# EXPLICIT ALGEBRAIC DEPENDENCE FORMULAE FOR INFINITE PRODUCTS RELATED WITH FIBONACCI AND LUCAS NUMBERS 

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Abstract. Let $d \geq 2$ be an integer. In [2], the second, third, and fourth authors gave necessary and sufficient conditions for the infinite products

$$
\prod_{\substack{k=1 \\ U_{d^{k}} \neq-a_{i}}}^{\infty}\left(1+\frac{a_{i}}{U_{d^{k}}}\right) \quad(i=1, \ldots, m) \quad \text { or } \quad \prod_{\substack{k=1 \\ V_{d^{k}} \neq-a_{i}}}^{\infty}\left(1+\frac{a_{i}}{V_{d^{k}}}\right)(i=1, \ldots, m)
$$

to be algebraically dependent, where $a_{i}$ are non-zero integers and $U_{n}$ and $V_{n}$ are generalized Fibonacci numbers and Lucas numbers, respectively. The purpose of this paper is to relax the condition on the non-zero integers $a_{1}, \ldots, a_{m}$ to non-zero real algebraic numbers, which gives new cases where the infinite products above are algebraically dependent.

## 1. Introduction

Let $\alpha$ and $\beta$ be real algebraic numbers with $|\alpha|>1$ and $\alpha \beta=-1$. Then the generalized Fibonacci numbers and Lucas numbers are expressed, respectively, as

$$
\begin{equation*}
U_{n}=\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta} \quad \text { and } \quad V_{n}=\alpha^{n}+\beta^{n} \quad(n \geq 0) \tag{1.1}
\end{equation*}
$$

If $\alpha=(1+\sqrt{5}) / 2$, we have $U_{n}=F_{n}$ and $V_{n}=L_{n}(n \geq 0)$, where $\left\{F_{n}\right\}_{n \geq 0}$ and $\left\{L_{n}\right\}_{n \geq 0}$ are the sequences of Fibonacci numbers and Lucas numbers defined, respectively, by $F_{n+2}=F_{n+1}+F_{n}(n \geq 0), F_{0}=0, F_{1}=1$ and by $L_{n+2}=L_{n+1}+L_{n}(n \geq 0), L_{0}=2, L_{1}=1$. Let $d \geq 2$ be an integer. In [2], the second, third, and fourth authors gave necessary and sufficient conditions for the infinite products

$$
\begin{equation*}
\prod_{\substack{k=1 \\ U_{d^{k}} \neq-a_{i}}}^{\infty}\left(1+\frac{a_{i}}{U_{d^{k}}}\right) \quad(i=1, \ldots, m) \tag{1.2}
\end{equation*}
$$

[^0]or
\[

$$
\begin{equation*}
\prod_{\substack{k=1 \\ V_{d^{k}} \neq-a_{i}}}^{\infty}\left(1+\frac{a_{i}}{V_{d^{k}}}\right) \quad(i=1, \ldots, m) \tag{1.3}
\end{equation*}
$$

\]

to be algebraically dependent, where $a_{i}$ are non-zero rational integers. In this paper, we relax the condition on the non-zero rational integers $a_{1}, \ldots, a_{m}$ to non-zero real algebraic numbers, which gives new cases where the infinite products (1.2) or (1.3) are algebraically dependent.

The algebraic independency of the infinite products above can be proved by using Mahler's method explained in Section 2; thereby, the algebraic dependency of the infinite products (1.3) with non-zero distinct real algebraic numbers $a_{1}, \ldots, a_{m}$ is reduced to the problem of determining whether the set of the roots of the quadratic polynomials $z^{2}+a_{i} z+1(1 \leq i \leq m)$ and $z^{2}+1$ includes subsets described by certain algorithm. If $\left|a_{i}\right|>2(1 \leq i \leq m)$, the method used in this paper is essentially similar to that of [2] dealt with the case where $a_{1}, \ldots, a_{m}$ are rational integers. If $a_{1}, \ldots, a_{m}$ are non-zero distinct real algebraic numbers including those with $\left|a_{i}\right| \leq 2$, it can arise that the infinite products (1.3) which were not treated in [2] are algebraically dependent (see Examples 2-6 below). In such a case, we establish the algorithm of selecting $d$-th roots to find subsets mentioned above, whose elements distribute on the unit circle with certain symmetry. For this purpose, Lemmas 4.1 and 4.2 proved in Section 4 play a crucial role. The necessary and sufficient conditions given in Theorems 1.1 and 1.3 of this paper are useful to obtain explicit algebraic dependence relations among the infinite products (1.2) or (1.3), whose transcendence degrees are just one less than the numbers of the infinite products appearing in each relation (see Examples 1-6 below).

We introduce the following notation which will be needed throughout this paper. Let $d \geq 2$ be a fixed integer. For $\tau \in \mathbb{C}$ with $|\tau|=1$ and $i=0,1, \ldots$, define $\Omega_{i}(\tau):=\left\{z \in \mathbb{C} \mid z^{d^{i}}=\tau\right.$ or $\left.z^{d^{i}}=\bar{\tau}\right\}$. Here and in what follows, for any $\gamma \in \mathbb{C}$ we denote by $\bar{\gamma}$ the complex conjugate of $\gamma$. Moreover, for $S \subset \mathbb{C}$ we denote $\bar{S}:=\{\bar{\gamma} \mid \gamma \in S\}$. Let $\zeta_{m}=\exp (2 \pi \sqrt{-1} / m)$. For any fixed integer $k \geq 1$, let $S_{k}(\tau)$ be a non-empty subset of $\Omega_{k}(\tau)$ such that for any $\gamma \in S_{k}(\tau)$ the numbers $\zeta_{d} \gamma$ and $\bar{\gamma}$ belong to $S_{k}(\tau)$. Namely, $S_{k}(\tau)$ satisfies

$$
\begin{equation*}
S_{k}(\tau)=\zeta_{d} S_{k}(\tau) \quad \text { and } \quad S_{k}(\tau)=\overline{S_{k}(\tau)} \tag{1.4}
\end{equation*}
$$

For example, if $k=3, d=2$, and $\tau=1$, we have $\Omega_{3}(1)=\left\{\zeta_{8}^{j} \mid 0 \leq j \leq 7\right\}$ and we can choose $S_{3}(1)=\left\{ \pm \zeta_{8}, \pm \zeta_{8}^{3}\right\}$. Note that the following sets are
determined depending only on $S_{k}(\tau)$ :

$$
\begin{gathered}
\Lambda_{i}(\tau)=\left\{\gamma^{d^{k-i}} \mid \gamma \in S_{k}(\tau)\right\} \subset \Omega_{i}(\tau) \quad(0 \leq i \leq k-1), \\
\Gamma_{i}(\tau)=\left\{\gamma \in \Omega_{i}(\tau) \mid \gamma^{d} \in \Lambda_{i-1}(\tau)\right\} \backslash \Lambda_{i}(\tau) \quad(1 \leq i \leq k-1) .
\end{gathered}
$$

Define

$$
\mathcal{E}_{k}(\tau)=\left(\bigcup_{i=1}^{k-1} \Gamma_{i}(\tau)\right) \bigcup S_{k}(\tau)
$$

and

$$
\mathcal{F}_{k}(\tau)= \begin{cases}\mathcal{E}_{k}(\tau) \cup\{\tau, \bar{\tau}\} & \text { if } \tau \notin \mathcal{E}_{k}(\tau), \\ \mathcal{E}_{k}(\tau) \backslash\{\tau, \bar{\tau}\} & \text { otherwise }\end{cases}
$$

Note that $\mathcal{E}_{1}(\tau)=S_{1}(\tau)$. The main results of this paper are as follows:
Theorem 1.1. Let $\left\{U_{n}\right\}_{n \geq 0}$ be the sequence defined by (1.1) and d an integer greater than 1. Let $a_{1}, \ldots, a_{m}$ be non-zero distinct real algebraic numbers. Then the numbers

$$
\prod_{\substack{k=0 \\ U_{d^{k} \neq-a_{i}}}}^{\infty}\left(1+\frac{a_{i}}{U_{d^{k}}}\right) \quad(i=1, \ldots, m)
$$

are algebraically dependent if and only if $d$ is odd and there exist $\tau_{1}, \tau_{2} \in \mathbb{C}$ with $\tau_{1} \neq \tau_{2},\left|\tau_{1}\right|=\left|\tau_{2}\right|=1$ and $\mathcal{F}_{k_{1}}\left(\tau_{1}\right), \mathcal{F}_{k_{2}}\left(\tau_{2}\right)$ with $k_{1}, k_{2} \geq 1$ such that $\mathcal{F}_{k_{1}}\left(\tau_{1}\right) \cap \mathcal{F}_{k_{2}}\left(\tau_{2}\right) \subset\left\{\tau_{1}, \overline{\tau_{1}}, \tau_{2}, \overline{\tau_{2}}\right\}$ and $\left\{a_{1}, \ldots, a_{m}\right\}$ contains

$$
-\frac{1}{\alpha-\beta}(\gamma+\bar{\gamma})
$$

for all $\gamma \in\left(\mathcal{F}_{k_{1}}\left(\tau_{1}\right) \cup \mathcal{F}_{k_{2}}\left(\tau_{2}\right)\right) \backslash\{ \pm \sqrt{-1}\}$.
Corollary 1.2. For any integer $d \geq 2$ and for any real algebraic number $a \neq 0$, the infinite product

$$
\prod_{\substack{k=0 \\ U_{d^{k} \neq-a}}}^{\infty}\left(1+\frac{a}{U_{d^{k}}}\right)
$$

is transcendental.
This follows from the fact that the algebraic dependence condition of Theorem 1.1 requires two non-empty sets $\mathcal{F}_{k_{1}}\left(\tau_{1}\right)$ and $\mathcal{F}_{k_{2}}\left(\tau_{2}\right)$. The transcendency of the numbers such as the infinite products in Corollary 1.2 was shown in [5].

Examples $1-6$ below are obtained by using Theorems 1.1 and 1.3 of this paper. For the details, see [3].

Example 1. Let a be a non-zero real algebraic number. The transcendental numbers

$$
s_{1}=\prod_{\substack{k=0 \\ F_{3^{k} \neq-a} \neq-}}^{\infty}\left(1+\frac{a}{F_{3^{k}}}\right) \quad \text { and } \quad s_{2}=\prod_{\substack{k=0 \\ F_{3^{k}} \neq a}}^{\infty}\left(1-\frac{a}{F_{3^{k}}}\right)
$$

are algebraically dependent if and only if $a= \pm 1 / \sqrt{5}$. If $a=1 / \sqrt{5}$, then $s_{1} s_{2}^{-1}=2+\sqrt{5}$.

Theorem 1.3. Let $\left\{V_{n}\right\}_{n \geq 0}$ be the sequence defined by (1.1) and d an integer greater than 1. Let $a_{1}, \ldots, a_{m}$ be non-zero distinct real algebraic numbers. Then the numbers

$$
\begin{equation*}
\prod_{\substack{k=0 \\ V_{d} \neq-a_{i}}}^{\infty}\left(1+\frac{a_{i}}{V_{d^{k}}}\right) \quad(i=1, \ldots, m) \tag{1.5}
\end{equation*}
$$

are algebraically dependent if and only if at least one of the following three properties is satisfied:
(i) $d=2$ and the set $\left\{a_{1}, \ldots, a_{m}\right\}$ contains $b_{1}, \ldots, b_{l}(l \geq 3)$ with $b_{1}<-2$ satisfying

$$
b_{2}=-b_{1}, \quad b_{j}=b_{j-1}^{2}-2 \quad(j=3, \ldots, l-1), \quad b_{l}=-b_{l-1}^{2}+2 .
$$

(ii) $d=2$ and there exist $\tau \in \mathbb{C}$ with $|\tau|=1$ and $\mathcal{F}_{k}(\tau)$ with $k \geq 1$ such that $\left\{a_{1}, \ldots, a_{m}\right\}$ contains

$$
-(\gamma+\bar{\gamma})
$$

for all $\gamma \in \mathcal{F}_{k}(\tau) \backslash\{ \pm \sqrt{-1}\}$.
(iii) $d \geq 4$ is even and there exist $\tau_{1}, \tau_{2} \in \mathbb{C}$ with $\tau_{1} \neq \tau_{2},\left|\tau_{1}\right|=\left|\tau_{2}\right|=1$ and $\mathcal{F}_{k_{1}}\left(\tau_{1}\right), \mathcal{F}_{k_{2}}\left(\tau_{2}\right)$ with $k_{1}, k_{2} \geq 1$ such that $\mathcal{F}_{k_{1}}\left(\tau_{1}\right) \cap \mathcal{F}_{k_{2}}\left(\tau_{2}\right) \subset$ $\left\{\tau_{1}, \overline{\tau_{1}}, \tau_{2}, \overline{\tau_{2}}\right\}$ and $\left\{a_{1}, \ldots, a_{m}\right\}$ contains

$$
-(\gamma+\bar{\gamma})
$$

for all $\gamma \in\left(\mathcal{F}_{k_{1}}\left(\tau_{1}\right) \cup \mathcal{F}_{k_{2}}\left(\tau_{2}\right)\right) \backslash\{ \pm \sqrt{-1}\}$.
Remark 1.4. In the case of $d=2$, putting $\tau_{1}=\zeta_{3}=\zeta_{6}^{2}, S_{1}\left(\tau_{1}\right)=$ $\left\{\zeta_{6}, \zeta_{6}^{2}, \zeta_{6}^{4}, \zeta_{6}^{5}\right\}, \tau_{2}=-1$, and $S_{1}\left(\tau_{2}\right)=\{\sqrt{-1},-\sqrt{-1}\}$, we have $\mathcal{F}_{1}\left(\tau_{1}\right)=$ $\left\{\zeta_{6}, \zeta_{6}^{5}\right\}$ and $\mathcal{F}_{1}\left(\tau_{2}\right)=\{-1, \sqrt{-1},-\sqrt{-1}\}$. Hence, using (ii) in Theorem 1.3 and noting that $-\left(\zeta_{6}+\zeta_{6}^{5}\right)=-1$ and $-(-1-1)=2$, we see that the corresponding infinite products (1.5) are algebraic numbers. Indeed, we have

$$
\prod_{k=1}^{\infty}\left(1-\frac{1}{V_{2^{k}}}\right)=\frac{\alpha^{4}-1}{\alpha^{4}+\alpha^{2}+1} \quad \text { and } \quad \prod_{k=1}^{\infty}\left(1+\frac{2}{V_{2^{k}}}\right)=\frac{\alpha^{2}+1}{\alpha^{2}-1}
$$

Corollary 1.5. Let $d \geq 2$ be an integer and $a \neq 0$ be a real algebraic number with $(d, a) \neq(2,-1),(2,2)$. Then the infinite product

$$
\prod_{\substack{k=0 \\ V_{d^{k}} \neq-a}}^{\infty}\left(1+\frac{a}{V_{d^{k}}}\right)
$$

is transcendental.
This corollary can be deduced from the following discussion: The case (iii) of Theorem 1.3 requires two non-empty sets $\mathcal{F}_{k_{1}}\left(\tau_{1}\right)$ and $\mathcal{F}_{k_{2}}\left(\tau_{2}\right)$. Hence, if $d \geq 4$, the infinite product in the corollary is transcendental. When $d=2$, the case (i) of Theorem 1.3 requires at least three numbers. Therefore only the case (ii) has a possibility for the infinite product to be algebraic. If the number of the elements in $\mathcal{F}_{k}(\tau) \backslash\{ \pm \sqrt{-1}\}$ is at most two, the infinite product is algebraic as is shown in Remark 1.4 above. The transcendency of the numbers such as the infinite products in the corollary was shown in [5].

Example 2. Let $a \neq \pm 1, \pm 2$ be a real algebraic number. The transcendental numbers

$$
s_{1}=\prod_{\substack{k=1 \\ L_{2} k \neq-a}}^{\infty}\left(1+\frac{a}{L_{2^{k}}}\right) \quad \text { and } \quad s_{2}=\prod_{\substack{k=1 \\ L_{2} \neq a}}^{\infty}\left(1-\frac{a}{L_{2^{k}}}\right)
$$

are algebraically dependent if and only if $a= \pm \sqrt{2}$. If $a= \pm \sqrt{2}$, then $s_{1} s_{2}=\sqrt{5} / 3$.

Example 3. The transcendental numbers

$$
\begin{array}{ll}
s_{1}=\prod_{k=1}^{\infty}\left(1-\frac{\sqrt{3}}{L_{4^{k}}}\right), \quad s_{2}=\prod_{k=1}^{\infty}\left(1+\frac{\sqrt{3}}{L_{4^{k}}}\right), \\
s_{3}=\prod_{k=1}^{\infty}\left(1-\frac{1}{L_{4^{k}}}\right), \quad s_{4}=\prod_{k=1}^{\infty}\left(1+\frac{2}{L_{4^{k}}}\right)
\end{array}
$$

satisfy

$$
s_{1} s_{2} s_{3} s_{4}^{-1}=\frac{5}{8}
$$

while trans. $\operatorname{deg}_{\mathbb{Q}} \mathbb{Q}\left(s_{1}, s_{2}, s_{3}, s_{4}\right)=3$.
Example 4. The transcendental numbers

$$
\begin{gathered}
s_{1}=\prod_{k=1}^{\infty}\left(1-\frac{1}{L_{6^{k}}}\right), \quad s_{2}=\prod_{k=1}^{\infty}\left(1+\frac{1}{L_{6^{k}}}\right), \quad s_{3}=\prod_{k=1}^{\infty}\left(1+\frac{2}{L_{6^{k}}}\right) \\
s_{4}=\prod_{k=1}^{\infty}\left(1+\frac{\sqrt{3}}{L_{6^{k}}}\right), \quad s_{5}=\prod_{k=1}^{\infty}\left(1-\frac{\sqrt{3}}{L_{6^{k}}}\right)
\end{gathered}
$$

satisfy

$$
s_{1} s_{2} s_{3} s_{4}^{-1} s_{5}^{-1}=\frac{\sqrt{5}}{2}
$$

while trans. $\operatorname{deg}_{\mathbb{Q}} \mathbb{Q}\left(s_{1}, s_{2}, s_{3}, s_{4}, s_{5}\right)=4$.
Example 5. The transcendental numbers

$$
s_{i}=\prod_{k=1}^{\infty}\left(1+\frac{a_{i}}{L_{4^{k}}}\right) \quad(i=1, \ldots, 8),
$$

where

$$
\begin{gathered}
a_{1}=-\left(\zeta_{16}^{1}+\zeta_{16}^{15}\right), a_{2}=-\left(\zeta_{16}^{5}+\zeta_{16}^{11}\right), a_{3}=-\left(\zeta_{16}^{7}+\zeta_{16}^{9}\right), a_{4}=-\left(\zeta_{64}^{3}+\zeta_{64}^{61}\right), \\
a_{5}=-\left(\zeta_{64}^{13}+\zeta_{64}^{51}\right), a_{6}=-\left(\zeta_{64}^{19}+\zeta_{64}^{45}\right), a_{7}=-\left(\zeta_{64}^{29}+\zeta_{64}^{35}\right), a_{8}=2,
\end{gathered}
$$

satisfy

$$
s_{1} s_{2} \cdots s_{7} s_{8}^{-2}=\frac{25}{7(7-\sqrt{2-\sqrt{2}})}
$$

while trans. $\operatorname{deg}_{\mathbb{Q}} \mathbb{Q}\left(s_{1}, s_{2}, \ldots, s_{8}\right)=7$.
Example 6. The transcendental numbers

$$
s_{i}=\prod_{k=1}^{\infty}\left(1+\frac{a_{i}}{L_{4^{k}}}\right) \quad(i=1, \ldots, 10),
$$

where

$$
\begin{aligned}
& a_{1}=-\frac{3}{2}, \quad a_{2}=\frac{\sqrt{7}}{2}, \quad a_{3}=\frac{3}{2}, \quad a_{4}=-\frac{\sqrt{7}}{2}, \quad a_{5}=\frac{31}{16}, \\
& a_{6}=-\frac{4}{\sqrt{5}}, \quad a_{7}=\frac{2}{\sqrt{5}}, \quad a_{8}=\frac{4}{\sqrt{5}}, \quad a_{9}=-\frac{2}{\sqrt{5}}, \quad a_{10}=\frac{14}{25},
\end{aligned}
$$

satisfy

$$
s_{1} s_{2} s_{3} s_{4} s_{5}^{-1} s_{6}^{-1} s_{7}^{-1} s_{8}^{-1} s_{9}^{-1} s_{10}=\frac{3024}{3575},
$$

while trans. $\operatorname{deg}_{\mathbb{Q}} \mathbb{Q}\left(s_{1}, s_{2}, \ldots, s_{10}\right)=9$.
The proofs of Theorems 1.1 and 1.3 will be given in Section 5 .

## 2. Functional Equations

In this section, we explain the Mahler's method mentioned in the introduction. Let $\boldsymbol{K}$ be an algebraic number field, $\boldsymbol{K}(z)$ the field of rational functions over $\boldsymbol{K}$, and $\boldsymbol{K}[[z]]$ the ring of formal power series with coefficients in $\boldsymbol{K}$. In what follows, let $d$ be an integer greater than 1 . We define the subgroup $H_{d}$ of the multiplicative group $\boldsymbol{K}(z)^{\times}$of non-zero elements of $\boldsymbol{K}(z)$ by

$$
\begin{equation*}
H_{d}=\left\{\left.\frac{g\left(z^{d}\right)}{g(z)} \right\rvert\, g(z) \in \boldsymbol{K}(z)^{\times}\right\} . \tag{2.1}
\end{equation*}
$$

The functions $c_{1}(z), \ldots, c_{m}(z) \in \boldsymbol{K}(z)^{\times}$are called multiplicatively dependent modulo $H_{d}$ if there exist rational integers $e_{1}, \ldots, e_{m}$, not all zero, such that

$$
\prod_{i=1}^{m} c_{i}(z)^{e_{i}} \in H_{d}
$$

If no such rational integers exist, then the functions $c_{1}(z), \ldots, c_{m}(z)$ are said to be multiplicatively independent modulo $H_{d}$.

We use the following lemmas for proving the theorems.
Lemma 2.1 (Kubota [1, Corollary 8]). Let $f_{1}(z), \ldots, f_{m}(z) \in \boldsymbol{K}[[z]] \backslash\{0\}$ satisfy the functional equations

$$
\begin{equation*}
f_{i}\left(z^{d}\right)=c_{i}(z) f_{i}(z), \quad c_{i}(z) \in \boldsymbol{K}(z)^{\times} \quad(i=1, \ldots, m) . \tag{2.2}
\end{equation*}
$$

Then $f_{1}(z), \ldots, f_{m}(z)$ are algebraically independent over $\boldsymbol{K}(z)$ if and only if the rational functions $c_{1}(z), \ldots, c_{m}(z)$ are multiplicatively independent modulo $H_{d}$.

Lemma 2.2 (Kubota [1], see also Theorem 3.6.4 in Nishioka [4]). Suppose that the functions $f_{1}(z), \ldots, f_{m}(z) \in \boldsymbol{K}[[z]]$ converge in $|z|<1$ and satisfy the functional equations (2.2) with $c_{i}(0) \neq 0$. Let $\gamma$ be an algebraic number with $0<|\gamma|<1$ such that $c_{i}\left(\gamma^{d^{k}}\right)$ are defined and non-zero for all $k \geq 0$. If $f_{1}(z), \ldots, f_{m}(z)$ are algebraically independent over $\boldsymbol{K}(z)$, then the values $f_{1}(\gamma), \ldots, f_{m}(\gamma)$ are algebraically independent.

Let $\left\{R_{n}\right\}_{n \geq 0}$ be the sequence $\left\{U_{n}\right\}_{n \geq 0}$ or $\left\{V_{n}\right\}_{n \geq 0}$ defined by (1.1). Then for any non-zero real algebraic numbers $a_{1}, \ldots, a_{m}$, we put

$$
\Phi_{i}(z)=\prod_{k=0}^{\infty}\left(1+\frac{p_{i} z^{d^{k}}}{1+b z^{2 d^{k}}}\right) \quad(i=1, \ldots, m)
$$

where

$$
\left(p_{i}, b\right)=\left\{\begin{array}{cl}
\left((\alpha-\beta) a_{i},-(-1)^{d}\right) & \text { if } R_{n}=U_{n}  \tag{2.3}\\
\left(a_{i},(-1)^{d}\right) & \text { if } R_{n}=V_{n}
\end{array}\right.
$$

Taking an integer $N \geq 1$ such that $\left|R_{d^{k}}\right|>\max \left\{\left|a_{1}\right|, \ldots,\left|a_{m}\right|\right\}$ for all $k \geq N$ and noting that $\alpha \beta=-1$, we have

$$
\begin{aligned}
\Phi_{i}\left(\alpha^{-d^{N}}\right) & =\prod_{k=N}^{\infty}\left(1+\frac{p_{i} \alpha^{-d^{k}}}{1+b \alpha^{-2 d^{k}}}\right) \\
& =\prod_{k=N}^{\infty}\left(1+\frac{p_{i}}{\alpha^{d^{k}}+b(-1)^{d^{k}} \beta^{d^{k}}}\right) \\
& =\prod_{k=N}^{\infty}\left(1+\frac{a_{i}}{R_{d^{k}}}\right) \quad(i=1, \ldots, m)
\end{aligned}
$$

so that

$$
\begin{equation*}
\prod_{\substack{k=0 \\ R_{d^{k}} \neq-a_{i}}}^{\infty}\left(1+\frac{a_{i}}{R_{d^{k}}}\right)=\Phi_{i}\left(\alpha^{-d^{N}}\right) \prod_{\substack{k=0 \\ R_{d^{k}} \neq-a_{i}}}^{N-1}\left(1+\frac{a_{i}}{R_{d^{k}}}\right) \quad(i=1, \ldots, m) \tag{2.4}
\end{equation*}
$$

Suppose that the numbers (2.4) are algebraically dependent. Then so are the values $\Phi_{1}\left(\alpha^{-d^{N}}\right), \ldots, \Phi_{m}\left(\alpha^{-d^{N}}\right)$. Since $\Phi_{1}(z), \ldots, \Phi_{m}(z)$ satisfy the functional equations

$$
\begin{equation*}
\Phi_{i}\left(z^{d}\right)=c_{i}(z) \Phi_{i}(z), \quad c_{i}(z)=\frac{1+b z^{2}}{1+p_{i} z+b z^{2}} \quad(i=1, \ldots, m) \tag{2.5}
\end{equation*}
$$

the functions $\Phi_{1}(z), \ldots, \Phi_{m}(z)$ are algebraically dependent over $\boldsymbol{K}(z)$ by Lemma 2.2 with $\boldsymbol{K}=\mathbb{Q}\left(\alpha, a_{1}, \ldots, a_{m}\right)$. Then by Lemma 2.1 the rational functions $c_{1}(z), \ldots, c_{m}(z)$ are multiplicatively dependent modulo $H_{d}$, namely, there exist integers $e_{1}, \ldots, e_{m}$, not all zero, and $g(z) \in \boldsymbol{K}(z)^{\times}$such that $\prod_{i=1}^{m} c_{i}(z)^{e_{i}}=g\left(z^{d}\right) / g(z)$. Then, renumbering the $p_{i}$, we may assume that there exist coprime polynomials $A(z), B(z) \in \boldsymbol{K}[z] \backslash\{0\}$ such that

$$
\begin{equation*}
A\left(z^{d}\right) B(z) \prod_{i=1}^{k} P_{i}(z)^{e_{i}}=\left(1+b z^{2}\right)^{e} A(z) B\left(z^{d}\right) \prod_{i=k+1}^{l} P_{i}(z)^{e_{i}} \tag{2.6}
\end{equation*}
$$

where $k, e_{i}, e$ are integers with $k, e_{i} \geq 1, e \geq 0$ and $P_{i}(z)=1+p_{i} z+b z^{2}$. We note that $\sum_{i=1}^{k} e_{i}=e+\sum_{i=k+1}^{l} e_{i}$.

We consider the functional equation (2.7) below, which is more general than (2.6). Let $P(z), Q(z) \in \mathbb{C}[z] \backslash\{0\}$ be coprime polynomials with $\operatorname{deg} P(z) Q(z)>0$ satisfying

$$
\begin{equation*}
A\left(z^{d}\right) B(z) P(z)=A(z) B\left(z^{d}\right) Q(z) \tag{2.7}
\end{equation*}
$$

where $d \geq 2$ is an integer and $A(z), B(z) \in \mathbb{C}[z] \backslash\{0\}$ are coprime. Note that the degrees of $P(z)$ and $Q(z)$ are not necessarily the same.

Let $\theta$ be a complex number and $\left\{\theta_{n}\right\}_{n \geq 1}$ a sequence of non-real numbers. We call $\left\{\theta_{n}\right\}_{n \geq 1}$ a compatible non-real sequence of roots of $\theta$ if $\theta_{1}^{d}=\theta$, $\theta_{n+1}^{d}=\theta_{n}$ for any $n \geq 1$, and if the set $\left\{\theta_{n} \mid n \geq 1\right\}$ is infinite. In particular, we have $\theta_{n}^{d^{n}}=\theta$ for any $n \geq 1$.

Lemma 2.3. Assume that $P(z)$ and $Q(z)$ satisfy (2.7). Let $\theta \in \mathbb{C}$.
(i) Suppose that there exists a compatible non-real sequence $\left\{\theta_{n}\right\}_{n \geq 1}$ of roots of $\theta$ satisfying $Q\left(\theta_{n}\right) \neq 0$ (resp. $P\left(\theta_{n}\right) \neq 0$ ) for any $n \geq 1$. Then $A(\theta) \neq 0($ resp. $B(\theta) \neq 0)$.
(ii) Let l be a positive integer. Assume that $Q\left(\theta^{d^{n}}\right) \neq 0$ for any $n$ with $1 \leq n \leq l$ and that $B\left(\theta^{d}\right)=0$. Then we have $B\left(\theta^{d^{n}}\right)=0$ for any $n$ with $1 \leq n \leq l+1$.
(iii) Suppose that $Q\left(\theta^{d^{n}}\right) \neq 0$ for any $n \geq 1$ and that the set $\left\{\theta^{d^{n}} \mid n \geq\right.$ $1\}$ is infinite. Then $B\left(\theta^{d}\right) \neq 0$.

Proof. For the proof of statement (i) we only check the case of

$$
\begin{equation*}
Q\left(\theta_{n}\right) \neq 0 \quad(n \geq 1) \tag{2.8}
\end{equation*}
$$

since that of $P\left(\theta_{n}\right) \neq 0(n \geq 1)$ is proved by the symmetry of (2.7). Suppose on the contrary that $A(\theta)=0$. By (2.8) and the fact that $A(z)$ and $B(z)$ are coprime, $B(\theta) Q\left(\theta_{1}\right) \neq 0$. Thus, substituting $z=\theta_{1}$ into (2.7), we get $A\left(\theta_{1}\right)=0$ because $\theta_{1}^{d}=\theta$. Next we suppose that $A\left(\theta_{n}\right)=0$ for some $n \geq 1$. In the same way as above, $B\left(\theta_{n}\right) Q\left(\theta_{n+1}\right) \neq 0$. Since $\theta_{n+1}^{d}=\theta_{n}$, putting $z=\theta_{n+1}$ into (2.7), we see that $A\left(\theta_{n+1}\right)=0$. Hence $A\left(\theta_{n}\right)=0$ for any $n \geq 1$, which is impossible since the set $\left\{\theta_{n} \mid n \geq 1\right\}$ is infinite and $A(z)$ is a polynomial. This completes the proof of statement (i).

Next we show statement (ii) by induction on $n$. The case of $n=1$ is trivial. Suppose that $B\left(\theta^{d^{n}}\right)=0$ for some $n$ with $1 \leq n \leq l$. Then we have $A\left(\theta^{d^{n}}\right) Q\left(\theta^{d^{n}}\right) \neq 0$ since $A(z)$ and $B(z)$ are coprime. Thus, substituting $z=\theta^{d^{n}}$ into (2.7), we get $B\left(\theta^{d^{n+1}}\right)=0$, and statement (ii) is proved.

Statement (iii) follows from (ii) since $B(z)$ is a polynomial.

## 3. The case where $P(z)$ and $Q(z)$ are the products of QUADRATIC POLYNOMIALS

Let $\boldsymbol{K} \subset \mathbb{R}$ be an algebraic number field. In this section, we consider the special case of $P(z)$ and $Q(z)$ involving (2.6), namely, $P(z), Q(z)$ are expressed as

$$
\begin{equation*}
P(z)=\prod_{i=1}^{s}\left(1+p_{i} z+b z^{2}\right), \quad Q(z)=\prod_{j=s+1}^{t}\left(1+q_{j} z+b z^{2}\right) \tag{3.1}
\end{equation*}
$$

with $b= \pm 1$ and $p_{i} \neq q_{j}$ for all $i, j$ and $P(z), Q(z)$ satisfy the functional equation (2.7) with $A(z), B(z) \in \boldsymbol{K}[z] \backslash\{0\}$. Note that $p_{1}, \ldots, p_{s}$ are not necessarily distinct and so are $q_{s+1}, \ldots, q_{t}$. First we show $b=1$ in Lemma 3.2 below, and then we investigate the properties of $P(z)$ and $Q(z)$ under the different situations (see Subsections 3.1 and 3.2).

Suppose that $P(z) Q(z)$ has real roots. Let $\alpha_{1}$ be one of the real roots of $P(z) Q(z)$ with the largest absolute value among its real roots, namely, $\alpha_{1} \in \mathbb{R}$ satisfies $P\left(\alpha_{1}\right) Q\left(\alpha_{1}\right)=0$ and

$$
\begin{equation*}
\left|\alpha_{1}\right|=\max \{|\gamma| \mid \gamma \in \mathbb{R}, P(\gamma) Q(\gamma)=0\} . \tag{3.2}
\end{equation*}
$$

Then, exchanging $A(z)$ and $B(z)$ in (2.7) if necessary, we may assume that

$$
P\left(\alpha_{1}\right)=0 .
$$

Then by (3.1), $\beta_{1}:=\left(b \alpha_{1}\right)^{-1}$ satisfies $P\left(\beta_{1}\right)=0$ and the absolute value of $\beta_{1}$ is the smallest among those of the real roots of $P(z) Q(z)$. Comparing the orders at $z=1$ of both sides of (2.7), we obtain $P(1) Q(1) \neq 0$, which yields $\alpha_{1}, \beta_{1} \neq 1$.

Lemma 3.1. Let $P(z)$ and $Q(z)$ be polynomials of the form (3.1) which satisfy (2.7). If the roots of $P(z) Q(z)$ are real, then $A(z) B(z)$ has no negative root.

Proof. For any negative number $\theta$, there exists a compatible non-real sequence $\left\{\theta_{n}\right\}_{n \geq 1}$ of roots of $\theta$. We see that $P\left(\theta_{n}\right) Q\left(\theta_{n}\right) \neq 0$ for any $n \geq 1$ by the assumption of the lemma. Thus $A(\theta) B(\theta) \neq 0$ by statement (i) of Lemma 2.3. Since $\theta$ is any negative number, the lemma is proved.

Lemma 3.2. If $b=-1$, then there are no polynomials $A(z)$ and $B(z)$ of the form (3.1) which satisfy (2.7).

Proof. Since $b<0$, the roots of $P(z) Q(z)$ are real. By the definition of $\alpha_{1}$ and $\beta_{1}$, we have $\alpha_{1} \beta_{1}=-1$. Hence we have $\alpha_{1}<-1$ or $-1<\beta_{1}<0$ because $\alpha_{1}, \beta_{1} \neq 1$. Suppose that $\alpha_{1}<-1$. Then we see that $Q\left(\alpha_{1}^{d^{n}}\right) \neq 0$ for any $n \geq 1$ by (3.2). Substituting $z=\alpha_{1}$ into (2.7), we get $A\left(\alpha_{1}\right) B\left(\alpha_{1}^{d}\right)=0$, which is a contradiction since $A\left(\alpha_{1}\right) \neq 0$ by Lemma 3.1 and $B\left(\alpha_{1}^{d}\right) \neq 0$ by statement (iii) of Lemma 2.3. Similarly we deduce a contradiction also in the case of $-1<\beta_{1}<0$, using the fact that $\left|\beta_{1}\right|$ is the smallest modulus among the roots of $P(z) Q(z)$.

By Lemma 3.2, we have $b=1$. Hence we only have to consider the equation

$$
\begin{equation*}
A\left(z^{d}\right) B(z) P(z)=A(z) B\left(z^{d}\right) Q(z) \tag{3.3}
\end{equation*}
$$

where $A(z), B(z) \in \boldsymbol{K}[z] \backslash\{0\}$ are coprime and

$$
P(z)=\prod_{i=1}^{s}\left(1+p_{i} z+z^{2}\right), \quad Q(z)=\prod_{j=s+1}^{t}\left(1+q_{j} z+z^{2}\right)
$$

with $p_{i} \neq q_{j}$ for all $i, j$.
3.1. The case where $d=2$ and $P(z) Q(z)$ has real roots. In this subsection, we consider the equation (3.3) with $d=2$ and $P(z) Q(z)$ has real roots.

Lemma 3.3. Let $P(z)$ and $Q(z)$ be polynomials satisfying (3.3) with $d=2$. Suppose that $P(z) Q(z)$ has a real root $\alpha_{1}<0$ with (3.2). Then $\alpha_{1}=-1$.

Proof. First we note that the non-real roots of $P(z) Q(z)$ are of absolute value 1, since $P(z) Q(z)$ is the product of quadratic self-reciprocal polynomials. Assume that $\alpha_{1} \neq-1$. Since $\alpha_{1}<0$ and $\beta_{1}=\alpha_{1}^{-1}$, we get $\left|\alpha_{1}\right|>1>\left|\beta_{1}\right|$ and so $Q\left(\alpha_{1}^{2^{n}}\right) \neq 0$ for any $n \geq 0$ by (3.2) and the fact that $P(z)$ and $Q(z)$ are coprime. Substituting $z=\alpha_{1}$ into (3.3), we have $A\left(\alpha_{1}\right)=0$, because $B\left(\alpha_{1}^{2}\right) \neq 0$ by statement (iii) of Lemma 2.3.

On the other hand, there exists a compatible non-real sequence $\left\{\theta_{n}\right\}_{n \geq 1}$ of roots of $\alpha_{1}$ by $\alpha_{1}<0$. Hence we see that $Q\left(\theta_{n}\right) \neq 0$ for any $n \geq 1$ by $\left|\theta_{n}\right|>1$. By statement (i) of Lemma 2.3 we get $A\left(\alpha_{1}\right) \neq 0$, which is a contradiction. Therefore $\alpha_{1}=\beta_{1}=-1$.

Lemma 3.4. Let $P(z)$ and $Q(z)$ be polynomials satisfying (3.3) with $d=2$. Suppose that $P(z) Q(z)$ has a real root $\alpha_{1}>0$ with (3.2). Then there exist $k \geq 1$ and $\alpha, \beta \in \mathbb{R}$ with $\alpha_{1}=\alpha^{2^{k}}$ and $\beta=\alpha^{-1}$ such that $P(z), Q(z)$, and $A(z)$ are divisible by

$$
\begin{align*}
\left(z-\alpha^{2^{k}}\right)\left(z-\beta^{2^{k}}\right), & (z-\alpha)(z-\beta) \prod_{i=0}^{k-1}\left(z+\alpha^{2^{i}}\right)\left(z+\beta^{2^{i}}\right)  \tag{3.4}\\
\text { and } & \prod_{i=1}^{k}\left(z-\alpha^{2^{i}}\right)\left(z-\beta^{2^{i}}\right)
\end{align*}
$$

respectively.
Proof. In the proof of the lemma, we take the positive $2^{j}$-th roots $\alpha_{1}^{2^{-j}}, \beta_{1}^{2^{-j}}$ for any integer $j \geq 1$. We note that $\alpha_{1}>1$. We first show that $A\left(-\alpha_{1}^{2^{-j}}\right) \neq 0$ for any $j \geq 1$. Suppose on the contrary that $A\left(-\alpha_{1}^{2^{-j}}\right)=0$ for some $j \geq 1$. Then there exists an integer $l \geq 1$ such that, for $\theta:=\left(-\alpha_{1}^{2^{-j}}\right)^{2^{-l}} \in \mathbb{C} \backslash \mathbb{R}$, $A\left(\theta^{2}\right)=0$ and $A(\theta) \neq 0$ since $A(z)$ is a polynomial. Substituting $z=\theta$ into (3.3) with $d=2$, we have $Q(\theta)=0$, which is impossible by $|\theta|>1$, since $Q(z)$ is the product of quadratic self-reciprocal polynomials and so its non-real roots are of absolute value 1 .

Suppose that there exists an integer $i \geq 1$ satisfying $Q\left(\alpha_{1}^{2^{-i}}\right)=0$. Then, for such an $i$, we denote the minimal value by $k$. Otherwise, let $k=\infty$. We verify

$$
A\left(\alpha_{1}^{2-j}\right)=0 \quad(0 \leq j \leq k-1)
$$

by induction on $j$, which implies that $k<\infty$ since $A(z)$ is a polynomial. For the case of $j=0$ we substitute $z=\alpha_{1}$ into (3.3) with $d=2$. Then $A\left(\alpha_{1}\right)=0$ because $B\left(\alpha_{1}^{2}\right) \neq 0$ by (3.2) and statement (iii) of Lemma 2.3. Next we show that $A\left(\alpha_{1}^{2^{-j}}\right)=0$ for $1 \leq j \leq k-1$ under the assumption
that $A\left(\alpha_{1}^{2^{-(j-1)}}\right)=0$. Then $B\left(\alpha_{1}^{2^{-j+1}}\right) \neq 0$ and by the minimality of $k$ we have $Q\left(\alpha_{1}^{2^{-j}}\right) \neq 0$. Substituting $z=\alpha_{1}^{2^{-j}}$ into (3.3), we obtain $A\left(\alpha_{1}^{2^{-j}}\right)=0$.

We see that $k$ is the minimal integer such that $Q\left(\beta_{1}^{2-k}\right)=0$ since $\beta_{1}=$ $\alpha_{1}^{-1}$ and $Q(z)$ is self-reciprocal. In the same way as the preceding paragraph, we obtain $A\left(\beta_{1}^{2^{-j}}\right)=0$ for $0 \leq j \leq k-1$. Letting $\alpha:=\alpha_{1}^{2^{-k}}$ and $\beta:=$ $\alpha^{-1}=\beta_{1}^{2^{-k}}$, we see that $P(z)$ and $A(z)$ are divisible by the corresponding polynomials in (3.4). For any $1 \leq j \leq k$, substituting $z=-\alpha_{1}^{2-j}$ into (3.3), we get $Q\left(-\alpha_{1}^{2^{-j}}\right)=0$ since $A\left(\alpha_{1}^{2^{-j+1}}\right)=0, B\left(\alpha_{1}^{2^{-j+1}}\right) \neq 0$, and $A\left(-\alpha_{1}^{2^{-j}}\right) \neq$ 0 by the first paragraph of the proof. Observing that $Q\left(\alpha_{1}^{2^{-k}}\right)=0$ and that $\beta_{1}=\alpha_{1}^{-1}$ and $Q(z)$ is self-reciprocal, we verified the lemma.

Remark 3.5. Let $P(z)$ and $Q(z)$ be polynomials satisfying (3.3) with $d=2$ and $\alpha, \beta$ as in Lemma 3.4. Then $P(z)$ and $Q(z)$ are divisible by

$$
z^{2}+b_{k+2} z+1 \quad \text { and } \quad \prod_{i=1}^{k+1}\left(z^{2}+b_{i} z+1\right)
$$

respectively, where $k \geq 1$ and

$$
\begin{aligned}
b_{1} & =-(\alpha+\beta)<-2 \sqrt{\alpha \beta}=-2 \\
b_{2} & =\alpha+\beta=-b_{1} \\
b_{i} & =\alpha^{2^{i-2}}+\beta^{2^{i-2}}=\left(\alpha^{2^{i-3}}+\beta^{2^{i-3}}\right)^{2}-2=b_{i-1}^{2}-2 \quad(3 \leq i \leq k+1) \\
b_{k+2} & =-\left(\alpha^{2^{k}}+\beta^{2^{k}}\right)=-b_{k+1}^{2}+2
\end{aligned}
$$

3.2. The case where $d \geq 3$ or $P(z) Q(z)$ has no real roots. In this subsection, we consider the equation (3.3) in the case where $d \geq 3$ or $P(z) Q(z)$ has no real roots. First we treat the latter case. Since $P(z) Q(z)$ is the product of quadratic self-reciprocal polynomials, the roots of $P(z) Q(z)$ are included in the set

$$
\begin{equation*}
\mathcal{M}:=\{\omega \in \mathbb{C}| | \omega \mid=1, \omega \neq 1\} . \tag{3.5}
\end{equation*}
$$

In the case of $d \geq 3$ we have the following:
Lemma 3.6. Let $P(z)$ and $Q(z)$ be polynomials satisfying (3.3). If $d \geq 3$, then the roots of $P(z) Q(z)$ are included in $\mathcal{M}$.

Proof. Suppose that $P(z) Q(z)$ has real roots and let $\alpha_{1}(\neq 1)$ be a real root of $P(z)$ as in (3.2). Assume that $\alpha_{1} \neq-1$. Then we get $\left|\alpha_{1}\right|>1>\left|\beta_{1}\right|$. In the same way as in the proof of Lemma 3.3 we deduce a contradiction by $d \geq 3$ since there exists a compatible non-real sequence $\left\{\theta_{n}\right\}_{n \geq 1}$ of roots of $\alpha_{1}$.

In any case stated above, the roots of $P(z) Q(z)$ are included in $\mathcal{M}$. In the next section we investigate such a case for more general polynomials $P(z)$ and $Q(z)$.
4. The case where $P(z) Q(z)$ has roots included in $\mathcal{M}$

Let $P(z)$ and $Q(z)$ be non-zero coprime polynomials with complex coefficients satisfying (2.7). We note that $P(z)$ and $Q(z)$ are not necessarily the products of quadratic polynomials. In this section, assume that $P(z) Q(z)$ has roots included in $\mathcal{M}$. Let $\alpha \in \mathbb{C}$ with $|\alpha|=1$ be the root of $P(z) Q(z)$ having the smallest positive argument among its roots in $\mathcal{M}$. Without loss of generality, we may assume that $P(\alpha)=0$ and $Q(\alpha) \neq 0$. Substituting $z=\alpha$ into (2.7), we get $A(\alpha) B\left(\alpha^{d}\right)=0$. Taking a compatible non-real sequence $\left\{\theta_{n}\right\}_{n \geq 1}$ of roots of $\alpha$ satisfying $0<\arg \left(\theta_{n}\right)<\arg (\alpha)$ for any $n \geq 1$, we get $Q\left(\theta_{n}\right) \neq 0$ and so $A(\alpha) \neq 0$ by statement (i) of Lemma 2.3. Therefore

$$
\begin{equation*}
B\left(\alpha^{d}\right)=0 . \tag{4.1}
\end{equation*}
$$

In this section we calculate the factors of $B(z), P(z)$, and $Q(z)$. First we consider the case where $Q\left(\alpha^{d^{m}}\right)=0$ for some $m \geq 1$, which corresponds to Lemma 4.1 below. Next we treat the case where $Q\left(\alpha^{d^{m}}\right) \neq 0$ for any integer $m \geq 1$, which corresponds to Lemma 4.2. We introduce the following notation. For $\tau \in \mathbb{C}$ with $|\tau|=1$, put

$$
\Theta_{i}(\tau)=\left\{\gamma \in \mathbb{C} \mid \gamma^{d^{i}}=\tau\right\} \quad(i=0,1, \ldots)
$$

We note that if $\pm 1 \in \Theta_{i}(\tau)$ for some $i \geq 0$, then $\tau= \pm 1$.
Let $k \geq 1$ be an integer and $M_{k}(\tau)$ a subset of $\Theta_{k}(\tau)$ satisfying $M_{k}(\tau)=$ $\zeta_{d} M_{k}(\tau)$. For any given $M_{k}(\tau)$ the following sets are uniquely determined:

$$
\begin{gathered}
N_{i}(\tau)=\left\{\gamma^{d^{k-i}} \mid \gamma \in M_{k}(\tau)\right\} \subset \Theta_{i}(\tau) \quad(0 \leq i \leq k-1), \\
M_{i}(\tau)=\left\{\gamma \in \Theta_{i}(\tau) \mid \gamma^{d} \in N_{i-1}(\tau)\right\} \backslash N_{i}(\tau) \quad(1 \leq i \leq k-1), \\
\tilde{\mathcal{E}}_{k}(\tau)=\bigcup_{i=1}^{k} M_{i}(\tau)
\end{gathered}
$$

and

$$
\tilde{\mathcal{F}}_{k}(\tau)= \begin{cases}\tilde{\mathcal{E}}_{k}(\tau) \cup\{\tau\} & \text { if } \tau \notin \tilde{\mathcal{E}}_{k}(\tau) \\ \tilde{\mathcal{E}}_{k}(\tau) \backslash\{\tau\} & \text { otherwise }\end{cases}
$$

We note that

$$
\begin{equation*}
N_{0}(\tau)=\{\tau\} \tag{4.2}
\end{equation*}
$$

Moreover, we use the notation

$$
N_{i}^{1 / d}(\tau):=\left\{\gamma \in \mathbb{C} \mid \gamma^{d} \in N_{i}(\tau)\right\}
$$

in the proof of Lemmas 4.1 and 4.2.

Let $F^{(\tau)}(z)$ be a polynomial defined by

$$
F^{(\tau)}(z)=\prod_{\gamma \in M_{1}(\tau)}(z-\gamma) \cdots \prod_{\gamma \in M_{k}(\tau)}(z-\gamma) .
$$

Lemma 4.1. Let $P(z)$ and $Q(z)$ satisfy (2.7). Let $\alpha \in \mathbb{C}$ with $|\alpha|=1$ be the root of $P(z) Q(z)$ with the smallest positive argument among its roots in $\mathcal{M}$. Assume that $P(\alpha)=0$ and $Q\left(\alpha^{d^{m}}\right)=0$ for some integer $m \geq 1$. Then there exist $k \geq 1, \tau \in \mathbb{C}$ with $|\tau|=1$, and $M_{k}(\tau)$ with $\tau \notin \tilde{\mathcal{E}}_{k}(\tau)$ such that $P(z)$ and $Q(z)$ are divisible by $F^{(\tau)}(z)$ and $z-\tau$, respectively.

Proof. Let $s \geq 1$ be an integer such that $Q\left(\alpha^{d^{s}}\right)=0$ and $Q\left(\alpha^{d j}\right) \neq 0$ for $j=0, \ldots, s-1$. Then we have $B\left(\alpha^{d^{j+1}}\right)=0$ for $j=0, \ldots, s-1$ by (4.1) and statement (ii) of Lemma 2.3. Putting $\tau=\alpha^{d^{s}}$, we have $|\tau|=1, B(\tau)=0$, and $A(\tau) \neq 0$. We give an algorithm how to find $M_{k}(\tau)$, defining $M_{i}(\tau)$ and $N_{i}(\tau)$ below for $i=1,2, \ldots, k$ inductively.

Let

$$
B_{1}(z):=\frac{B(z)}{z-\tau} \in \mathbb{C}[z] \quad \text { and } \quad Q_{1}(z):=\frac{Q(z)}{z-\tau} \in \mathbb{C}[z]
$$

Then we get

$$
\begin{equation*}
A\left(z^{d}\right) B_{1}(z) P(z)=\left(z^{d}-\tau\right) A(z) B_{1}\left(z^{d}\right) Q_{1}(z) \tag{4.3}
\end{equation*}
$$

Define
$N_{1}(\tau):=\left\{\gamma \in \Theta_{1}(\tau) \mid B_{1}(\gamma)=0\right\}$ and $M_{1}(\tau):=\left\{\gamma \in \Theta_{1}(\tau) \mid B_{1}(\gamma) \neq 0\right\}$.
Note that $\Theta_{1}(\tau)=N_{1}(\tau) \cup M_{1}(\tau)$ and $N_{1}(\tau) \cap M_{1}(\tau)=\phi$. Substituting $z=$ $\gamma \in \Theta_{1}(\tau)$ into (4.3), we have $B_{1}(\gamma) P(\gamma)=0$ because $A\left(\gamma^{d}\right)=A(\tau) \neq 0$. Hence, putting

$$
B_{2}(z):=\frac{B_{1}(z)}{\prod_{\gamma \in N_{1}(\tau)}(z-\gamma)} \quad \text { and } \quad P_{1}(z):=\frac{P(z)}{\prod_{\gamma \in M_{1}(\tau)}(z-\gamma)} \in \mathbb{C}[z]
$$

we see

$$
\begin{align*}
& A\left(z^{d}\right)\left(B_{2}(z) \prod_{\gamma \in N_{1}(\tau)}(z-\gamma)\right)\left(P_{1}(z) \prod_{\gamma \in M_{1}(\tau)}(z-\gamma)\right)  \tag{4.4}\\
= & \left(z^{d}-\tau\right) A(z)\left(B_{2}\left(z^{d}\right) \prod_{\gamma \in N_{1}(\tau)}\left(z^{d}-\gamma\right)\right) Q_{1}(z) .
\end{align*}
$$

Noting that

$$
\prod_{\gamma \in N_{1}(\tau)}(z-\gamma) \prod_{\gamma \in M_{1}(\tau)}(z-\gamma)=z^{d}-\tau
$$

and dividing both sides of (4.4) by $z^{d}-\tau$, we get

$$
\begin{equation*}
A\left(z^{d}\right) B_{2}(z) P_{1}(z)=A(z) B_{2}\left(z^{d}\right) Q_{1}(z) \prod_{\gamma \in N_{1}(\tau)}\left(z^{d}-\gamma\right) \tag{4.5}
\end{equation*}
$$

If $N_{1}(\tau)=\phi$, then $\Theta_{1}(\tau)=M_{1}(\tau)$ and hence $M_{1}(\tau)=\zeta_{d} M_{1}(\tau)$. Otherwise, for any $\gamma \in N_{1}^{1 / d}(\tau)$, we have $B_{1}\left(\gamma^{d}\right)=0$ and hence $A\left(\gamma^{d}\right) \neq 0$. Then, substituting $z=\gamma \in N_{1}^{1 / d}(\tau)$ into (4.5), we have $B_{2}(\gamma) P_{1}(\gamma)=0$. Define $N_{2}(\tau):=\left\{\gamma \in N_{1}^{1 / d}(\tau) \mid B_{2}(\gamma)=0\right\}$ and $M_{2}(\tau):=\left\{\gamma \in N_{1}^{1 / d}(\tau) \mid B_{2}(\gamma) \neq 0\right\}$.
We note that $N_{1}^{1 / d}(\tau)=N_{2}(\tau) \cup M_{2}(\tau)$ and $N_{2}(\tau) \cap M_{2}(\tau)=\phi$. Hence, putting

$$
B_{3}(z):=\frac{B_{2}(z)}{\prod_{\gamma \in N_{2}(\tau)}(z-\gamma)} \quad \text { and } \quad P_{2}(z):=\frac{P_{1}(z)}{\prod_{\gamma \in M_{2}(\tau)}(z-\gamma)} \in \mathbb{C}[z]
$$

we have

$$
\begin{align*}
& A\left(z^{d}\right)\left(B_{3}(z) \prod_{\gamma \in N_{2}(\tau)}(z-\gamma)\right)\left(P_{2}(z) \prod_{\gamma \in M_{2}(\tau)}(z-\gamma)\right)  \tag{4.6}\\
= & A(z)\left(B_{3}\left(z^{d}\right) \prod_{\gamma \in N_{2}(\tau)}\left(z^{d}-\gamma\right)\right) Q_{1}(z) \prod_{\gamma \in N_{1}(\tau)}\left(z^{d}-\gamma\right) .
\end{align*}
$$

Dividing both sides of (4.6) by

$$
\prod_{\gamma \in N_{2}(\tau)}(z-\gamma) \prod_{\gamma \in M_{2}(\tau)}(z-\gamma)=\prod_{\gamma \in N_{1}(\tau)}\left(z^{d}-\gamma\right),
$$

we get

$$
A\left(z^{d}\right) B_{3}(z) P_{2}(z)=A(z) B_{3}\left(z^{d}\right) Q_{1}(z) \prod_{\gamma \in N_{2}(\tau)}\left(z^{d}-\gamma\right)
$$

If $N_{2}(\tau)=\phi$, then $N_{1}^{1 / d}(\tau)=M_{2}(\tau)$ and hence $\zeta_{d} M_{2}(\tau)=M_{2}(\tau)$. Otherwise, in the same way as above, we have

$$
A\left(z^{d}\right) B_{4}(z) P_{3}(z)=A(z) B_{4}\left(z^{d}\right) Q_{1}(z) \prod_{\gamma \in N_{3}(\tau)}\left(z^{d}-\gamma\right)
$$

We repeat this process, which terminates in a finite number of steps since $B(z)$ is a polynomial. Namely, there exists $k \geq 1$ such that $N_{k}(\tau)=\phi$, and so $N_{k-1}^{1 / d}(\tau)=M_{k}(\tau)$. This implies $M_{k}(\tau)=\zeta_{d} M_{k}(\tau)$ and

$$
A\left(z^{d}\right) B_{k+1}(z) P_{k}(z)=A(z) B_{k+1}\left(z^{d}\right) Q_{1}(z)
$$

Since $P(z)$ and $Q(z)$ are coprime and $Q(\tau)=0$, we deduce $\tau \notin \tilde{\mathcal{E}}_{k}(\tau)$. This completes the proof of the lemma.

Lemma 4.2. Let $P(z)$ and $Q(z)$ satisfy (2.7). Let $\alpha \in \mathbb{C}$ with $|\alpha|=1$ be the root of $P(z) Q(z)$ with the smallest positive argument among its roots in $\mathcal{M}$. Assume that $P(\alpha)=0$ and $Q\left(\alpha^{d^{m}}\right) \neq 0$ for any integer $m \geq 1$. Then there exist $k \geq 1, \tau \in \mathbb{C}$ with $|\tau|=1$, and $M_{k}(\tau)$ with $\tau \in \tilde{\mathcal{E}}_{k}(\tau)$ such that $P(z)$ is divisible by $F^{(\tau)}(z) /(z-\tau)$.

Proof. We give an algorithm how to find $M_{k}(\tau)$, defining $M_{i}(\tau)$ and $N_{i}(\tau)$ below for $i=1,2, \ldots, k$ inductively. We see that $B\left(\alpha^{d^{m}}\right)=0$ for any $m \geq 1$ by (4.1) and statement (ii) of Lemma 2.3. Hence there exist integers $r, s$ with $1 \leq r<s$ such that $\alpha^{d^{r}}=\alpha^{d^{s}}$, since $B(z)$ is a polynomial. We take the smallest $l=s-r \geq 1$. Note that $B\left(\alpha^{d^{r+1}}\right)=B\left(\alpha^{d^{r+2}}\right)=\cdots=B\left(\alpha^{d^{s}}\right)=0$. Put $\tau:=\alpha^{d^{r}}=\alpha^{d^{s}}$. Since $Q(\tau) \neq 0$, we need the following discussion different from the proof of Lemma 4.1.

Set

$$
B_{0}(z):=B(z), \quad B_{1}(z):=\frac{B(z)}{z-\tau} \in \mathbb{C}[z], \quad \text { and } \quad P_{0}^{\dagger}(z):=(z-\tau) P(z)
$$

For $i=1, \ldots, l-1$ we define the sets $N_{i}(\tau), M_{i}(\tau) \subset \Theta_{i}(\tau)$ and the polynomials $B_{i+1}(z)$ and $P_{i}^{\dagger}(z)$, which are factors of $B(z)$ and $(z-\tau) P(z)$, respectively. Hence $A(z)$ and $B_{i}(z)$ are coprime for $i=0,1, \ldots, l$. To proceed the inductive steps, we simultaneously check the following for $i=0,1, \ldots, l-1$.
(i) For any $\gamma \in N_{i}(\tau)$ we have

$$
\begin{equation*}
B_{i}(\gamma)=0 \tag{4.7}
\end{equation*}
$$

(ii) We have

$$
\begin{equation*}
\alpha^{d^{s-i}} \in N_{i}(\tau) \tag{4.8}
\end{equation*}
$$

In particular, $N_{i}(\tau) \neq \phi$.
(iii) It follows that

$$
\begin{equation*}
A\left(z^{d}\right) B_{i+1}(z) P_{i}^{\dagger}(z)=A(z) B_{i+1}\left(z^{d}\right) Q(z) \prod_{\gamma \in N_{i}(\tau)}\left(z^{d}-\gamma\right) \tag{4.9}
\end{equation*}
$$

Then (4.7) and (4.8) with $i=0$ is clear by (4.2). By (2.7) we have

$$
A\left(z^{d}\right) B_{1}(z) P_{0}^{\dagger}(z)=A(z) B_{1}\left(z^{d}\right) Q(z)\left(z^{d}-\tau\right)
$$

which implies (4.9) with $i=0$.
Suppose that there exists an integer $j$ with $1 \leq j \leq l-1$ such that $N_{i}(\tau)$, $B_{i+1}(z)$, and $P_{i}^{\dagger}(z)$ satisfy (4.7), (4.8), and (4.9) for $i=0,1, \ldots, j-1$. Set $N_{j}(\tau):=\left\{\gamma \in N_{j-1}^{1 / d}(\tau) \mid B_{j}(\gamma)=0\right\}$ and $M_{j}(\tau):=\left\{\gamma \in N_{j-1}^{1 / d}(\tau) \mid B_{j}(\gamma) \neq 0\right\}$.
Then (4.7) holds for $i=j$. Since $N_{j-1}^{1 / d}(\tau) \subset \Theta_{j}(\tau)$ by $N_{j-1}(\tau) \subset \Theta_{j-1}(\tau)$, we get $N_{j}(\tau), M_{j}(\tau) \subset \Theta_{j}(\tau)$. For any $\gamma \in N_{j-1}^{1 / d}(\tau)$, we have $B_{j-1}\left(\gamma^{d}\right)=0$
by (4.7) with $i=j-1$ and so $A\left(\gamma^{d}\right) \neq 0$ since $B_{j-1}(z)$ and $A(z)$ are coprime. Thus, substituting $z=\gamma \in N_{j-1}^{1 / d}(\tau)$ into (4.9) with $i=j-1$, we get $B_{j}(\gamma) P_{j-1}^{\dagger}(\gamma)=0$. In particular, all the elements of the set $M_{j}(\tau)$ are the roots of $P_{j-1}^{\dagger}(z)$. Put
$B_{j+1}(z):=\frac{B_{j}(z)}{\prod_{\gamma \in N_{j}(\tau)}(z-\gamma)} \in \mathbb{C}[z] \quad$ and $\quad P_{j}^{\dagger}(z):=\frac{P_{j-1}^{\dagger}(z)}{\prod_{\gamma \in M_{j}(\tau)}(z-\gamma)} \in \mathbb{C}[z]$.
Note that $\alpha^{d^{s-j}} \in N_{j-1}^{1 / d}(\tau)$ by (4.8) with $i=j-1$ and

$$
B_{j}(z)=\frac{B(z)}{\prod_{i=0}^{j-1} \prod_{\gamma \in N_{i}(\tau)}(z-\gamma)}
$$

Recall that $B\left(\alpha^{d^{s-j}}\right)=0$. For the proof of (4.8) with $i=j$, it suffices to show that $\alpha^{d^{s-j}} \notin N_{h}(\tau)$ for any $h=0,1, \ldots, j-1$. Suppose on the contrary that $\alpha^{d^{s-j}} \in N_{h}(\tau) \subset \Theta_{h}(\tau)$. Then $\alpha^{d^{s-j+h}}=\tau=\alpha^{d^{s}}$, which contradicts the minimality of $l$. Hence we showed (4.8) with $i=j$. We rewrite (4.9) with $i=j-1$ as

$$
\begin{aligned}
& A\left(z^{d}\right)\left(B_{j+1}(z) \prod_{\gamma \in N_{j}(\tau)}(z-\gamma)\right)\left(P_{j}^{\dagger}(z) \prod_{\gamma \in M_{j}(\tau)}(z-\gamma)\right) \\
= & A(z)\left(B_{j+1}\left(z^{d}\right) \prod_{\gamma \in N_{j}(\tau)}\left(z^{d}-\gamma\right)\right) Q(z) \prod_{\gamma \in N_{j-1}(\tau)}\left(z^{d}-\gamma\right) .
\end{aligned}
$$

Dividing both sides of this equality by

$$
\prod_{\gamma \in N_{j}(\tau)}(z-\gamma) \prod_{\gamma \in M_{j}(\tau)}(z-\gamma)=\prod_{\gamma \in N_{j-1}(\tau)}\left(z^{d}-\gamma\right)
$$

we get

$$
A\left(z^{d}\right) B_{j+1}(z) P_{j}^{\dagger}(z)=A(z) B_{j+1}\left(z^{d}\right) Q(z) \prod_{\gamma \in N_{j}(\tau)}\left(z^{d}-\gamma\right),
$$

which implies (4.9) with $i=j$. Therefore, we have defined $N_{i}(\tau), M_{i}(\tau)$, $B_{i+1}(z)$, and $P_{i}^{\dagger}(z)$ for $i=1, \ldots, l-1$.

We show that $z-\tau$ divides both $\prod_{\gamma \in N_{l-1}(\tau)}\left(z^{d}-\gamma\right)$ and

$$
P_{l-1}^{\dagger}(z)=\frac{(z-\tau) P(z)}{\prod_{i=1}^{l-1} \prod_{\gamma \in M_{i}(\tau)}(z-\gamma)} .
$$

First by (4.8) with $i=l-1$ we have

$$
\begin{equation*}
\tau^{d}=\alpha^{d^{r+1}}=\alpha^{d^{s-(l-1)}} \in N_{l-1}(\tau) \tag{4.10}
\end{equation*}
$$

Hence $z-\tau$ divides $\prod_{\gamma \in N_{l-1}(\tau)}\left(z^{d}-\gamma\right)$. Next if $P_{l-1}^{\dagger}(\tau) \neq 0$, then $\tau \in M_{i}(\tau) \subset$ $\Theta_{i}(\tau)$ for some $i$ with $1 \leq i \leq l-1$ and so $\tau^{d^{i}}=\tau$, which contradicts the
minimality of $l$. Dividing both sides of (4.9) with $i=l-1$ by $z-\tau$ and putting $P_{l-1}(z):=P_{l-1}^{\dagger}(z) /(z-\tau)$, we have

$$
\begin{equation*}
A\left(z^{d}\right) B_{l}(z) P_{l-1}(z)=A(z) B_{l}\left(z^{d}\right) Q(z) \frac{\prod_{\gamma \in N_{l-1}(\tau)}\left(z^{d}-\gamma\right)}{z-\tau} \tag{4.11}
\end{equation*}
$$

Define

$$
N_{l}(\tau):=\left\{\gamma \in N_{l-1}^{1 / d}(\tau) \backslash\{\tau\} \mid B_{l}(\gamma)=0\right\}
$$

and

$$
M_{l}(\tau):=\left\{\gamma \in N_{l-1}^{1 / d}(\tau) \backslash\{\tau\} \mid B_{l}(\gamma) \neq 0\right\} \bigcup\{\tau\} .
$$

If $\gamma \in N_{l-1}^{1 / d}(\tau) \backslash\{\tau\}$, then $A\left(\gamma^{d}\right) \neq 0$ by (4.7) with $i=l-1$. Substituting $z=\gamma$ into (4.11), we have $B_{l}(\gamma) P_{l-1}(\gamma)=0$. Hence, putting
$B_{l+1}(z):=\frac{B_{l}(z)}{\prod_{\gamma \in N_{l}(\tau)}(z-\gamma)} \in \mathbb{C}[z]$ and $P_{l}(z):=\frac{P_{l-1}(z)}{\prod_{\gamma \in M_{l}(\tau) \backslash\{\tau\}}(z-\gamma)} \in \mathbb{C}[z]$
and dividing both sides of (4.11) by

$$
\prod_{\gamma \in N_{l}(\tau)}(z-\gamma) \prod_{\gamma \in M_{l}(\tau) \backslash\{\tau\}}(z-\gamma)=\frac{\prod_{\gamma \in N_{l-1}(\tau)}\left(z^{d}-\gamma\right)}{z-\tau}
$$

we have

$$
\begin{equation*}
A\left(z^{d}\right) B_{l+1}(z) P_{l}(z)=A(z) B_{l+1}\left(z^{d}\right) Q(z) \prod_{\gamma \in N_{l}(\tau)}\left(z^{d}-\gamma\right) \tag{4.12}
\end{equation*}
$$

Since $\tau \in N_{l-1}^{1 / d}(\tau)$ by (4.10), if $N_{l}(\tau)=\phi$, then $N_{l-1}^{1 / d}(\tau)=M_{l}(\tau)$ and hence $M_{l}(\tau)=\zeta_{d} M_{l}(\tau)$. Then we put $k=l$, which implies the lemma because

$$
P_{l}(z)=\frac{(z-\tau) P(z)}{\prod_{i=1}^{l} \prod_{\gamma \in M_{i}(\tau)}(z-\gamma)} \in \mathbb{C}[z]
$$

If $N_{l}(\tau) \neq \phi$, for $i(\geq l+1)$, we define inductively

$$
\begin{gathered}
N_{i}(\tau):=\left\{\gamma \in N_{i-1}^{1 / d}(\tau) \mid B_{i}(\gamma)=0\right\}, M_{i}(\tau):=\left\{\gamma \in N_{i-1}^{1 / d}(\tau) \mid B_{i}(\gamma) \neq 0\right\}, \\
B_{i+1}(z):=\frac{B_{i}(z)}{\prod_{\gamma \in N_{i}(\tau)}(z-\gamma)}, \quad \text { and } \quad P_{i}(z):=\frac{P_{i-1}(z)}{\prod_{\gamma \in M_{i}(\tau)}(z-\gamma)}
\end{gathered}
$$

unless $N_{i-1}(\tau)$ is empty. Note that $B_{i+1}(z), P_{i}(z) \in \mathbb{C}[z]$, since for any $\gamma \in$ $N_{i-1}^{1 / d}(\tau)$ we have $B_{i}(\gamma) P_{i-1}(\gamma)=0$ by (4.12) and $A\left(\gamma^{d}\right) \neq 0$. By the same way as above, we have

$$
A\left(z^{d}\right) B_{l+2}(z) P_{l+1}(z)=A(z) B_{l+2}\left(z^{d}\right) Q(z) \prod_{\gamma \in N_{l+1}(\tau)}\left(z^{d}-\gamma\right)
$$

We repeat this process, which terminates in a finite number of steps since $B(z)$ is a polynomial. Thus there exists an integer $k \geq l$ such that

$$
A\left(z^{d}\right) B_{k+1}(z) P_{k}(z)=A(z) B_{k+1}\left(z^{d}\right) Q(z)
$$

and $N_{k}(\tau)=\phi$, which implies $N_{k-1}^{1 / d}(\tau)=M_{k}(\tau)$ and hence $M_{k}(\tau)=$ $\zeta_{d} M_{k}(\tau)$.

Remark 4.3. The case where $\tau=-1$ and $d$ is even corresponds to Lemma 4.1. The case where $\tau=-1$ and $d$ is odd and that of $\tau=1$ correspond to Lemma 4.2. We also note that the case where $-1 \in \tilde{\mathcal{F}}_{k}(\tau)$ occurs when $d$ is even and $\tau= \pm 1$.

Let $H^{(\tau)}(z)$ be a polynomial defined by

$$
H^{(\tau)}(z)=\prod_{\gamma \in N_{k-1}(\tau)}(z-\gamma) \cdots \prod_{\gamma \in N_{0}(\tau)}(z-\gamma)
$$

where $N_{i}(\tau)(0 \leq i \leq k-1)$ are defined in the proof of either Lemma 4.1 or 4.2.

Lemma 4.4. The polynomial $B(z)$ is divisible by $H^{(\tau)}(z)$ and by factoring out we have an equation of the same form as (2.7), namely,

$$
A\left(z^{d}\right) B^{\dagger}(z) P^{\dagger}(z)=A(z) B^{\dagger}\left(z^{d}\right) Q^{\dagger}(z)
$$

where

$$
P^{\dagger}(z)=\frac{P(z)}{F^{(\tau)}(z)}, \quad Q^{\dagger}(z)=\frac{Q(z)}{z-\tau}, \quad \text { and } \quad B^{\dagger}(z)=\frac{B(z)}{H^{(\tau)}(z)}
$$

if $\tau \notin \tilde{\mathcal{E}}_{k}(\tau)$, or

$$
P^{\dagger}(z)=\frac{P(z)}{F^{(\tau)}(z) /(z-\tau)}, \quad Q^{\dagger}(z)=Q(z), \quad \text { and } \quad B^{\dagger}(z)=\frac{B(z)}{H^{(\tau)}(z)}
$$

if $\tau \in \tilde{\mathcal{E}}_{k}(\tau)$.

Proof. We see that $B(z)$ is divisible by $H^{(\tau)}(z)$ as is shown in the proof of Lemma 4.1 or 4.2. By the definition of the sets therein, we have

$$
\begin{align*}
& F^{(\tau)}(z)  \tag{4.13}\\
& =\prod_{\gamma \in M_{k}(\tau)}(z-\gamma) \prod_{\gamma \in M_{k-1}(\tau)}(z-\gamma) \cdots \prod_{\gamma \in M_{1}(\tau)}(z-\gamma) \\
& =\prod_{\gamma \in N_{k-1}(\tau)}\left(z^{d}-\gamma\right) \prod_{\gamma \in N_{k-2}^{1 / d}(\tau) \backslash N_{k-1}(\tau)}(z-\gamma) \cdots \prod_{\gamma \in N_{0}^{1 / d}(\tau) \backslash N_{1}(\tau)}(z-\gamma) \\
& =\prod_{\gamma \in N_{k-1}(\tau)} \frac{z^{d}-\gamma}{z-\gamma} \prod_{\gamma \in N_{k-2}(\tau)}\left(z^{d}-\gamma\right) \prod_{\substack{ \\
\gamma \in N_{k-3}^{1 / d}(\tau) \backslash N_{k-2}(\tau)}}(z-\gamma) \\
& \text { … } \Pi^{(z-\gamma)} \\
& \gamma \in N_{0}^{1 / d}(\tau) \backslash N_{1}(\tau) \\
& =\prod_{\gamma \in N_{k-1}(\tau)} \frac{z^{d}-\gamma}{z-\gamma} \prod_{\gamma \in N_{k-2}(\tau)} \frac{z^{d}-\gamma}{z-\gamma} \cdots \prod_{\gamma \in N_{0}(\tau)} \frac{z^{d}-\gamma}{z-\gamma} \prod_{\gamma \in N_{0}(\tau)}(z-\gamma) \\
& =\frac{H^{(\tau)}\left(z^{d}\right)}{H^{(\tau)}(z)}(z-\tau) .
\end{align*}
$$

Hence the lemma is proved by dividing both sides of (2.7) by $H^{(\tau)}(z) F^{(\tau)}(z)=$ $H^{(\tau)}\left(z^{d}\right)(z-\tau)$ in the case of Lemma 4.1 and by $H^{(\tau)}(z) F^{(\tau)}(z) /(z-\tau)=$ $H^{(\tau)}\left(z^{d}\right)$ in the case of Lemma 4.2.

## 5. Proof of the theorems

Lemma 5.1 (Nishioka [4, Lemma 2.3.3]). Let $\boldsymbol{L}$ be a subfield of $\mathbb{C}$ and suppose that

$$
f(z) \in \mathbb{C}[[z]] \cap \boldsymbol{L}(z) .
$$

If $f(z)$ converges at $z=\alpha$, then $f(\alpha) \in \boldsymbol{L}(\alpha)$.
Proof of Theorem 1.3. First we check the necessary conditions for algebraic dependence. Assume that the values $\Phi_{1}\left(\alpha^{-d^{N}}\right), \ldots, \Phi_{m}\left(\alpha^{-d^{N}}\right)$ in Section 2 are algebraically dependent. Then there exist coprime polynomials $A(z), B(z) \in \boldsymbol{K}[z] \backslash\{0\}$ satisfying the functional equation (2.6) with $b=1$ by Lemma 3.2. We define $P(z):=\prod_{i=1}^{k} P_{i}(z)^{e_{i}}$ and $Q(z):=$ $\left(1+z^{2}\right)^{e} \prod_{i=k+1}^{l} P_{i}(z)^{e_{i}}$ as in (2.6). We note that $\operatorname{deg} P(z)=\operatorname{deg} Q(z)$. If $\gamma \in \mathbb{C}$ is a zero of $P(z) Q(z)$, then $\gamma= \pm \sqrt{-1}$ or $-(\gamma+\bar{\gamma}) \in\left\{a_{1}, \ldots, a_{m}\right\}$ by (2.3).

First we consider the case of $d=2$. If $P(z)$ or $Q(z)$ has a real root, we take a real root $\alpha_{1}$ of $P(z) Q(z)$ with the largest absolute value among its real roots, namely, $\alpha_{1}$ satisfies (3.2). Exchanging the above definition of $P(z)$ and $Q(z)$ if necessary, we may assume that $P\left(\alpha_{1}\right)=0$. If $\alpha_{1}$ is positive,
then the case (i) of Theorem 1.3 holds by Lemma 3.4 and Remark 3.5. If $\alpha_{1}$ is negative, then we have $\alpha_{1}=-1$ by Lemma 3.3, namely, $P(-1)=0$. Thus we see that $a_{i}=2$ for some $i$ and the case (ii) of Theorem 1.3 holds (see the latter case of Remark 1.4).

Next we suppose that $P(z) Q(z)$ has non-real roots, which are included in the set $\mathcal{M}$ defined by (3.5) as is shown in Subsection 3.2. Exchanging the above definition of $P(z)$ and $Q(z)$ if necessary, we may assume that $P(z)$ has the non-real root with the smallest positive argument among the roots of $P(z) Q(z)$ in $\mathcal{M}$. Then the assumptions of either Lemma 4.1 or Lemma 4.2 are satisfied. Putting $\mathcal{E}_{k}(\tau):=\tilde{\mathcal{E}}_{k}(\tau) \cup \overline{\tilde{\mathcal{E}}_{k}(\tau)}$, we have

$$
\mathcal{E}_{k}(\tau)=\Gamma_{1}(\tau) \cup \cdots \cup \Gamma_{k-1}(\tau) \cup S_{k}(\tau)
$$

where $S_{k}(\tau)=M_{k}(\tau) \cup \overline{M_{k}(\tau)}, \Lambda_{i}(\tau)=N_{i}(\tau) \cup \overline{N_{i}(\tau)}(0 \leq i \leq k-1)$, and $\Gamma_{i}(\tau)=M_{i}(\tau) \cup \overline{M_{i}(\tau)}(1 \leq i \leq k-1)$. Using the conditions on $M_{i}(\tau)(1 \leq i \leq k)$, we see that the assumptions on $\mathcal{E}_{k}(\tau)$ stated in the introduction of this paper are satisfied. In what follows, we show that the set of the roots of $P(z) Q(z)$ contains $\mathcal{F}_{k}(\tau)$. Note that if $\gamma \in \mathbb{C}$ is a zero of $P(z)$ (resp. $Q(z))$, then $\bar{\gamma}$ is also a zero of $P(z)($ resp. $Q(z))$. If the assumptions in Lemma 4.1 are satisfied, then the set of the roots of $P(z)$ (resp. $Q(z)$ ) contains $\mathcal{E}_{k}(\tau)$ (resp. $\{\tau, \bar{\tau}\}$ ). Since $P(z)$ and $Q(z)$ are coprime, $\tau \notin \mathcal{E}_{k}(\tau)$ and so $\mathcal{F}_{k}(\tau)=\mathcal{E}_{k}(\tau) \cup\{\tau, \bar{\tau}\}$. Thus the set of the roots of $P(z) Q(z)$ contains $\mathcal{F}_{k}(\tau)$ in this case. On the other hand, if the assumptions of Lemma 4.2 are satisfied, then we get $\mathcal{E}_{k}(\tau) \supset\{\tau, \bar{\tau}\}$ and $\mathcal{F}_{k}(\tau)=\mathcal{E}_{k}(\tau) \backslash\{\tau, \bar{\tau}\}$. Moreover, the set of the roots of $P(z)$ contains $\mathcal{F}_{k}(\tau)$. Hence the case (ii) of Theorem 1.3 holds in both cases.

We now consider the case of $d \geq 3$. By (2.3) and Lemma 3.2, we get $b=1$ and so $d$ is even. By Lemma 3.6, the roots of $P(z) Q(z)$ are included in $\mathcal{M}$. By Lemma 4.1 or Lemma 4.2, there exist $\tau_{1} \in \mathbb{C}$ with $\left|\tau_{1}\right|=1$ and $\tilde{\mathcal{E}}_{k_{1}}\left(\tau_{1}\right)$ with $k_{1} \geq 1$ such that
(i) $\tau_{1} \notin \tilde{\mathcal{E}}_{k_{1}}\left(\tau_{1}\right)$ and $P(z), Q(z)$ are divisible by $F^{\left(\tau_{1}\right)}(z), z-\tau_{1}$, respectively, or
(ii) $\tau_{1} \in \tilde{\mathcal{E}}_{k_{1}}\left(\tau_{1}\right)$ and $P(z)$ is divisible by $F^{\left(\tau_{1}\right)}(z) /\left(z-\tau_{1}\right)$.

Dividing (2.7) by these terms, by Lemma 4.4 we have

$$
A\left(z^{d}\right) B^{\dagger}(z) P^{\dagger}(z)=A(z) B^{\dagger}\left(z^{d}\right) Q^{\dagger}(z)
$$

which is the same form as (2.7). For the later convenience, denote $\eta^{(1)}(z):=$ $P^{\dagger}(z)$ and $\xi^{(1)}(z):=Q^{\dagger}(z)$. Since the number of the elements in $\tilde{\mathcal{E}}_{k_{1}}\left(\tau_{1}\right)$ is not less than $d>2$, we have $\operatorname{deg} \eta^{(1)}(z)<\operatorname{deg} \xi^{(1)}(z)$. In particular, $\operatorname{deg} \eta^{(1)}(z) \xi^{(1)}(z)>0$. Let $\alpha^{(1)} \in \mathbb{C}$ with $\left|\alpha^{(1)}\right|=1$ be the root of $\eta^{(1)}(z) \xi^{(1)}(z)$
having the smallest positive argument among its roots. If $\xi^{(1)}\left(\alpha^{(1)}\right) \neq 0$, then $\eta^{(1)}\left(\alpha^{(1)}\right)=0$. We apply Lemma 4.4 with $P(z)=\eta^{(1)}(z)$ and $Q(z)=\xi^{(1)}(z)$. We write the polynomials corresponding to $P^{\dagger}(z)$ and $Q^{\dagger}(z)$ therein as $\eta^{(2)}(z)$ and $\xi^{(2)}(z)$, respectively. Then we see $\operatorname{deg} \eta^{(2)}(z)<\operatorname{deg} \xi^{(2)}(z)$. Repeating this process, we can define $\eta^{(i)}(z), \xi^{(i)}(z)$, and $\alpha^{(i)}(i=2,3, \ldots)$ inductively whenever $\xi^{(i-1)}\left(\alpha^{(i-1)}\right) \neq 0$. This process terminates in a finite number of steps since $P^{\dagger}(z)$ is a polynomial. Thus there exists an integer $k \geq 1$ such that $\xi^{(k)}\left(\alpha^{(k)}\right)=0$. Since $\eta^{(k)}(z)$ and $\xi^{(k)}(z)$ are the factors of $P^{\dagger}(z)$ and $Q^{\dagger}(z)$, respectively, Lemma 4.1 or 4.2 implies the following: There exist $\tau_{2} \in \mathbb{C}$ with $\left|\tau_{2}\right|=1$ and $\tilde{\mathcal{E}}_{k_{2}}\left(\tau_{2}\right)$ with $k_{2} \geq 1$ such that
(i) $\tau_{2} \notin \tilde{\mathcal{E}}_{k_{2}}\left(\tau_{2}\right)$ and $Q^{\dagger}(z), P^{\dagger}(z)$ are divisible by $F^{\left(\tau_{2}\right)}(z), z-\tau_{2}$, respectively, or
(ii) $\tau_{2} \in \tilde{\mathcal{E}}_{k_{2}}\left(\tau_{2}\right)$ and $Q^{\dagger}(z)$ is divisible by $F^{\left(\tau_{2}\right)}(z) /\left(z-\tau_{2}\right)$.

We note that $\tau_{1} \neq \tau_{2}$, since $B\left(\tau_{1}\right)=A\left(\tau_{2}\right)=0$ and since $A(z)$ and $B(z)$ are coprime. For $j=1,2$, we put $\mathcal{E}_{k_{j}}\left(\tau_{j}\right):=\tilde{\mathcal{E}}_{k_{j}}\left(\tau_{j}\right) \cup \overline{\tilde{\mathcal{E}}_{k_{j}}\left(\tau_{j}\right)}$. In the same way as in the case where $d=2$ and $P(z) Q(z)$ has non-real roots, we see that the set of the roots of $P(z)$ (resp. $Q(z)$ ) contains $\mathcal{E}_{k_{1}}\left(\tau_{1}\right) \backslash\left\{\tau_{1}, \overline{\tau_{1}}\right\}$ (resp. $\left.\mathcal{E}_{k_{2}}\left(\tau_{2}\right) \backslash\left\{\tau_{2}, \overline{\tau_{2}}\right\}\right)$ both in the case of Lemmas 4.1 and 4.2. Since $P(z)$ and $Q(z)$ are coprime, we obtain

$$
\left(\mathcal{E}_{k_{1}}\left(\tau_{1}\right) \backslash\left\{\tau_{1}, \overline{\tau_{1}}\right\}\right) \cap\left(\mathcal{E}_{k_{2}}\left(\tau_{2}\right) \backslash\left\{\tau_{2}, \overline{\tau_{2}}\right\}\right)=\phi
$$

and so

$$
\begin{aligned}
\mathcal{F}_{k_{1}}\left(\tau_{1}\right) \cap \mathcal{F}_{k_{2}}\left(\tau_{2}\right) & \subset\left(\mathcal{E}_{k_{1}}\left(\tau_{1}\right) \cap \mathcal{E}_{k_{2}}\left(\tau_{2}\right)\right) \cup\left\{\tau_{1}, \overline{\tau_{1}}, \tau_{2}, \overline{\tau_{2}}\right\} \\
& \subset\left\{\tau_{1}, \overline{\tau_{1}}, \tau_{2}, \overline{\tau_{2}}\right\} .
\end{aligned}
$$

Hence we obtain the case (iii) of Theorem 1.3.
In what follows, we show that $\Phi_{1}\left(\alpha^{-d^{N}}\right), \ldots, \Phi_{m}\left(\alpha^{-d^{N}}\right)$ are algebraically dependent under the assumption that the case (i), (ii), or (iii) in Theorem 1.3 holds. Recall by (2.3) that $p_{i}=a_{i}(i=1, \ldots, m)$ and $b=1$ since $d$ is even in every case. It suffices to show that there exist a non-empty subset $I$ of $\{1, \ldots, m\}$ and non-zero integers $e_{i}(i \in I)$ satisfying

$$
\begin{equation*}
\prod_{i \in I} c_{i}(z)^{e_{i}}=\prod_{i \in I}\left(\frac{z^{2}+1}{z^{2}+a_{i} z+1}\right)^{e_{i}} \in H_{d} \tag{5.1}
\end{equation*}
$$

where $H_{d}$ is the subgroup of the multiplicative group $\boldsymbol{K}(z)^{\times}$defined by (2.1), or there exists a $g(z) \in \boldsymbol{K}(z)^{\times}$such that

$$
\prod_{i \in I} c_{i}(z)^{e_{i}}=\frac{g\left(z^{d}\right)}{g(z)} .
$$

Here, if $z=0$ is a zero or a pole of $g(z)$, then it is a zero or a pole of $g\left(z^{d}\right) / g(z)$, respectively. Hence $g(0) \neq 0$ because $c_{i}(0)=1(i \in I)$. Then we see by (2.5) that $F(z):=g(z)^{-1} \prod_{i \in I} \Phi_{i}(z)^{e_{i}} \in \boldsymbol{K}[[z]]$ satisfies $F\left(z^{d}\right)=$ $F(z)$, which holds only if $F(z)=\lambda \in \boldsymbol{K}$. In fact, if $l(\geq 1)$ is the lowest degree of the non-constant terms of $F(z)$, then that of $F\left(z^{d}\right)$ is $d l$, which contradicts $F\left(z^{d}\right)=F(z)$. Hence

$$
\prod_{i \in I} \Phi_{i}(z)^{e_{i}}=\lambda g(z) \in \boldsymbol{K}[[z]] \cap \boldsymbol{K}(z) .
$$

By Lemma 5.1 we have

$$
\prod_{i \in I} \Phi_{i}\left(\alpha^{-d^{N}}\right)^{e_{i}} \in \boldsymbol{K}
$$

which implies that $\Phi_{1}\left(\alpha^{-d^{N}}\right), \ldots, \Phi_{m}\left(\alpha^{-d^{N}}\right)$ are algebraically dependent and thus we only have to prove (5.1).

Note that, for any $h \geq 1$ and $g(z) \in \boldsymbol{K}(z)^{\times}$,

$$
\begin{equation*}
\frac{g\left(z^{d^{h}}\right)}{g(z)}=\frac{g\left(z^{d}\right)}{g(z)} \frac{g\left(z^{d^{2}}\right)}{g\left(z^{d}\right)} \cdots \frac{g\left(z^{d^{h}}\right)}{g\left(z^{d^{h-1}}\right)} \in H_{d} . \tag{5.2}
\end{equation*}
$$

If $d=2$, then, for the proof of (5.1), it suffices to check that

$$
\begin{equation*}
\prod_{i \in I}\left(z^{2}+a_{i} z+1\right)^{e_{i}} \in H_{2} \tag{5.3}
\end{equation*}
$$

because

$$
\begin{equation*}
z^{2}+1=\frac{z^{4}-1}{z^{2}-1} \in H_{2} . \tag{5.4}
\end{equation*}
$$

First we suppose that the case (i) of Theorem 1.3 holds. Since $b_{1}=-b_{2}$, we have

$$
\left(z^{2}+b_{1} z+1\right)\left(z^{2}+b_{2} z+1\right)=z^{4}-\left(b_{2}^{2}-2\right) z^{2}+1
$$

and then

$$
\begin{equation*}
\left(z^{2}+b_{1} z+1\right)\left(z^{2}+b_{2} z+1\right) \prod_{j=3}^{l-1}\left(z^{2^{j-1}}+b_{j} z^{2^{j-2}}+1\right)=z^{2^{l-1}}+b_{l} z^{2^{l-2}}+1 \tag{5.5}
\end{equation*}
$$ by $b_{j}=b_{j-1}^{2}-2(j=3, \ldots, l-1)$ and $b_{l}=-b_{l-1}^{2}+2$. Thus by (5.2) and (5.5) we obtain

$$
\begin{aligned}
& \left(z^{2}+b_{l} z+1\right)^{-1} \prod_{j=1}^{l-1}\left(z^{2}+b_{j} z+1\right) \\
& =\frac{z^{2 l-1}+b_{l} z^{2-2}+1}{z^{2}+b_{l} z+1} \prod_{j=3}^{l-1}\left(\frac{z^{2}+b_{j} z+1}{z^{2 j-1}+b_{j} z^{2 j-2}+1}\right) \in H_{2},
\end{aligned}
$$

which implies (5.3).

Here we suspend the proof of the theorem and investigate the properties of the sets defined in Section 1. For the later convenience, denote $\Gamma_{k}(\tau):=$ $S_{k}(\tau)$. Then $\mathcal{E}_{k}(\tau)=\cup_{i=1}^{k} \Gamma_{i}(\tau)$.

Lemma 5.2. Let $\tau \in \mathbb{C}$ with $|\tau|=1, k \geq 1$, and $S_{k}(\tau) \subset \Omega_{k}(\tau)$ satisfy (1.4). Suppose that $\tau \in \mathcal{E}_{k}(\tau)$. Then we have

$$
\operatorname{Card}\left\{i \mid 1 \leq i \leq k, \tau \in \Gamma_{i}(\tau)\right\}=\operatorname{Card}\left\{i \mid 1 \leq i \leq k, \bar{\tau} \in \Gamma_{i}(\tau)\right\}=1
$$

where Card denotes the cardinality.
Proof. Since $\overline{\Gamma_{i}(\tau)}=\Gamma_{i}(\tau)$ for $i=1, \ldots, k$, it suffices to show

$$
\begin{equation*}
\operatorname{Card}\left\{i \mid 1 \leq i \leq k, \tau \in \Gamma_{i}(\tau)\right\}=1 \tag{5.6}
\end{equation*}
$$

For $x, y \in \mathbb{C}$, we write $x \sim y$ if $x=y$ or if $\bar{x}=y$. Noting that $\tau \in$ $\mathcal{E}_{k}(\tau) \subset \cup_{i=1}^{k} \Omega_{i}(\tau)$, we take $l:=\min \left\{i \geq 1 \mid \tau^{d^{i}} \sim \tau\right\}(\leq k)$. Suppose that $\tau \in \Gamma_{j}(\tau) \subset \Omega_{j}(\tau)$ for some $j \geq 1$. Put $j=q l+r$, where $q$ and $r$ are integers with $q \geq 0$ and $0 \leq r \leq l-1$. Then we get $\tau \sim \tau^{d^{j}}=\tau^{d^{q l+r}} \sim \tau^{d^{r}}$ and so $r=0$ by the minimality of $l$. We take $b:=\min \left\{q \geq 1 \mid \tau \in \Gamma_{q l}(\tau)\right\}$. For the proof of (5.6), it suffices to show that $\tau \notin \Gamma_{b l+c l}(\tau)$ for any $c \geq 1$.

Suppose on the contrary that $\tau \in \Gamma_{b l+c l}(\tau)$. Then $\tau^{d} \in \Lambda_{b l+c l-1}(\tau)$. Note that for any $i, j$ with $i \geq j$, if $\gamma \in \Lambda_{i}(\tau)$, then $\gamma^{d^{i-j}} \in \Lambda_{j}(\tau)$. Thus $\tau \sim$ $\tau^{d^{c l}}=\left(\tau^{d}\right)^{d^{c l-1}} \in \Lambda_{b l}(\tau)$. Since $\overline{\Lambda_{b l}(\tau)}=\Lambda_{b l}(\tau)$, we obtain $\tau \in \Lambda_{b l}(\tau)$, which contradicts the fact that $\Gamma_{b l}(\tau) \cap \Lambda_{b l}(\tau)=\phi$. This completes the proof of the lemma.

Put

$$
\begin{equation*}
g_{\gamma}(z)=(z-\gamma)(z-\bar{\gamma}) \tag{5.7}
\end{equation*}
$$

for $\gamma \in \mathbb{C}$.
Lemma 5.3. Let $\tau \in \mathbb{C}$ with $|\tau|=1, k \geq 1$, and $S_{k}(\tau) \subset \Omega_{k}(\tau)$ satisfy (1.4). Then there exists a mapping $e: \mathcal{F}_{k}(\tau) \rightarrow \mathbb{Z} \backslash\{0\}$ such that

$$
\begin{equation*}
e(\gamma)=e(\bar{\gamma}) \tag{5.8}
\end{equation*}
$$

for any $\gamma \in \mathcal{F}_{k}(\tau)$ and

$$
\begin{equation*}
\prod_{\gamma \in \mathcal{F}_{k}(\tau)} g_{\gamma}(z)^{e(\gamma)} \in H_{d} \tag{5.9}
\end{equation*}
$$

where $H_{d}$ is the subgroup of $\boldsymbol{K}(z)^{\times}$defined by (2.1). In particular, there exists an integer $p$ such that

$$
\begin{equation*}
\left(z^{2}+1\right)^{p} \prod_{\gamma \in \mathcal{F}_{k}(\tau) \backslash\{ \pm \sqrt{-1}\}} g_{\gamma}(z)^{e(\gamma)} \in H_{d} . \tag{5.10}
\end{equation*}
$$

Proof. It suffices to show (5.9) because $g_{\sqrt{-1}}(z)=g_{-\sqrt{-1}}(z)=z^{2}+1$. Set $\Lambda_{i}^{1 / d}(\tau)=\left\{\gamma \in \mathbb{C} \mid \gamma^{d} \in \Lambda_{i}(\tau)\right\}$ for $i=0,1, \ldots, k-2$ and

$$
g\left(\mathcal{E}_{k}(\tau) ; z\right)=\prod_{\gamma \in S_{k}(\tau)} g_{\gamma}(z) \prod_{\gamma \in \Gamma_{k-1}(\tau)} g_{\gamma}(z) \cdots \prod_{\gamma \in \Gamma_{1}(\tau)} g_{\gamma}(z) .
$$

In the same way as (4.13), noting that $S_{k}(\tau)=\Lambda_{k-1}^{1 / d}(\tau)$ by $S_{k}(\tau)=M_{k}(\tau) \cup$ $\overline{M_{k}(\tau)}, M_{k}(\tau)=N_{k-1}^{1 / d}(\tau)$, and $\Lambda_{k-1}(\tau)=N_{k-1}(\tau) \cup \overline{N_{k-1}(\tau)}$, we see that

$$
\begin{aligned}
& g\left(\mathcal{E}_{k}(\tau) ; z\right) \\
&=\prod_{\gamma \in \Lambda_{k-1}(\tau)} g_{\gamma}\left(z^{d}\right) \prod_{\gamma \in \Lambda_{k-2}^{1 / d}(\tau) \backslash \Lambda_{k-1}(\tau)} g_{\gamma}(z) \cdots \prod_{\gamma \in \Lambda_{0}^{1 / d}(\tau) \backslash \Lambda_{1}(\tau)} g_{\gamma}(z) \\
&=\prod_{\gamma \in \Lambda_{k-1}(\tau)} \frac{g_{\gamma}\left(z^{d}\right)}{g_{\gamma}(z)} \prod_{\gamma \in \Lambda_{k-2}(\tau)} g_{\gamma}\left(z^{d}\right) \prod_{\gamma \in \Lambda_{k-3}^{1 / d}(\tau) \backslash \Lambda_{k-2}(\tau)} g_{\gamma}(z) \cdots \prod_{\gamma \in \Lambda_{0}^{1 / d}(\tau) \backslash \Lambda_{1}(\tau)} g_{\gamma}(z) \\
&=\prod_{\gamma \in \Lambda_{k-1}(\tau)} \frac{g_{\gamma}\left(z^{d}\right)}{g_{\gamma}(z)} \prod_{\gamma \in \Lambda_{k-2}(\tau)} \frac{g_{\gamma}\left(z^{d}\right)}{g_{\gamma}(z)} \cdots \prod_{\gamma \in \Lambda_{0}(\tau)} \frac{g_{\gamma}\left(z^{d}\right)}{g_{\gamma}(z)} \prod_{\gamma \in \Lambda_{0}(\tau)} g_{\gamma}(z) .
\end{aligned}
$$

Since $\Lambda_{0}(\tau)=\{\tau, \bar{\tau}\}$, we obtain

$$
\begin{equation*}
g^{*}(z):=g\left(\mathcal{E}_{k}(\tau) ; z\right) \prod_{\gamma \in\{\tau, \bar{\tau}\}} g_{\gamma}(z)^{-1} \in H_{d} . \tag{5.11}
\end{equation*}
$$

Note that for $\gamma \in \mathbb{C}$,

$$
\begin{equation*}
\gamma \in \mathcal{E}_{k}(\tau) \text { if and only if } g\left(\mathcal{E}_{k}(\tau) ; \gamma\right)=0 \tag{5.12}
\end{equation*}
$$

Suppose first that $\tau \notin \mathcal{E}_{k}(\tau)$. Then (5.7) and (5.11) imply (5.8) and (5.9) because $\mathcal{F}_{k}(\tau)=\mathcal{E}_{k}(\tau) \cup\{\tau, \bar{\tau}\}$. Noting that $\bar{\tau} \notin \mathcal{E}_{k}(\tau)$ by $\overline{\mathcal{E}_{k}(\tau)}=\mathcal{E}_{k}(\tau)$, we get $e(\gamma) \neq 0$ for any $\gamma \in \mathcal{F}_{k}(\tau)$ by (5.12). Next assume that $\tau \in \mathcal{E}_{k}(\tau)$. Then Lemma 5.2 implies that $g^{*}(z)$ is a polynomial with $g^{*}(\tau) \neq 0$ and $g^{*}(\bar{\tau}) \neq 0$. Thus (5.7) and (5.11) implies (5.8) and (5.9) by $\mathcal{F}_{k}(\tau)=\mathcal{E}_{k}(\tau) \backslash\{\tau, \bar{\tau}\}$. Moreover, $e(\gamma) \neq 0$ for any $\gamma \in \mathcal{F}_{k}(\tau)$ by (5.12).

Continuation of the proof of Theorem 1.3. Suppose that the case (ii) of Theorem 1.3 holds. Namely, for any $\gamma \in \mathcal{F}_{k}(\tau) \backslash\{ \pm \sqrt{-1}\}$ we have $a_{i(\gamma)}=-(\gamma+\bar{\gamma})$ for some $1 \leq i(\gamma) \leq m$. Using (5.4) and (5.10), we obtain

$$
\prod_{\gamma \in \mathcal{F}_{k}(\tau) \backslash\{ \pm \sqrt{-1}\}}\left(z^{2}+a_{i(\gamma)} z+1\right)^{e(\gamma)} \in H_{2}, \quad e(\gamma) \neq 0,
$$

which implies (5.3) with a non-empty subset $I$ of $\{1, \ldots, m\}$ and integers $e_{i}(i \in I)$. Note that for $\gamma, \eta \in \mathcal{F}_{k}(\tau) \backslash\{ \pm \sqrt{-1}\}, a_{i(\gamma)}=a_{i(\eta)}$ if and only if $\gamma \sim \eta$. Moreover, if $\gamma \sim \eta$, then $e(\gamma)=e(\eta)$ by (5.8). Hence $e_{i} \neq 0$ for any $i \in I$.

Next suppose that the case (iii) of Theorem 1.3 holds. Then, for any $\gamma \in$ $\mathcal{F}_{k_{1}}\left(\tau_{1}\right) \backslash\{ \pm \sqrt{-1}\}$ (resp. $\gamma \in \mathcal{F}_{k_{2}}\left(\tau_{2}\right) \backslash\{ \pm \sqrt{-1}\}$ ), we have $a_{i(\gamma)}=-(\gamma+\bar{\gamma})$
for some $i(\gamma)$ (resp. $a_{j(\gamma)}=-(\gamma+\bar{\gamma})$ for some $\left.j(\gamma)\right)$. Combining (2.5) and (5.10), we get
$\left(z^{2}+1\right)_{\gamma \in \mathcal{F}_{k_{1}}}^{q_{1}} \prod_{\left(\tau_{1}\right) \backslash\{ \pm \sqrt{-1}\}} c_{i(\gamma)}(z)^{e(\gamma)} \in H_{d}, \quad\left(z^{2}+1\right)^{q_{2}} \prod_{\gamma \in \mathcal{F}_{k_{2}}\left(\tau_{2}\right) \backslash\{ \pm \sqrt{-1\}}} c_{j(\gamma)}(z)^{e^{\prime}(\gamma)} \in H_{d}$,
where $q_{1}, q_{2}, e(\gamma)=e\left(\mathcal{F}_{k_{1}}\left(\tau_{1}\right) ; \gamma\right)$, and $e^{\prime}(\gamma)=e\left(\mathcal{F}_{k_{2}}\left(\tau_{2}\right) ; \gamma\right)$ are integers with $e(\gamma), e^{\prime}(\gamma) \neq 0$.

We show that (5.1) is satisfied with a non-empty subset $I$ of $\{1, \ldots, m\}$ and integers $e_{i}(i \in I)$. The case where $q_{1}=0$ or $q_{2}=0$ is clear. If $q_{1} \neq 0$ and $q_{2} \neq 0$, then (5.1) follows from

$$
\prod c_{i(\gamma)}(z)^{-q_{2} e(\gamma)} \quad \prod \quad c_{j(\gamma)}(z)^{q_{1} e^{\prime}(\gamma)} \in H_{d}
$$

By (5.8), to prove the existence of the subset $I$ such that $e_{i} \neq 0(i \in I)$, we only have to show that

$$
\begin{equation*}
\mathcal{F}_{k_{1}}\left(\tau_{1}\right) \backslash\{ \pm \sqrt{-1}\} \neq \mathcal{F}_{k_{2}}\left(\tau_{2}\right) \backslash\{ \pm \sqrt{-1}\} \tag{5.13}
\end{equation*}
$$

Suppose on the contrary

$$
\begin{equation*}
\mathcal{F}_{k_{1}}\left(\tau_{1}\right) \backslash\{ \pm \sqrt{-1}\}=\mathcal{F}_{k_{2}}\left(\tau_{2}\right) \backslash\{ \pm \sqrt{-1}\} . \tag{5.14}
\end{equation*}
$$

Thus, using (5.14) and the assumptions on $\mathcal{F}_{k_{i}}\left(\tau_{i}\right)$ for $i=1,2$, we get

$$
\begin{align*}
\mathcal{E}_{k_{i}}\left(\tau_{i}\right) & \subset \mathcal{F}_{k_{i}}\left(\tau_{i}\right) \cup\left\{\tau_{i}, \overline{\tau_{i}}\right\} \subset\left(\mathcal{F}_{k_{1}}\left(\tau_{1}\right) \cap \mathcal{F}_{k_{2}}\left(\tau_{2}\right)\right) \cup\left\{\tau_{i}, \overline{\tau_{i}}, \sqrt{-1},-\sqrt{-1}\right\} \\
(5.15) & \subset\left\{\tau_{1}, \overline{\tau_{1}}, \tau_{2}, \overline{\tau_{2}}, \sqrt{-1},-\sqrt{-1}\right\} . \tag{5.15}
\end{align*}
$$

Suppose that there exists an $i \in\{1,2\}$ such that $\tau_{i} \notin \mathbb{R}$. Then $\mathcal{E}_{k_{i}}\left(\tau_{i}\right)$ contains at least $2 d \geq 8$ elements by (1.4). This contradicts (5.15). Hence we see $\tau_{1}, \tau_{2} \in\{1,-1\}$ by $\left|\tau_{1}\right|=\left|\tau_{2}\right|=1$ and so $\tau_{h}=-1$ for some $h \in\{1,2\}$ by $\tau_{1} \neq \tau_{2}$. Therefore $\mathcal{E}_{k_{h}}(-1) \subset\{1,-1, \sqrt{-1},-\sqrt{-1}\}$ by (5.15). Since $\mathcal{E}_{k_{h}}(-1)$ contains at least $d \geq 4$ elements by (1.4), we obtain $\mathcal{E}_{k_{h}}(-1)=$ $\{1,-1, \sqrt{-1},-\sqrt{-1}\}$, which is impossible because $1 \notin \Omega_{i}(-1)$ for any $i \geq 1$. This completes the proof of the theorem.

Proof of Theorem 1.1. If the values $\Phi_{1}\left(\alpha^{-d^{N}}\right), \ldots, \Phi_{m}\left(\alpha^{-d^{N}}\right)$ in Section 2 are algebraically dependent, then we see that $b=1$ and $d$ is odd by (2.3) and Lemma 3.2. The theorem can be proved by a similar way to the proof of Theorem 1.3 only except the following: We show that the sets $\mathcal{F}_{k_{1}}\left(\tau_{1}\right)$ and $\mathcal{F}_{k_{2}}\left(\tau_{2}\right)$ satisfy (5.13). Suppose on the contrary that (5.13) does not hold. Then

$$
\begin{equation*}
\mathcal{F}_{k_{1}}\left(\tau_{1}\right) \backslash\{ \pm \sqrt{-1}\}=\mathcal{F}_{k_{2}}\left(\tau_{2}\right) \backslash\{ \pm \sqrt{-1}\} . \tag{5.16}
\end{equation*}
$$

Thus, using (5.16) and the assumptions on $\mathcal{F}_{k_{i}}\left(\tau_{i}\right)$ for $i=1,2$, we get

$$
\begin{aligned}
S_{k_{i}}\left(\tau_{i}\right) & \subset \mathcal{F}_{k_{i}}\left(\tau_{i}\right) \cup\left\{\tau_{i}, \overline{\tau_{i}}\right\} \subset\left(\mathcal{F}_{k_{1}}\left(\tau_{1}\right) \cap \mathcal{F}_{k_{2}}\left(\tau_{2}\right)\right) \cup\left\{\tau_{i}, \overline{\tau_{i}}, \sqrt{-1},-\sqrt{-1}\right\} \\
(5.17) & \subset\left\{\tau_{1}, \overline{\tau_{1}}, \tau_{2}, \overline{\tau_{2}}, \sqrt{-1},-\sqrt{-1}\right\} .
\end{aligned}
$$

Suppose that there exists an $i \in\{1,2\}$ such that $\tau_{i} \notin \mathbb{R}$. By the assumptions on $S_{k_{i}}\left(\tau_{i}\right)$ we see that $S_{k_{i}}\left(\tau_{i}\right)$ contains at least $2 d$ elements. Thus (5.17) implies that $d=3$ and

$$
S_{k_{i}}\left(\tau_{i}\right)=\left\{\tau_{1}, \overline{\tau_{1}}, \tau_{2}, \overline{\tau_{2}}, \sqrt{-1},-\sqrt{-1}\right\}
$$

Hence we get

$$
\sqrt{-1}^{3^{k_{i}}}=\tau_{i} \text { or } \sqrt{-1}^{3^{k_{i}}}=\overline{\tau_{i}} .
$$

Consequently, we obtain $\tau_{i}=\sqrt{-1}$ or $\tau_{i}=-\sqrt{-1}$, and so $6 \leq \operatorname{Card} S_{k}\left(\tau_{i}\right) \leq$ 4 by (5.17), a contradiction.

We now assume that $\tau_{1}, \tau_{2} \in \mathbb{R}$. Since $\left|\tau_{1}\right|=\left|\tau_{2}\right|=1$, (5.17) implies that, for $i=1,2$,

$$
S_{k_{i}}\left(\tau_{i}\right) \subset\{1,-1, \sqrt{-1},-\sqrt{-1}\}
$$

which contradicts the fact that $S_{k_{i}}\left(\tau_{i}\right)=\zeta_{d} S_{k_{i}}\left(\tau_{i}\right)$ since $d$ is odd. This completes the proof of (5.13) and the theorem is proved.

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