$ satisfies the assumptions of Propositions 5.2. We have

$$\begin{aligned}\xi(\boldsymbol{x}) &= \frac{1}{\alpha}\sum_{i=-\infty}^{\infty}\sum_{j=0}^{\infty}\alpha^i\rho_j(\alpha_2,\ldots,\alpha_d)x_{i+j} \\ &= \frac{1}{\alpha}\sum_{j=0}^{\infty}\alpha^{-j}\rho_j(\alpha_2,\ldots,\alpha_d)\sum_{h=-\infty}^{\infty}\alpha^h x_h.\end{aligned}$$

The transcendency of $\xi(\boldsymbol{x})$ has been proved for instance in [13]. By proposition 5.2 we get

$$\xi(\boldsymbol{x})\alpha^n \equiv \frac{1}{\alpha}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\alpha^{-i}\rho_j(\alpha_2,\ldots,\alpha_d)x_{j-i-n-1} \bmod \mathbb{Z},$$

and so

$$\xi(\boldsymbol{x})\alpha^n \equiv \frac{1}{\alpha}\sum_{l=-\infty}^{\infty}\eta_l x_{l-n-1} \bmod \mathbb{Z}.$$

Note that there exists an N with $\eta_N = \sup_{l\in\mathbb{Z}}\eta_l$. Put $\Lambda = \{m! + N - 1 | m \geq 1\}$. Then we get

$$\lim_{n\to\infty, n\in\Lambda}\{\xi(\boldsymbol{x})\alpha^n\} = \frac{\eta_N}{\alpha} = \delta_2$$

and

$$\limsup_{n\to\infty, n\notin\Lambda}\{\xi(\boldsymbol{x})\alpha^n\} < \delta_2.$$

Thus,

$$\limsup_{n\to\infty}\{\xi(\boldsymbol{x})\alpha^n\} = \delta_2.$$

□

Proof of Corollary 2.6. δ_1, δ_2, which are defined in Theorem 2.4, are rewritten by

$$\delta_1 = \frac{1}{a_2 + a_0} = \frac{1}{1 + \alpha\alpha_2}$$

and

$$\delta_2 = \frac{1}{\alpha - \alpha_2}.$$

It suffices to show that

$$\delta_1 - \delta_2 = \frac{\alpha - 1 - (\alpha+1)\alpha_2}{(1+\alpha\alpha_2)(\alpha-\alpha_2)} \geq 0. \tag{6.8}$$

First, we assume $\alpha > 1 + 2\sqrt{2}$. Then

$$\delta_1 - \delta_2 > \frac{\alpha - 1 + (-2+\sqrt{2})(\alpha+1)}{(1+\alpha\alpha_2)(\alpha-\alpha_2)} > 0.$$

On the other hand, it is easily seen that if $\alpha \leq 1 + 2\sqrt{2}$ and $\alpha_2 < 2 - \sqrt{2}$, then $\alpha = 2 + \sqrt{3}$ or $\alpha = (3+\sqrt{5})/2$. (6.8) holds in each case. □

7 Note on Mahler's Z-numbers

Mahler conjectured that there does not exist a positive number ξ satisfying

$$\left\{\xi\left(\frac{3}{2}\right)^n\right\} < \frac{1}{2}$$

for all integers $n \geq 0$. Such a ξ is called a Z-number. Mahler's First Theorem [8, 12] implies for any $u \geq 0$ that there exists at most one Z-number whose integral part coincides with u. Flatto [8] generalized the theorem above as follows.
Let $u \geq 0$ and $a > b \geq 1$ be integers. Assume that a and b are coprime. Then there exists at most one positive ξ satisfying

$$[\xi] = u$$

and, for any $n \geq 0$,

$$\left\{\xi\left(\frac{a}{b}\right)^n\right\} < \min\left\{\frac{1}{b}, \frac{b}{a}\right\}.$$

In this section we introduce generalization of these results to the powers of algebraic numbers.

THEOREM 7.1. *Let $\alpha > 1$ be an algebraic number and let $a_d(> 0)$ be the leading coefficient of the minimal polynomial of α. Suppose that α has no conjugate on the unit circle. Let y be a positive number. If $L_-(\alpha) \geq L_+(\alpha)$, then assume that*

$$L_+(\alpha)y + [L_-(\alpha)y] \leq a_d. \tag{7.1}$$

Otherwise, suppose that

$$L_-(\alpha)y + [L_+(\alpha)y] \leq a_d. \tag{7.2}$$

Then there exist at most countably many nonzero ξ such that

$$\{\xi\alpha^n\} < y$$

for any n.

Example 7.1. Let us recall that $\theta_1(= 24.97\ldots)$ is the unique zero of the polynomial $2X^2 - 50X + 1$ with $X > 1$. We have

$$L_+(\theta_1) = 3,\ L_-(\theta_1) = 50.$$

Put

$$S_y = \{\xi \neq 0 | \{\xi\theta_1^n\} < y \text{ for any } n \geq 0\}$$

for positive y. If $y < 1/25 = 0.04$, then (7.1) holds. Thus the cardinality of S_y is at most countable by Theorem 7.1. Assume further $y \geq 1/47 = 0.02127\ldots$. Then S_y is not empty by Example 2.1. Moreover, S_y is a countably infinite set. In fact, take an element $\xi = \xi(\theta_1) \in S_y$. So we have

$$S_y \supset \{\xi\theta_1^m | m \geq 0\}.$$

Proof of Theorem 7.1. Suppose

$$L_-(\alpha) \geq L_+(\alpha). \tag{7.3}$$

First, note that the set S of ξ satisfying $\{\xi\alpha^n\} = 0$ for some $n \geq 0$ is countable. In fact,

$$S \subset \{k\alpha^l | k, l \in \mathbb{Z}\}.$$

Next, let S' be the set of ξ such that

$$0 < \{\xi\alpha^n\} < y \tag{7.4}$$

for any $n \geq 0$. In what follows, we prove that the cardinality of S' is at most countable. Put

$$S_+ = S' \cap (0, \infty),\ S_- = S' \cap (-\infty, 0).$$

Take any $\xi \in S_+$ and $n \geq d$. Let $a_d X^d + \cdots + a_0 \in \mathbb{Z}[X]$ be the minimal polynomial of α. Since

$$\sum_{i=0}^{d} a_{d-i}\xi\alpha^{n-i} = \sum_{i=0}^{d} a_{d-i}([\xi\alpha^{n-i}] + \{\xi\alpha^{n-i}\}) = 0,$$

we get

$$[\xi\alpha^n] = -\frac{1}{a_d}\sum_{i=1}^{d} a_{d-i}[\xi\alpha^{n-i}] - \frac{1}{a_d}\sum_{i=0}^{d} a_{d-i}\{\xi\alpha^{n-i}\}. \tag{7.5}$$

By putting

$$I_h = I_h(y) = \left(\frac{h}{a_d} - \frac{L_+(\alpha)}{a_d}y, \frac{h}{a_d} + \frac{L_-(\alpha)}{a_d}y\right)\ (0 \leq h \leq a_d - 1),$$

we have

$$-\frac{1}{a_d}\sum_{i=0}^{d} a_{d-i}\{\xi\alpha^{n-i}\} \in I_0. \tag{7.6}$$

We now verify for any integer h with $0 \leq h \leq a_d - 1$ that I_h contains at most one integer. If such an integer exists, we denote it by w_h. By putting

$$R = \left\lceil \frac{h + L_-(\alpha)y}{a_d} \right\rceil,$$

we get

$$Ra_d - L_-(\alpha)y \leq h < (R+1)a_d - L_-(\alpha)y.$$

Since h is a rational integer, by (7.1)

$$h \geq Ra_d - [L_-(\alpha)y] \geq (R-1)a_d + L_+(\alpha)y,$$

and so $I_h \subset (R-1, R+1)$.

By (7.5), (7.6), $[\xi\alpha^n]$ is calculated as follows:

$$[\xi\alpha^n] = -\frac{1}{a_d}\sum_{i=1}^{d}[\xi\alpha^{n-i}] - \frac{h}{a_d} + w_h,$$

where

$$-\sum_{i=1}^{d} a_{d-i}[\xi\alpha^{n-i}] \equiv h \bmod a_d \text{ with } h \in \{0, 1, \ldots, -1 + a_d\}.$$

Thus, if $\xi \in S_+$ and $n \geq d$, then $[\xi\alpha^n]$ depends only on $[\xi\alpha^{n-1}], \ldots, [\xi\alpha^{n-d}]$. Therefore, the two-sided sequences $([\xi\alpha^m])_{m=-\infty}^{\infty}$ and $(s_m(\alpha;\xi))_{m=-\infty}^{\infty}$ are obtained by $([\xi\alpha^m])_{m=-\infty}^{d-1}$. Note that the cardinality of the set

$$\left\{([\xi\alpha^m])_{m=-\infty}^{d-1} | \xi \in S_+\right\}$$

is at most countable because $[\xi\alpha^{-m}] = 0$ for all sufficiently large m. By Proposition 4.1, $\xi \in S_+$ is uniquely determined by the sequence $(s_m(\alpha;\xi))_{m=-\infty}^{\infty}$, and so by $([\xi\alpha^m])_{m=-\infty}^{d-1}$. Consequently, the cardinality of S_+ is at most countable.

Next, we verify that S_- is a countable set. Let $\xi \in S_-$. Note for $m \geq 0$ that

$$1 - \{-\xi\alpha^m\} = \{\xi\alpha^m\}$$

since $\xi\alpha^m \notin \mathbb{Z}$. If $n \geq d$, then

$$[-\xi\alpha^n] = -\frac{1}{a_d}\sum_{i=1}^{d} a_{d-i}[-\xi\alpha^{n-i}] - \frac{1}{a_d}\sum_{i=0}^{d} a_i + \frac{1}{a_d}\sum_{i=0}^{d} a_{d-i}\{\xi\alpha^{n-i}\}$$

and

$$\frac{1}{a_d}\sum_{i=0}^{d} a_{d-i}\{\xi\alpha^{n-i}\} \in I'_0,$$

where

$$I'_h = I'_h(y) = \left(\frac{h}{a_d} - \frac{L_-(\alpha)}{a_d}y, \frac{h}{a_d} + \frac{L_+(\alpha)}{a_d}y\right) \ (0 \leq h \leq a_d - 1).$$

The interval I'_h has at most one integer point. If such an integer exists, we denote it by w'_h. In fact, by putting

$$R' = 1 + \left[\frac{h - L_-(\alpha)y}{a_d}\right],$$

we get $I'_h \subset (R' - 1, R' + 1)$. Thus, if $n \geq d$, we calculate the value $[-\xi\alpha^n]$ by using $[-\xi\alpha^{n-1}], \ldots, [-\xi\alpha^{n-d}]$ as follows:

$$[-\xi\alpha^n] = -\frac{1}{a_d}\sum_{i=1}^{d} a_{d-i}[-\xi\alpha^{n-i}] - \frac{1}{a_d}\sum_{i=0}^{d} a_i - \frac{h}{a_d} + w'_h,$$

where

$$-\sum_{i=1}^{d} a_{d-i}[-\xi\alpha^{n-i}] - \sum_{i=0}^{d} a_i \equiv h \bmod a_d \text{ with } h \in \{0, 1, \ldots, -1 + a_d\}.$$

Finally, by Proposition 4.1 $-\xi$ depends only on $([-\xi\alpha^m])_{m=-\infty}^{d-1}$, which implies that the cardinality of S_- is at most countable. We can also verify the theorem in the case of $L_-(\alpha) < L_+(\alpha)$ in the same way as above by showing that $I_h \subset (R^{(2)} - 1, R^{(2)} + 1)$ for $0 \le h \le a_d - 1$, where

$$R^{(2)} = 1 + \left[\frac{h - L_+(\alpha)y}{a_d}\right]$$

and that $I'_h \subset (R^{(3)} - 1, R^{(3)} + 1)$ for $0 \le h \le a_d - 1$, where

$$R^{(3)} = \left[\frac{h + L_+(\alpha)y}{a_d}\right].$$

□

Let $\alpha > 1$ be an algebraic number and y a positive number. Suppose that y satisfies the assumption of Theorem 7.1. Then by Theorem 7.1 there exist at most countably many nonzero ξ such that all limit points of the sequence $\xi\alpha^n \bmod \mathbb{Z}$ $(n = 0, 1, \ldots)$ lie in $\tau([0, y])$. We now consider the cardinality of the set of real ξ such that all limit points of $\xi\alpha^n \bmod \mathbb{Z}$ $(n = 0, 1, \ldots)$ lie in a given interval in $\mathbb{R}/\mathbb{Z}$.

THEOREM 7.2. *Let $\alpha > 1$ be an algebraic number and $a_d(> 0)$ the leading coefficient of the minimal polynomial of α. Suppose that α does not have a conjugate on the unit circle. Let J be any interval in $\mathbb{R}/\mathbb{Z}$ such that its Haar measure satisfies*

$$\mu(J) < \frac{a_d}{L(\alpha)}. \tag{7.7}$$

Then there exist at most countably many real ξ such that all limit points of $\xi\alpha^n \bmod \mathbb{Z}$ $(n = 0, 1, \ldots)$ lie in J.

REMARK 7.1. Let $J = \tau([0, y])$ $(y > 0)$. Then (7.7) is rewritten by

$$L(\alpha)y < a_d.$$

The assumption above is stronger than (7.1) and (7.2). In fact,

$$L_+(\alpha)y + [L_-(\alpha)y] \le L(\alpha)y$$

and

$$L_+(\alpha)y + [L_-(\alpha)y] \le L(\alpha)y.$$

Example 7.2. We consider the case of $\alpha = \theta_1$ again. For any interval J in $\mathbb{R}/\mathbb{Z}$ with $\mu(J) < 2/53 = 0.03773\ldots(< 1/25)$, there exist at most countably many real ξ such that all limit points of $\xi\alpha^n \bmod \mathbb{Z}$ $(n = 0, 1, \ldots)$ lie in J.

Proof of Theorem 7.2. It suffices to prove the following:

LEMMA 7.1. *Let J' be any interval in $\mathbb{R}/\mathbb{Z}$ with length*

$$\mu(J') < \frac{a_d}{L(\alpha)}.$$

Then there are at most countably many real ξ such that

$$\xi\alpha^n \bmod \mathbb{Z} \in J'$$

for any $n \geq 0$.

We check that Lemma 7.1 implies Theorem 7.2. Without loss of generality, we may assume that J is closed. Write J by

$$J = \tau([y_1, y_2]),$$

where $y_1 < y_2$ are real numbers with $y_2 - y_1 < a_d/L(\alpha)$. Take a sufficiently small $\varepsilon > 0$ such that

$$y_2 - y_1 + 2\varepsilon < \frac{a_d}{L(\alpha)}.$$

Put

$$J' = \tau([y_1 - \varepsilon, y_2 + \varepsilon]).$$

Let S (resp. S') be the set of ξ satisfying the properties of Theorem 7.2 (resp. Lemma 7.1). Then, since

$$S \subset \{\xi\alpha^m | m \in \mathbb{Z}, \xi \in S'\},$$

the cardinality of S is at most countable.

Let us verify Lemma 7.1. It suffices to prove the lemma in the case where J' is denoted as

$$J' = \tau([y, y + \delta]),$$

where $\delta < a_d/L(\alpha)$ and $-1 < y \leq 0$. We choose a real η with $-1 < \eta < y$. Then, for any real x there exist a unique integer $\varphi(x)$ and a real number $\psi(x)$ with $\psi(x) \in [\eta, \eta + 1)$ satisfying

$$x = \varphi(x) + \psi(x).$$

Note that 0 is an inner point of $[\eta, \eta + 1)$ since $-1 < \eta < 0$. Thus, if ξ is a real number, then we have $\psi(\xi\alpha^{-n}) = \xi\alpha^{-n}$ and $\varphi(\xi\alpha^{-n}) = 0$ for all sufficiently large n.

In the rest of the proof, we show that $\xi \in S'$ is uniquely determined by a sequence $(\varphi(\xi\alpha^m))_{m=-\infty}^{d-1}$. The cardinality of the set of such sequences is at most countable since $\varphi(\xi\alpha^{-n}) = 0$ for all sufficiently large n. Hence the theorem follows.

Let $p, \alpha_1, \ldots, \alpha_d$, and $a_d X^d + \cdots + a_0 \in \mathbb{Z}[X]$ be defined as Section 4. By putting

$$s'_m(\alpha; \xi) = \sum_{i=0}^{d} a_{d-i}\varphi(\xi\alpha^{-m-i}),$$

we obtain

$$\xi = \frac{1}{a_d} \sum_{i=-\infty}^{\infty} \sum_{j=0}^{\infty} \rho_i(\alpha_1, \dots, \alpha_p) \rho_j(\alpha_{1+p}, \dots, \alpha_d) s'_{i+j}(\alpha; \xi). \tag{7.8}$$

The proof of (7.8) is the same as that of (4.4).

We prove for $\xi \in S'$ that $\varphi(\xi\alpha^n)$ depends only on $\varphi(\xi\alpha^{n-1}), \dots, \varphi(\xi\alpha^{n-d})$ for $n \geq d$. By

$$0 = \frac{1}{a_d} \sum_{i=0}^{d} a_{d-i} \xi \alpha^{n-i} = \frac{1}{a_d} \sum_{i=0}^{d} a_{d-i} \Big(\varphi(\xi\alpha^{n-i}) + \psi(\xi\alpha^{n-i}) \Big),$$

we get

$$\varphi(\xi\alpha^n) + \frac{1}{a_d} \sum_{i=0}^{d} a_{d-i} \psi(\xi\alpha^{n-i}) = -\frac{1}{a_d} \sum_{i=1}^{d} a_{d-i} \varphi(\xi\alpha^{n-i}). \tag{7.9}$$

Thus

$$\frac{1}{a_d} \sum_{i=0}^{d} a_{d-i} \psi(\xi\alpha^{n-i}) \in K,$$

where the interval K is defined by

$$K = \left[\frac{y}{a_d} \sum_{i=0}^{d} a_i - \frac{L_-(\alpha)\delta}{a_d}, \frac{y}{a_d} \sum_{i=0}^{d} a_i + \frac{L_+(\alpha)\delta}{a_d} \right].$$

Note that $[y, y+\delta] \subset [\eta, \eta+1)$. So $y \leq \psi(\xi\alpha^n) \leq y + \delta$ for any $n \geq 0$ by the definition of $\psi(x)$ for a real x. Thus the length of K is less than 1 by the assumption of Lemma 7.1. Hence, since $\varphi(\xi\alpha^n)$ is a rational integer, $\varphi(\xi\alpha^n)$ is calculated by (7.9).

Therefore, if $\xi \in S'$, then the sequence $(\varphi(\xi\alpha^m))_{m=-\infty}^{\infty}$ and the value ξ depend only on the sequence $(\varphi(\xi\alpha^m))_{m=-\infty}^{d-1}$. $\square$

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References

[1] S. Akiyama, C. Frougny, J. Sakarovitch, Powers of rationals modulo 1 and rational base number systems, Israel J. Math. **168** (2008), 53-91.

[2] M. J. Bertin, A. Decomps-Guilloux, M. Grandet-Hugot, M. Pathiaux-Delefosse, and J. P. Schreiber, Pisot and Salem numbers, Birkhäuser Verlag, Basel, 1992.

[3] Y. Bugeaud, Linear mod one transformations and the distribution of fractional parts $\{\xi(p/q)^n\}$, Acta Arith. **114** (2004), 301-311.

[4] J. W. S. Cassels, An introduction to Diophantine approximation, Cambridge University Press, 1957.

[5] A. Dubickas, Arithmetical properties of powers of algebraic numbers, Bull. London Math. Soc. **38** (2006), 70-80.

[6] A. Dubickas, On the distance from a rational power to the nearest integer, J. Number Theory. **117** (2006), 222-239.

[7] A. Dubickas, On the fractional parts of lacunary sequences. Math. Scand. **99** (2006), 136-146.

[8] L. Flatto, Z-numbers and β-transformations, in: P. Walters (Ed.), Symbolic Dynamics and its Applications, New Haven, 1991, Contemp. Math. **135** (1992), 181-201

[9] L. Flatto, J. C. Lagarias, A. D. Pollington, On the range of fractional parts $\{\xi(p/q)^n\}$, Acta Arith. **70** (1995), 125-147.

[10] H. Kaneko, Distribution of geometric sequences modulo 1, Result. Math. **52** (2008), 91-109.

[11] J. F. Koksma, Ein mengen-theoretischer Satz über Gleichverteilung modulo eins, Compositio Math. **2** (1935), 250-258.

[12] K. Mahler, An unsolved problems on the powers of 3/2, J. Austral. Math. Soc. **8** (1968), 313-321.

[13] K. Nishioka, Mahler Functions and Transcendence, in: Lecture Notes in Mathematics, Vol. 1631, Springer, Berlin, 1996.

[14] Ch. Pisot, Répartition (mod 1) des puissances successives des nombres réels, Comm. Math. Helv. **19** (1946), 153-160.

[15] R. Tijdeman, Note on Mahler's $\frac{3}{2}$-problem, K. Norske Vidensk. Selsk. Skr. **16** (1972), 1-4.

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