Applications of numerical systems to transcendental number theory

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1 Introduction

There are close relations between numerical systems and number theory. For example, let b be an integer greater than 1. Then base-b expansions of real numbers are related to uniform distribution theory. Let ξ be a nonnegative real number. We write the integral and fractional parts of ξ by $\lfloor \xi \rfloor$ and $\{\xi\}$, respectively. Then ξ is a normal number in base-b if and only if ξb^n $(n = 0, 1, \ldots)$ is uniformly distributed modulo 1. Borel [7] conjectured that any algebraic irrational number is normal in every integral base-b. If Borel's conjecture is true, then it gives strong criteria for transcendence of real numbers. In Section 2 we introduce criteria for transcendence related to Borel's conjecture. In Section 3 we consider transcendence of the values of power series at algebraic points, which is related to the β-expansion of real numbers. In Section 4 we study algebraic independence of the values of lacunary series. In Section 5 we give algebraic independence related to the base-b expansions of real numbers. For references on base-b expansions, β-expansions, and more general numerical systems, see [4, 18]. There are a number of excellent books on uniform distribution theory [8, 12, 17]. In particular, see [8] for more details on relations between numerical systems and number theory. In this paper we denote the set of nonnegative integers by $\mathbb{N}$. We use the Landau symbols o and O with its usual meaning. Namely, we write $f = o(g)$ if f/g tends to zero. Moreover, $f = O(g)$ implies that $|f| \le cg$ with certain positive constant c.

2 Transcendence of the values of power series at certain rational points

Let $w(n)$ $(n = 0, 1, \ldots)$ be a strictly increasing sequence of nonnegative integers. Put

$$f(w(n); X) := \sum_{n=0}^{\infty} X^{w(n)}.$$

Bugeaud [9] conjectured that if $w(n)$ $(n = 0, 1, \ldots)$ increases sufficiently rapidly, then $f(w(n); \alpha)$ is transcendental for any algebraic α with $0 < |\alpha| < 1$. If b is

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an integer greater than 1, then the equality

$$\xi_b(w(n)) := f(w(n); b^{-1}) = \sum_{n=0}^{\infty} b^{-w(n)} \tag{2.1}$$

gives the base-b expansion of $\xi_b(w(n))$. Suppose that $w(n)$ $(n = 0, 1, \ldots)$ fulfills

$$\lim_{n\to\infty} \frac{w(n)}{n} = \infty. \tag{2.2}$$

Then $\xi_b(w(n))$ is neither rational nor normal. So, if $w(n)$ $(n = 0, 1, \ldots)$ satisfies (2.2) and if Borel's conjecture is true, then $\xi_b(w(n))$ is transcendental. Note for algebraic α with $0 < |\alpha| < 1$ that if subsums of $f(w(n); \alpha) = \sum_{n=0}^{\infty} \alpha^{w(n)}$ vanish, then $f(w(n); \alpha)$ is not generally transcendental. In fact, let α_0 be a unique zero of $X^3 + X + 1$ on the interval $(-1, 0)$. Then we have

$$0 = \sum_{n=2}^{\infty} \alpha_0^{n!}(1 + \alpha_0 + \alpha_0^3) = \alpha_0^2 + \alpha_0^3 + \alpha_0^5 + \alpha_0^6 + \alpha_0^7 + \alpha_0^9 + \cdots .$$

Next we consider the case of $b \geq 3$. Then the digits greater than 1 do not appear in the base-b expansion of $\xi_b(w(n))$. In particular, $\xi_b(w(n))$ is not normal in base-b. Thus, if Borel's conjecture holds, then $\xi_b(w(n))$ is rational or transcendental.

However, we know little on the base-b expansions of algebraic irrational numbers. For instance, we cannot prove that 1 appears infinitely many times in the decimal expansion of $\sqrt{2}$. There is no algebraic number whose normality was proved. There is also no known counter example on Borel's conjecture. Here we introduce known partial results on Borel's conjecture. In particular, we study the numbers of nonzero digits. Let η be a real number whose base-b expansion is written as

$$\eta = \lfloor \eta \rfloor + \sum_{n=1}^{\infty} s_n(b; \eta) b^{-n},$$

where $s_n(b; \eta) \in \{0, 1, \ldots, b-1\}$ for any $n \geq 1$ and $s_n(b; \eta) \leq b - 2$ for infinitely many n's. We write the number of nonzero digits among the first N digits of η by

$$\nu_b(\eta; N) := \mathrm{Card}\{n \in \mathbb{N} \mid n \leq N, s_n(b; \eta) \neq 0\},$$

where Card denotes the cardinality. Consider the case of $b = 2$. Let η be an algebraic irrational number of degree D. Then Bailey, Borwein, Crandall, and Pomerance [5] proved that there exist positive constants $C_1(\eta), C_2(\eta)$ (depending only on η) satisfying the following: for any integer N with $N \geq C_2(\eta)$ we have

$$\nu_2(\eta; N) \geq C_1(\eta) N^{1/D}. \tag{2.3}$$

In the proof of (2.3), the Thue—Siegel—Roth theorem [25] was applied. We can verify analogies of (2.3) in the same way as the proof of Theorem 7.1 in [5]. Moreover, applying Liouville's inequality instead of the Thue—Siegel—

Roth theorem and modifying the proof, we obtain an effective version of lower bounds. Namely, there are positive constants $C_3(b, \eta), C_4(b, \eta)$ depending only on b and η such that

$$\nu_b(\eta; N) \geq C_3(b, \eta) N^{1/D} \tag{2.4}$$

for any integer N with $N \geq C_4(b, \eta)$. In the case of $b = 2$, Rivoal [24] improved $C_1(\eta)$ for certain classes of algebraic irrational η. Adamczewski, Faverjon [3], and Bugeaud [8] independently calculated explicit formulae for $C_3(b, \eta)$ and $C_4(b, \eta)$ in (2.4). Here we introduce the formulae by Bugeaud as follows: Let $A_D X^D + A_{D-1} X^{D-1} + \cdots + A_0 \in \mathbb{Z}[X]$, where $A_D > 0$, be the minimal polynomial of $1 + \{\xi\}$. Let

$$H := \max\{|A_i| \mid 0 \leq i \leq D\}.$$

Then, for any integer N with $N > (20 b^D D^2 H)^{2D}$, we have

$$\nu_b(\eta; N) \geq \frac{1}{b-1} \left(\frac{N}{2(D+1) A_D} \right)^{1/D}. \tag{2.5}$$

Using (2.4) or (2.5), we obtain criteria for transcendence related to the base-b expansions of real numbers. Suppose that $w(n)$ $(n = 0, 1, \ldots)$ satisfies

$$\lim_{n \to \infty} \frac{w(n)}{n^R} = \infty \tag{2.6}$$

for any positive real number R. Then we have

$$\nu_b(\xi_b(w(n)); N) = o(N^{\varepsilon})$$

as N tends to infinity, where ε is an arbitrary positive real number. We now give examples. Let y be a positive real number. Put

$$\tau_y(n) := \left\lfloor \exp\left((\log y)^{1+y} \right) \right\rfloor \ (n = 1, 2, \ldots) \tag{2.7}$$

and

$$\mu_y(X) := f(\tau_y(n) : X) = \sum_{n=1}^{\infty} X^{\tau_y(n)}. \tag{2.8}$$

It is easily seen that $\tau_y(n)$ $(n = 1, 2, \ldots)$ satisfies (2.6) because $n = \exp(R \log n)$. Hence, $\mu_y(b^{-1})$ is transcendental for any integer b greater than 1. However, it is still unknown whether $\mu_y(-b^{-1})$ is transcendental or not.

3 Transcendence of the values of lacunary series at algebraic points

Let β be a real number greater than 1. The β-expansions of real numbers are introduced by Rényi [22] in 1957. Recall that β transformation is defined on the interval $[0, 1]$ by $T_\beta : x \longmapsto \beta x \mod 1$. Let x be a real number with $0 \leq x < 1$. Then the β-expansion of x is denoted by

$$x = \sum_{n=1}^{\infty} \frac{t_n(\beta; x)}{\beta^n},$$

where $t_n(\beta; x) = \lfloor \beta T_\beta^{n-1}(x) \rfloor \in \mathbb{Z} \cap [0, \beta)$ for $n = 1, 2, \ldots$. In the case where x is a general nonnegative real number, we define the β-expansion of x by using the β-expansion of $\beta^{-k}x$, where k is an integer with $0 \leq \beta^{-k}x < 1$. A sequence $s_1 s_2 \ldots$ is called β-admissible if there exists an $x \in [0, 1)$ such that $s_n = t_n(\beta; x)$ for any positive integer n. Here, we put

$$a_n(\beta) := \lim_{x \to 1-} t_n(\beta; x)$$

for $n = 1, 2, \ldots$. Then Parry [21] showed that $s_1 s_2 \ldots$ is β-admissible if and only if

$$00 \ldots \leq_{lex} s_k s_{k+1} \ldots <_{lex} a_1(\beta) a_2(\beta) \ldots$$

for any $k = 1, 2, \ldots$, where $<_{lex}$ denotes the lexicographical order. β-expansions are natural generalizations of base-b expansions for integral base $b \geq 2$. In particular, consider the case of $\beta > 2$. Then every sequence $s_1 s_2 \ldots$, where $s_n \in \{0, 1\}$ for $n = 1, 2, \ldots$, is β-admissible because $a_1(\beta) \geq 2$. Let again $w(n)(n = 0, 1, \ldots)$ be a strictly increasing sequence of nonnegative integers. Then

$$\xi_\beta(w(n)) := f(w(n); \beta^{-1}) = \sum_{n=0}^{\infty} \beta^{-w(n)}$$

gives the β-expansion of $\xi_\beta(w(n))$. We discuss the transcendence of $\xi_\beta(w(n))$. We now recall the following results by Corvaja and Zannier [11]: Assume that $w(n)(n = 0, 1, \ldots)$ satisfies

$$\liminf_{n \to \infty} \frac{w(n+1)}{w(n)} > 1. \tag{3.1}$$

Then, for any algebraic α with $0 < |\alpha| < 1$, we get that $f(w(n); \alpha)$ is transcendental. If $w(n)$ $(n = 0, 1, \ldots)$ fulfills (3.1), then we say that $w(n)$ $(n = 0, 1, \ldots)$ is lacunary. In particular, $\xi_\beta(w(n))$ is transcendental for any real algebraic number $\beta > 1$. The proof of the criteria above is based on the Schmidt subspace theorem. As we mentioned in Section 2, if $\beta = b$ is an integer greater than 1, then the transcendental results on $\xi_b(w(n))$ hold under weaker assumptions than (3.1). Here we introduce other criteria for transcendence of $\xi_b(w(n))$. Using Ridout's theorem [23], we deduce the following: Suppose that

$$\limsup_{n \to \infty} \frac{w(n+1)}{w(n)} > 1, \tag{3.2}$$

which is weaker than (3.1). Then $\xi_b(w(n))$ is transcendental for any integer $b \geq 2$. Recall that a Pisot number is an algebraic integer greater than 1 such that the conjugates except itself have absolute values less than 1. Moreover, a Salem number is an algebraic integer greater than 1 such that the conjugates except itself have absolute values at most 1 and that at least one conjugate has absolute value 1. Adamczewski [1] showed for any Pisot or Salem number β that if $w(n)$ $(n = 0, 1, \ldots)$ satisfies (3.2), then $\xi_\beta(w(n))$ is transcendental.

Investigating the digits of β-expansions of algebraic numbers, we obtain criteria for transcendence of real numbers. However, β-expansion of algebraic numbers are mysterious. Bugeaud [10] studied digits of β-expansions of algebraic numbers, giving lower bounds of the number of nonzero digits denoted as

$$\gamma_\beta(x;N) := \mathrm{Card}\{n \in \mathbb{Z} \mid 1 \le n \le N,\ t_n(\beta;x) \ne t_{n+1}(\beta;x)\},$$

where x is a nonnegative real number and N is a positive integer. He proved the following: Let η be an algebraic number such that $t_n(\beta;x) \ne 0$ for infinitely many positive integer n. Then there exists an effectively computable positive constant $C_5(\beta,\eta)$, depending only on β and η, such that

$$\gamma_\beta(\eta;N) \ge C_5(\beta,\eta)(\log N)^{3/2}(\log\log N)^{-3/2}$$

for any sufficiently large N. Consequently, we obtain the following results on transcendence: Let again β be a Pisot or Salem number. Let y be a real number with $y > 2/3$. Put

$$\rho_y(n) := 2^{\lfloor n^y \rfloor}$$

for $n \ge 1$. Then $\xi_\beta(\rho_y(n))$ is transcendental.

4 Algebraic independence of the values of lacunary series at algebraic points

In Sections 1 and 2 we introduced the transcendence of $f(w(n);\alpha)$ related to the rate of increase of $w(n)$ $(n = 0,1,\ldots)$. In particular, recall that if $w(n)$ $(n = 0,1,\ldots)$ is lacunary, then $f(w(n);\alpha)$ is transcendental for any algebraic α with $0 < |\alpha| < 1$. In this section we study the algebraic independence of $f(w(n);\alpha)$ in the case where the rates of increases of the sequences $w(n)$ $(n = 0,1,\ldots)$ are different. We first consider the case of

$$\lim_{n\to\infty} \frac{w(n+1)}{w(n)} = \infty. \tag{4.1}$$

Schmidt [26] gave criteria for algebraic independence, generalizing Liouville's inequality. Using his criteria, we deduce that if $\alpha = b$ is an integer greater than 1, then the set

$$\left\{ \xi_b((kn)!) = \sum_{n=0}^{\infty} b^{-(kn)!} \;\middle|\; k = 1,2,\ldots \right\}$$

is algebraically independent. In the case where α is a general algebraic number with $0 < |\alpha| < 1$, Shiokawa [27] gave criteria for algebraic independence. For instance, applying his criteria, we obtain that the continuum set

$$\left\{ f(\lfloor \lambda(n!) \rfloor;\alpha) = \sum_{n=0}^{\infty} \alpha^{\lfloor \lambda(n!) \rfloor} \;\middle|\; \lambda \in \mathbb{R},\ \lambda > 0 \right\} \tag{4.2}$$

is algebraically independent. Note that the algebraic independence of the set (4.2) was proved by Durand [13] in the case where α is a real algebraic number with $0 < \alpha < 1$.

Next we consider the case where $w(n)$ $(n = 0, 1, \ldots)$ does not satisfy (4.1). Mahler's method for algebraic independence is applicable to power series satisfying certain functional equations. For instance, let k be an integer greater than 1. Then $f(k^n; X) = \sum_{n=0}^{\infty} X^{k^n}$ satisfies

$$f(k^n; X^k) = \sum_{n=0}^{\infty} X^{k^{n+1}} = \sum_{n=0}^{\infty} X^{k^n} - X = f(k^n; X) - X.$$

Using Mahler's method, Nishioka [19] proved for any algebraic α with $0 < |\alpha| < 1$ that the set

$$\left\{ f(k^n; \alpha) = \sum_{n=0}^{\infty} \alpha^{k^n} \;\middle|\; k = 2, 3, \ldots \right\}$$

is algebraically independent. For more details on Mahler's method, see [20].

5 Main results

We recall that $\mu_y(X)$ is defined by (2.7) and (2.8) for a positive real number y and that transcendental results in Section 2 is applicable even to the case of

$$\lim_{n\to\infty} \frac{w(n+1)}{w(n)} = 1.$$

In fact, for a positive real y and a positive integer n, put

$$\widetilde{\tau_y}(n) := \exp\left((\log y)^{1+y}\right).$$

Then we have $\tau_y(n) = \lfloor \widetilde{\tau_y}(n) \rfloor$. Observe that

$$\log \widetilde{\tau_y}(n+1) - \log \widetilde{\tau_y} = \big(\log(n+1)\big)^{1+y} - (\log n)^{1+y}.$$

The mean value theorem implies that there exists a real number δ with $n < \delta < n + 1$ satisfying

$$\log \widetilde{\tau_y}(n+1) - \log \widetilde{\tau_y} = (1+y)\frac{(\log \delta)^{1+y}}{\delta},$$

which tends to zero as n tends to infinity. Thus, we obtain

$$\lim_{n\to\infty} \frac{\tau_y(n+1)}{\tau_y(n)} = \lim_{n\to\infty} \frac{\widetilde{\tau_y}(n+1)}{\widetilde{\tau_y}(n)} = 1.$$

We introduce algebraic independence of $\mu_y(b^{-1})$ for distinct y.

THEOREM 5.1 ([15]). *Let b be an integer greater than 1. Then the continuum set*

$$\{\mu_y(b^{-1}) \mid y \in \mathbb{R},\ y \geq 1\}$$

is algebraically independent.

We recall that the algebraic independence of $\{\mu_y(b^{-1}) \mid y \in \mathbb{R},\ y \geq 1\}$ implies the following: If we take arbitrary number of distinct real numbers $y_1, \ldots, y_r \geq 1$, then $\mu_{y_1}(b^{-1}), \ldots, \mu_{y_r}(b^{-1})$ are algebraically independent. It is unknown whether the set $\{\mu_y(b^{-1}) \mid y \in \mathbb{R},\ y > 0\}$ is algebraically independent or not. However, we have the following:

THEOREM 5.2 ([15])**.** *Let b be an integer greater than 1 and x, y distinct positive real numbers. Then $\mu_x(b^{-1})$ and $\mu_y(b^{-1})$ are algebraically independent.*

The lower bounds (2.4) or (2.5) implies the following: Let D be an integer and p a real number with $2 \leq D < p$. Then

$$\zeta_p(b^{-1}) := \sum_{n=0}^{\infty} b^{-\lfloor n^p \rfloor}$$

is not an algebraic number of degree at most D. The result above holds even in the case of $D = 1$. In fact, if $p > 1$, then $\zeta_p(b^{-1})$ is irrational because its base-b expansion is not periodic. If $p = 2$, then it is known that $\zeta_2(b^{-1})$ is transcendental (see [6, 14]). However, if p is a real number greater than 1, the transcendence of $\zeta_p(b^{-1})$ has not been proved.

Here we study further arithmetical properties on $\zeta_p(b^{-1})$. We introduce some notation to state the results. Let D be an integer greater than 2. Then it is easily seen that the polynomial

$$(1 - X)^D + (D - 1)X - 1$$

has a unique zero σ_D on the interval $(0, 1)$. Recall that $\xi_b(w(n))$ is defined by (2.1).

THEOREM 5.3. *Let b be an integer greater than 1 and $w(n)$ $(n = 0, 1, \ldots)$ a sequence of strictly increasing nonnegative integers. Suppose that $w(n)$ $(n = 0, 1, \ldots)$ satisfies the following two assumptions:*

1. *For any positive real number R, we have*

$$\lim_{n\to\infty} \frac{w(n)}{n^R} = \infty.$$

2. $$\limsup_{n\to\infty} \frac{w(n+1)}{w(n)} < \infty.$$

Let D be a positive integer and p a real number. If $1 \leq D \leq 3$, then assume that $p > D$. If $D \geq 4$, then suppose that $p > \sigma_D^{-1}$. Then the set

$$\{\zeta_p(b^{-1})^i \xi_b(w(n))^j \mid 0 \leq i \leq D,\ 0 \leq j\}$$

is linearly independent over $\mathbb{Q}$.

For example, we have $\sigma_4^{-1} = 5.278\ldots$, $\sigma_5^{-1} = 8.942\ldots$, $\sigma_6^{-1} = 13.60\ldots$. Note that Theorem 5.3 gives partial results on algebraic independence. In fact, two complex numbers x and y are algebraically independent if and only if the set

$$\{x^i y^j \mid 0 \leq i, j\}$$

is linearly independent over $\mathbb{Q}$.

6 Sketch of the proof of Theorem 5.3

In this section we provide a sketch of the proof of Theorem 5.3 without technical details. For simplicity, we put

$$\zeta := \zeta_p(b^{-1}),\ \xi := \xi_b(w(n)).$$

Then we have $1 \le \zeta < 2$. If necessary, changing ξ by $\{\xi\}+1$, we may assume that $1 \le \xi < 2$. We write the base-b expansions of ζ and ξ by

$$\zeta =: \sum_{m=0}^{\infty} s(m)b^{-m},\ \xi =: \sum_{n=0}^{\infty} t(n)b^{-n},$$

respectively, where $s(0) = \lfloor \zeta \rfloor = 1$ and $t(0) = \lfloor \xi \rfloor = 1$. In particular, $0 \le s(m), t(m) \le b-1$ for any nonnegative integer m. Let D be defined as in Theorem 5.3. Let $P(X,Y)$ be a non-constant polynomial with integral coefficients such that the degree in X is not greater than D. For the proof of Theorem 5.3, we show that $P(\zeta,\xi) \neq 0$ for such a polynomial. If necessary, changing $P(X,Y)$ by $YP(X,Y)$, we may assume that Y divides $P(X,Y)$. We denote the coefficients of $P(X,Y)$ by

$$P(X,Y) =: \sum_{\mathbf{k}=(k,l)\in\Lambda} A_{\mathbf{k}} X^k Y^l,$$

where Λ is a nonempty finite subset of $([0,D]\cap\mathbb{N})\times\mathbb{N}$ and $A_{\mathbf{k}}$ is a nonzero integer for each $\mathbf{k} \in \Lambda$. In order to show that $P(\zeta,\xi) \neq 0$, we search nonzero digits of the base-b expansion of $P(\zeta,\xi)$, using the assumptions on D and $w(n)$ $(n = 0,1,\ldots)$ in Theorem 5.3. The idea was inspired by the paper by Knight [16]. For any $\mathbf{k} = (k,l) \in \Lambda$, we get

$$\begin{aligned}
\zeta^k\xi^l &= \left(\sum_{m=0}^{\infty} s(m)b^{-m}\right)^k \left(\sum_{n=0}^{\infty} t(n)b^{-n}\right)^l \\
&= \sum_{i=0}^{\infty} b^{-i} \sum_{\substack{m_1,\ldots,m_k,n_1,\ldots,n_l\ge 0\\ m_1+\cdots+m_k+n_1+\cdots+n_l=i}} s(m_1)\cdots s(m_k)t(n_1)\cdots t(n_l) \\
&=: \sum_{i=0}^{\infty} b^{-i}\rho(\mathbf{k};i). \qquad (6.1)
\end{aligned}$$

It is easily seen that $\rho(\mathbf{k};i)$ is a nonnegative integer. Moreover, let δ be the total degree of $P(X,Y)$. Then

$$\begin{aligned}
\rho(\mathbf{k};i) &\le \sum_{\substack{m_1,\ldots,m_k,n_1,\ldots,n_l\ge 0\\ m_1+\cdots+m_k+n_1+\cdots+n_l=i}} (b-1)^{k+l} \\
&\le (b-1)^{k+l}(i+1)^{k+l} \le (b-1)^{\delta}(i+1)^{\delta}.
\end{aligned}$$

In particular, if i is greater than 1, then

$$\log\big(\rho(\mathbf{k};i)\big) = O(\log i). \qquad (6.2)$$

We study the conditions of positivity of $\rho(\mathbf{k};i)$. Set

$$S := \{m \in \mathbb{N} \mid s(m) \neq 0\},\ T := \{n \in \mathbb{N} \mid t(n) \neq 0\}.$$

Then we have $S, T \ni 0$ because $s(0) = t(0) = 1$. Moreover, put

$$kS + lT := \{m_1 + \cdots + m_k + n_1 + \cdots + n_l \mid m_1, \ldots, m_k \in S, n_1, \ldots, n_l \in T\}.$$

Let $(k,l), (k',l') \in \Lambda$ with $k \geq k'$ and $l \geq l'$. Then we have

$$kS + lT \supset k'S + l'T \tag{6.3}$$

because $S, T \ni 0$. We rewrite the conditions of positivity of $\rho(\mathbf{k};i)$, using the set $kS + lT$. Observe that

$$\rho(\mathbf{k};i) = \sum_{\substack{m_1,\ldots,m_k \in S, n_1,\ldots,n_l \in T \\ m_1+\cdots+m_k+n_1+\cdots+n_l = i}} s(m_1)\cdots s(m_k)t(n_1)\cdots t(n_l).$$

Thus, $\rho(\mathbf{k};i)$ is positive if and only if $i \in kS + lT$.

Using (6.1), we obtain

$$\begin{aligned} P(\zeta,\xi) &= \sum_{\mathbf{k}=(k,l)\in\Lambda} A_{\mathbf{k}}\zeta^k\xi^l = \sum_{\mathbf{k}=(k,l)\in\Lambda} A_{\mathbf{k}} \sum_{i=0}^{\infty} b^{-i}\rho(\mathbf{k};i) \\ &= \sum_{i=0}^{\infty} b^{-i} \sum_{\mathbf{k}=(k,l)\in\Lambda} A_{\mathbf{k}}\rho(\mathbf{k};i). \end{aligned} \tag{6.4}$$

Note that $\sum_{\mathbf{k}=(k,l)\in\Lambda} A_{\mathbf{k}}\rho(\mathbf{k};i)$ is not generally nonnegative. Let $\succ$ be the lexicographical order in $\mathbb{N}^2$, that is, $(k,l) \succ (k',l')$ if either $k > k'$, or $k = k'$ and $l > l'$. We write by $\mathbf{g} = (g,h)$ the maximal element of Λ with respect to $\succ$. Then h is positive because $P(X,Y)$ is divisible by Y. We may assume that $A_{\mathbf{g}} > 0$. In what follows, we search an integer i such that $\rho(\mathbf{g};i) > 0$ and that $\rho(\mathbf{k};i) = 0$ for any $\mathbf{k} \in \Lambda\backslash\{\mathbf{g}\}$. Put

$$\theta_g(R) := \max\{n \in gS \mid n < R\}.$$

Moreover, let

$$\begin{aligned} \lambda_1(R) &:= \{m \in \mathbb{N} \mid m \in S, m \leq R\}, \\ \lambda_2(R) &:= \{n \in \mathbb{N} \mid n \in T, n \leq R\}. \end{aligned}$$

Let $(k,l) \in \mathbb{N}^2$ with $k < g$. We use the assumptions on D and the first assumption on $w(n)$ $(n = 0, 1, \ldots)$ in order to check that

$$R - \theta_g(R) = o\left(\frac{R}{\lambda_1(R)^k\lambda_2(R)^l}\right) \tag{6.5}$$

as R tends to infinity and that, for any nonnegative integer m,

$$\lim_{R\to\infty} \frac{R}{\lambda_1(R)^g\lambda_2(R)^m \log R} = \infty. \tag{6.6}$$

For any interval $I = [x,y) \subset \mathbb{R}$, we write the length by $|I| = y - x$. In what follows, N is a sufficiently large integer. First we construct an interval $J(N) = [\alpha_1, \alpha_2) \subset [0, N)$ satisfying the following:

1. $\alpha_1 \in r_1 S + u_1 T$ for $(r_1, u_1) \in \Lambda$ with $r_1 < g$.
2. If $\alpha_2 < N$, then $\alpha_2 \in r_2 S + u_2 T$ for $(r_2, u_2) \in \Lambda$ with $r_2 < g$.
3. Let m be any integer with $\alpha_1 < m < \alpha_2$ and $(k, l) \in \Lambda$ with $k < g$. Then $m \notin kS + lT$.
4.
$$|J(N)| \geq C_6 \frac{N}{\lambda_1(N)^{r_3} \lambda_2(N)^{u_3}}, \tag{6.7}$$
where C_6 is a positive constant and $(r_3, u_3) \in \Lambda$ with $r_3 < g$.

Combining (6.5) and (6.7), we deduce that α_1 and α_2 are approximated by the elements in gS. Namely, we get the nonempty subinterval $J'(N) = [\beta_1, \beta_2) \subset J(N)$ defined by

$$\begin{aligned} \beta_1 &:= \min\{m \in gS \mid m > \alpha_1\}, \\ \beta_2 &:= \max\{m \in gS \mid m < \alpha_2\}, \end{aligned}$$

respectively. We divide $J'(N)$ into subintervals. Recall that $h \geq 1$. Using (6.3), we get a subinterval $I(N) = [\gamma_1, \gamma_2) \subset J'(N)$ satisfying the following:

1. $\gamma_1, \gamma_2 \in gS + (h-1)T$.
2. Let m be any integer with $\gamma_1 < m < \gamma_2$ and $\mathbf{k} = (k, l) \in \Lambda$ with $\mathbf{g} \succ \mathbf{k}$. Then
$$m \notin kS + lT. \tag{6.8}$$
3.
$$|I(N)| \geq C_7 \frac{N}{\lambda_1(N)^{g} \lambda_2(N)^{h-1}}, \tag{6.9}$$
where C_7 is a positive constant.

Combining (6.6) and (6.9), we obtain

$$\frac{|I(N)|}{\log N} = \infty. \tag{6.10}$$

The second assumption on $w(n)$ $(n = 0, 1, \ldots)$ implies that there exists a positive constant C_8 satisfying

$$T \cap (R, C_8 R) \neq \emptyset$$

for any sufficiently large R. In particular, there exists an $m_0 = m_0(N) \in T$ with

$$\frac{1}{1+C_8}|I(N)| \leq m_0 \leq \frac{C_8}{1+C_8}|I(N)| = \gamma_2 - \gamma_1 - \frac{1}{1+C_8}|I(N)|.$$

Put $U := \gamma_1 + m_0$. Then

$$\gamma_1 + \frac{1}{1+C_8}|I(N)| \leq U \leq \gamma_2 - \frac{1}{1+C_8}|I(N)|. \tag{6.11}$$

Moreover, $U \in gS + hT$ because $\gamma_1 \in gS + (h-1)T$ and $m_0 \in T$. Namely,

$$\rho(\mathbf{g}; U) > 0. \tag{6.12}$$

We consider the base-b expansion of (6.4). Then (6.2) and (6.12) mean that

$$b^{-U} A_{\mathbf{g}} \rho(\mathbf{g}; U)$$

causes $O(\log(A_{\mathbf{g}}\rho(\mathbf{g}; U))) = O(\log U) = O(\log N)$ carries to higher digits because $A_{\mathbf{g}} > 0$. Hence, combining (6.8), (6.10), and (6.11), we conclude that there are positive digits left in the base-b expansion of (6.4), which implies that $P(\eta, \xi) \neq 0$.

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