ALGEBRAIC INDEPENDENCE OF REAL NUMBERS WITH LOW DENSITY OF NONZERO DIGITS

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Abstract. Let α be an integer with $\alpha \geq 2$. In this paper we will give new criteria for algebraic independence of numbers $\xi = \sum_{n=1}^{\infty} \alpha^{-w(n)}$, where $(w(n))_{n=1}^{\infty}$ is a strictly increasing sequence of nonnegative integers. Applying our criteria, we deduce algebraic independence of such ξ also for suitable sequences $(w(n))_{n=1}^{\infty}$ with $\lim_{n\to\infty} w(n+1)/w(n) = 1$, which was impossible by early methods. For instance, let l be a positive real number and put

$$\eta_l = \sum_{n=1}^{\infty} \alpha^{-[f_l(n)]},$$

where $[x]$ is the integral part of a real number x and

$$f_l(n) = \exp\left((\log n)^{1+l}\right).$$

We prove, using Theorem 2.1, that the uncountable set $\{\eta_l \mid l \geq 1\}$ is algebraically independent. Moreover, if h and l are distinct positive real numbers, then η_h and η_l are algebraically independent.

1. Introduction

Let $\alpha \geq 2$ be an integer. A normal number in base α is a positive number whose base-α digits show a uniform distribution. Namely, all finite words with letters from the alphabet $\{0, 1, \ldots, \alpha-1\}$ occur with the proper frequency. Borel [3] showed that almost all positive numbers are normal in each integral base. However, it is generally difficult to check whether a given number is normal or not.

Borel [4] conjectured that all algebraic irrational numbers are normal in every integral base. This conjecture is still open. There exists no algebraic number that has been proven to be normal. Moreover, no counterexample is known. If Borel's conjecture is true, then nonzero digits in base-α expansions of algebraic irrational numbers appear with average frequency tending to $(\alpha - 1)/\alpha$. Consequently, for any irrational ξ, if Borel's conjecture is true and if nonzero digits of ξ in base-α occur with average frequency tending to 0, then ξ is transcendental.

We now introduce known results about transcendency and algebraic independence of positive numbers whose densities of nonzero digits are low.

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In this paper, let $\mathbb{N}$ be the set of nonnegative integers and $\mathbb{Z}_{\geq 1}$ the set of positive integers. Moreover, we denote the integral part of a real number ξ by $[\xi]$. Let us use the Vinogradov symbols $\gg$ and $\ll$, as well as the Landau symbols O and o with their regular meanings. Recall that $f \ll g$, $g \gg f$ and $f = O(g)$ are all equivalent and mean that $|f| \leq c|g|$ holds with some positive constant c. Moreover, $f = o(g)$ (resp. $f \sim g$) implies that the ratio f/g tends to zero (resp. 1). All implied constants may depend on the given data.

We consider the number

$$\xi = \sum_{n=1}^{\infty} \alpha^{-w(n)}, \tag{1.1}$$

where $\alpha \geq 2$ is an integer and $(w(n))_{n=1}^{\infty}$ is a strictly increasing sequence of nonnegative integers. Liouville [9, 10] first showed the existence of transcendental numbers in 1844. He obtained the transcendency of the number $\sum_{n=1}^{\infty} \alpha^{-n!}$ by proving what is nowadays called Liouville's inequality. Schmidt [16] generalized this inequality and showed the numbers $\gamma_1, \gamma_2, \ldots$ defined by

$$\gamma_l = \sum_{n=1}^{\infty} \alpha^{-(ln)!} \ (l = 1, 2, \ldots)$$

are algebraically independent. Durand [7] verified for each real algebraic number z with $0 < z < 1$ that the uncountable set

$$\left\{ \zeta_h = \sum_{n=0}^{\infty} z^{[hn!]} \mid h > 0 \right\}$$

is algebraically independent. Shiokawa [17] established algebraic independence of the values of gap series at algebraic points containing those appeared in [7] and [16]. However, we can not apply Liouville's method in the case of

$$\limsup_{n\to\infty} \frac{w(n+1)}{w(n)} < \infty.$$

Let $k \geq 2$ be an integer. Mahler [11] verified that the number $\sum_{n=0}^{\infty} \alpha^{-k^n}$ is transcendental. More generally, he proved for each algebraic number z with $0 < |z| < 1$ that $\Phi_k(z) = \sum_{n=0}^{\infty} z^{k^n}$ is transcendental by using the functional equation

$$\Phi_k(z^k) = \Phi_k(z) - z. \tag{1.2}$$

Using the Schmidt Subspace Theorem, Corvaja and Zannier [5] generalized Mahler's results above as follows: Assume that $(w(n))_{n=1}^{\infty}$ is lacunary, that is, satisfies

$$\liminf_{n\to\infty} \frac{w(n+1)}{w(n)} > 1. \tag{1.3}$$

Then, for every algebraic z with $0 < |z| < 1$, the number $\sum_{n=1}^{\infty} z^{w(n)}$ is transcendental. Mahler's method is also applicable to algebraic independence theory. Using (1.2), Nishioka [12] showed for each algebraic number z with $0 < |z| < 1$ that the values $\Phi_2(z), \Phi_3(z), \ldots$ are algebraically independent. For detailed information concerning Mahler's method for transcendence and algebraic independence, see [13].

Now we return to the base-α expansions of algebraic numbers. For positive numbers ξ and R, let $\lambda(\alpha, \xi, R)$ be the number of nonzero digits among the first $(1+[R])$ digits of the base-α expansion of ξ. Namely,

$$\lambda(\alpha, \xi, R) = \mathrm{Card}\{n \in \mathbb{N} \mid n \le [R], [\xi\alpha^n] - \alpha[\xi\alpha^{n-1}] \neq 0\},$$

where Card denotes the cardinality. Assume that $\alpha = 2$. Bailey, Borwein, Crandall, and Pomerance [1] showed for any algebraic irrational ξ that there exists a positive computable constant $C(\xi)$ depending only on ξ satisfying

$$\lambda(2, \xi, N) \ge C(\xi) N^{1/(\deg \xi)} \tag{1.4}$$

for all sufficiently large N. With a suitable positive $C(\alpha, \xi)$ in place of $C(\xi)$ we can prove (1.4) for any integral base $\alpha \ge 2$ in the same way.

Theorem 1.1. *Let α be an integer greater than 1 and $\xi > 0$ an algebraic irrational number. Then there exist effectively computable positive constants $C(\alpha, \xi)$ and $C'(\alpha, \xi)$ depending only on α and ξ such that, for any integer N with $N \ge C'(\alpha, \xi)$,*

$$\lambda(\alpha, \xi, N) \ge C(\alpha, \xi) N^{1/(\deg \xi)}. \tag{1.5}$$

The idea of the proof of Theorem 1.1 was inspired by the paper of Knight [8]. Let $A_D X^D + A_{D-1} X^{D-1} + \cdots + A_0 \in \mathbb{Z}[X]$ be the minimal polynomial of ξ, where $A_D > 0$. In the rest of this section, $C_1(\alpha, \xi)$ and $C_2(\alpha, \xi)$ denote effectively computable positive constants depending only on α and ξ. We have

$$A_D \xi^D + A_{D-1} \xi^{D-1} + \cdots + A_0 = 0. \tag{1.6}$$

We explain the notion of nonzero islands introduced by Knight for another proof of the transcendency of $\xi_0 = \sum_{n=0}^{\infty} \alpha^{-2^n}$. Let $D', A'_0, A'_1, \ldots, A'_{D'}$ be integers with $D' \geq 1$ and $A'_{D'} \geq 1$. We show that

$$\omega := \sum_{k=0}^{D'} A'_k \xi_0^k \neq 0.$$

For any k with $1 \leq k \leq D'$ we have

$$\xi_0^k = \sum_{m=0}^{\infty} \tau(m,k)\alpha^{-m},$$

where $\tau(m,k)$ denotes the number of ways that m can be written as a sum of k powers of 2. Let b be a sufficiently large integer. Put $N = (2^{D'}-1)2^b$. Let m be an integer with

$$N - 2^{b-1} + 1 \leq m \leq N + 2^b - 1.$$

Then Lemma 1 in [8] implies that

$$\tau(m,k) = \begin{cases} D'! & (\text{ if } m = N \text{ and } k = D'), \\ 0 & (\text{otherwise}). \end{cases}$$

Hence, considering the carries of the base-α expansion of $D'!A'_{D'}\alpha^{-N}$, we deduce the following: there exists an integer m with $N \leq m \leq N+O(1)$ such that the m-th digit of the base-α expansion of ω is not zero. In particular, $\omega \neq 0$. Knight used the term nonzero islands to refer nonzero digits which occur from the carries of the base-α expansion of $D'!A'_{D'}\alpha^{-N}$.

In [1], the Thue-Siegel-Roth theorem [15] was used in order to find nonzero islands. However, the Thue-Siegel-Roth theorem is ineffective. So, in this paper we use Liouville's inequality instead of the Thue-Siegel-Roth theorem. As a consequence, we obtain the effective lower bounds $C'(\alpha,\xi)$ in Theorem 1.1.

We give a sketch of the proof of Theorem 1.1 without technical details. For simplicity, assume that $1 \leq \xi < 2$ and write the α-ary expansion of ξ by

$$\xi = \sum_{m=0}^{\infty} t(\xi,m)\alpha^{-m}.$$

Note that $t(\xi,0) = 1$. For any k with $1 \leq k \leq D$,

$$(1.7)\ \xi^k = \sum_{m=0}^{\infty} \alpha^{-m} \sum_{\substack{i_1,\ldots,i_k \geq 0 \\ i_1+\cdots+i_k=m}} t(\xi,i_1)\cdots t(\xi,i_k) =: \sum_{m=0}^{\infty} \alpha^{-m}\rho(k,m).$$

Let $k \geq 2$. Then, putting $i_k = 0$, we get

$$(1.8)\ \rho(k,m) \geq \sum_{\substack{i_1,\ldots,i_{k-1} \geq 0 \\ i_1+\cdots+i_{k-1}=m}} t(\xi,i_1)\cdots t(\xi,i_{k-1})t(\xi,0) = \rho(k-1,m).$$

Let N be a positive integer. In the same way as the proof of Theorem 7.1 in [1], we can show that there exists an interval $I=[U_1,U_2)\subset[0,N)$ satisfying the following four conditions:

(1) $\rho(D-1,U_1)>0$.
(2) If $U_2<N$, then $\rho(D-1,U_2)>0$.
(3)

$$\rho(D-1,m)=0 \tag{1.9}$$

for any m with $U_1<m<U_2$.

(4)

$$|I|\geq C_1(\alpha,\xi)N^{1/D}, \tag{1.10}$$

where $|I|=U_2-U_1$ is the length of I.

Using (1.8) and (1.9), we get

$$\rho(k,m)=0, \tag{1.11}$$

where k and m are integers with $1\leq k\leq D-1$ and $U_1<m<U_2$. Liouville's inequality implies the following: By (1.10), if $N\geq C_2(\alpha,\xi)$, then there exists an m_0 satisfying $t(\xi,m_0)>0$ and

$$\frac{1}{D+2}|I|\leq m_0\leq\frac{D+1}{D+2}|I|.$$

In fact, suppose that $t(\xi,m)=0$ for any m with

$$\frac{1}{D+2}|I|\leq m\leq\frac{D+1}{D+2}|I|.$$

Put

$$\begin{aligned} m_1 &:= \max\left\{m\in\mathbb{N}\,\middle|\,m<\frac{1}{D+2}|I|,t(\xi,m)\neq0\right\},\\ m_2 &:= \min\{m\in\mathbb{N}\mid m_2>m_1,t(\xi,m)\neq0\}.\end{aligned}$$

Then we have

$$m_2\geq(D+1)m_1.$$

Let

$$p:=\sum_{m=0}^{m_1}t(\xi,m)\alpha^{m_1-m},\ q:=\alpha^{m_1}.$$

Then p and q are integers. Thus,

$$\begin{aligned}\xi-\frac{p}{q} &= \sum_{m=m_2}^{\infty}t(\xi,m)\alpha^{-m}\\ &\leq \alpha^{1-m_2}\leq\alpha^{1-(D+1)m_1}=\alpha q^{D+1},\end{aligned}$$

which contradicts Liouville's inequality in the case of $N \geq C_2(\alpha, \xi)$ because we have (1.10).

Hence, putting $U := U_1 + m_0$, we obtain

$$U_1 + \frac{1}{D+2}|I| \leq U \leq U_1 + \frac{D+1}{D+2}|I| \tag{1.12}$$

and

$$\rho(D, U) \geq \rho(D-1, U_1)t(\xi, m_0) > 0. \tag{1.13}$$

In what follows, we observe the α-ary expansion of the left-hand side of (1.6), using (1.7). Note that (1.7) is not generally the α-ary expansion of ξ^k because $\alpha^{-m}\rho(k,m)$ causes carry, $O(\log \rho(k,m))$ to the higher digits. Recall that $A_D > 0$. Combining (1.10), (1.11), (1.12), and (1.13), we conclude that positive digits left in the α-ary expansion of (1.6), which is a contradiction. To explain the details of remaining positive digits, Bailey, Borwein, Crandall, and Pomerance [1] introduced BBP tails. Note that the concept of BBP tails is defined in the paper [2] in order to give rapid algorithms for the computation of the n-th digits of certain transcendental numbers.

In the case of $\alpha = 2$, Rivoal [14] improved the constant $C(\xi)$ for certain classes of algebraic irrational ξ. For example, let ε be an arbitrary positive number and $\xi' = 0.558\ldots$ the unique positive zero of the polynomial $8X^3 - 2X^2 + 4X - 3$. Theorem 7.1 in [1] implies for any sufficiently large N that

$$\lambda(2, \xi', N) \geq (1-\varepsilon)16^{-1/3}N^{1/3}.$$

On the other hand, using Corollary 2 in [14], we obtain

$$\lambda(2, \xi', N) \geq (1-\varepsilon)N^{1/3}$$

for all sufficiently large N.

Let us consider applications of Theorem 1.1. For each real number k with $k > 1$, put

$$\nu_k = \sum_{n=0}^{\infty} \alpha^{-[n^k]}.$$

Let d be a natural number with $2 \leq d < k$. (1.5) implies that ν_k is not an algebraic number of degree at most d. Moreover, using Theorem 1.1, we deduce criteria for transcendence.

Corollary 1.2. *Let α be an integer greater than 1 and ξ a positive irrational number. Assume for an arbitrary positive number ε that*

$$\lambda(\alpha, \xi, N) = o(N^{\varepsilon}). \tag{1.14}$$

Then ξ is transcendental.

For instance, the numbers

$$\sum_{n=1}^{\infty} \alpha^{-n!}, \; \sum_{n=0}^{\infty} \alpha^{-2^n}$$

are transcendental by Corollary 1.2 because these numbers fulfill (1.14). Moreover, for positive numbers l and x with $x \geq 1$, let

$$f_l(x) = \exp\left((\log x)^{1+l}\right).$$

Then the number

$$\eta_l = \sum_{n=1}^{\infty} \alpha^{-[f_l(n)]}$$

is transcendental by Corollary 1.2 because η_l satisfies (1.14). In fact, it is easily seen that, for any $\varepsilon > 0$,

$$\begin{aligned}\lambda(\alpha, \eta_l, R) &\sim \exp\left((\log R)^{1/(1+l)}\right)\\ &= o\Big(\exp(\varepsilon \log R)\Big) = o\left(R^{\varepsilon}\right)\end{aligned}$$

as R tends to infinity. Note that η_l does not satisfy inequality (1.3). Thus, we cannot prove, using the result of Corvaja and Zannier, that the number η_l is transcendental. In fact, for a real number x with $x > 1$, we have

$$\log\left(\frac{f_l(x+1)}{f_l(x)}\right) = (\log(x+1))^{1+l} - (\log x)^{1+l}.$$

By the mean value theorem, there exists $\sigma = \sigma(l, x) \in (0, 1)$ such that

$$\log\left(\frac{f_l(x+1)}{f_l(x)}\right) = (1+l)\frac{(\log(x+\sigma))^l}{x+\sigma}.$$

Since the right-hand side of the equality above converges to zero as x tends to infinity, we obtain

$$\lim_{x\to\infty} \frac{f_l(x+1)}{f_l(x)} = 1.$$

The main purpose of this paper is to deduce algebraic independence of certain classes of numbers ξ which satisfy (1.14). We will introduce criteria for algebraic independence in Theorem 2.1. We prove this theorem in Section 4. Our method is quite flexible because we do not use functional equation. As a consequence of Theorem 2.1 we deduce algebraic independence of the values η_l for real numbers l with $l \geq 1$.

Theorem 1.3. *The uncountable set* $\{\eta_l \mid l \geq 1\}$ *is algebraically independent.*

We can not prove, using Theorem 2.1, that the uncountable set $\{\eta_l \mid l > 0\}$ is algebraically independent. However, we deduce that any two elements of this set are algebraically independent.

Theorem 1.4. *Let h and l be distinct positive real numbers. Then η_h and η_l are algebraically independent.*

We verify Theorems 1.3 and 1.4 in Section 3.

2. Criteria for algebraic independence

Let $\alpha \geq 2$ be an integer and ξ a positive number. For each integer m, put

$$t(\xi, m) = [\xi\alpha^m] - \alpha[\xi\alpha^{m-1}] \in \{0, 1, \ldots, \alpha - 1\}.$$

Note that $t(\xi, -m) = 0$ for all sufficiently large $m \in \mathbb{N}$. Then ξ is written as

$$\xi = \sum_{m=-\infty}^{\infty} t(\xi, m)\alpha^{-m},$$

which is the α-ary expansion of ξ. Set

$$S(\xi) = \{m \in \mathbb{N} \mid t(\xi, m) \neq 0\}.$$

Recall for $R > 0$ that

$$\lambda(\alpha, \xi, R) = \mathrm{Card}\{n \in S(\xi) \mid n \leq R\}.$$

Note that if $1 \leq \xi < \alpha$, then $0 \in S(\xi)$. For each positive integer a, let

$$aS(\xi) = \{n_1 + \cdots + n_a \mid n_1, \ldots, n_a \in S(\xi)\}.$$

For convenience, let $0S(\xi) = \{0\}$. Moreover, for any positive numbers $\xi_1, \ldots, \xi_r$ and nonnegative integers $a_1, \ldots, a_r$, let

$$\sum_{i=1}^{r} a_i S(\xi_i) = \{s_1 + \cdots + s_r \mid s_i \in a_i S(\xi_i) \text{ for } 1 \leq i \leq r\}.$$

For a nonempty subset $\mathcal{A}$ of $\mathbb{N}$, let us define the function $\theta(R; \mathcal{A})$ by

$$\theta(R; \mathcal{A}) = \max\{n \in \mathcal{A} \mid n < R\},$$

where R is a real number with

$$R > \min\{n \in \mathcal{A}\}.$$

Assume that $1 \leq \xi_1, \ldots, \xi_r < \alpha$. Let $(a_1, \ldots, a_r), (a'_1, \ldots, a'_r) \in \mathbb{N}^r$, where $a_i \geq a'_i$ for every i with $1 \leq i \leq r$. Then we have

$$\sum_{i=1}^{r} a_i S(\xi_i) \supset \sum_{i=1}^{r} a'_i S(\xi_i)$$

because $S(\xi_1), \ldots, S(\xi_r) \ni 0$. We now state criteria for algebraic independence.

Theorem 2.1. *Let $\xi_1, \ldots, \xi_r$ be positive irrational numbers. Suppose that these numbers satisfy the following three assumptions:*

(1) For an arbitrary positive ε, we have

$$\lambda(\alpha, \xi_1, R) = o\left(R^{\varepsilon}\right)$$

and, for $h = 2, \dots, r$,

$$\lambda(\alpha, \xi_h, R) = o\big(\lambda(\alpha, \xi_{h-1}, R)^{\varepsilon}\big).$$

as R tends to infinity.

(2) There exists a positive constant C_1 such that

$$S(\xi_r) \cap [C_1 R, R] \neq \emptyset$$

for every sufficiently large real number R.

(3) Let $a_1, \dots, a_{r-1}, a_r$ be any nonnegative integers. If $r \geq 2$, then there exist a positive integer $\kappa = \kappa(a_1, \dots, a_{r-1})$ and a positive constant $C_2(a_1, \dots, a_r)$, where κ depends only on $a_1, \dots, a_{r-1}$ and $C_2(a_1, \dots, a_r)$ only on $a_1, \dots, a_r$, such that

$$R - \theta\left(R; \sum_{i=1}^{r-2} a_i S(\xi_i) + \kappa S(\xi_{r-1})\right) < R \prod_{i=1}^{r} \lambda(\alpha, \xi_i, R)^{-a_i}$$

for each real number R with $R \geq C_2(a_1, \dots, a_r)$.

Then $\xi_1, \dots, \xi_r$ are algebraically independent.

Remark 2.2. In the case of $r = 1$, Theorem 2.1 follows from Corollary 1.2.

We verify Theorem 2.1 in Section 4. In the rest of this section we give a sketch of the proof of Theorem 2.1 without technical details in the case where $r = 2$ and $\kappa(a_1) = 1 + a_1$ for all $a_1 \geq 0$, where $\kappa(a_1)$ is defined in the third assumption of Theorem 2.1. For simplicity, suppose that $1 \leq \xi_1, \xi_2 < 2$. If ξ_1 and ξ_2 are algebraically dependent, then there exists a nonzero polynomial $P(X_1, X_2) \in \mathbb{Z}[X]$ such that

$$P(\xi_1, \xi_2) = 0. \tag{2.1}$$

Let

$$P(X_1, X_2) := \sum_{\mathbf{k}=(a_1, a_2) \in \Lambda} A_{\mathbf{k}} X_1^{a_1} X_2^{a_2},$$

where Λ is a nonempty finite subset of $\mathbb{N}^2$ and $A_{\mathbf{k}}$ a nonzero integer for each $\mathbf{k} \in \Lambda$. We search nonzero islands of the α-ary expansion of the left-hand side of (2.1). For any $\mathbf{k} = (a, b) \in \Lambda$, we get

$$\begin{aligned}
\xi_1^a \xi_2^b &= \left(\sum_{x=0}^{\infty} t(\xi_1, x) \alpha^{-x} \right)^a \left(\sum_{y=0}^{\infty} t(\xi_2, y) \alpha^{-y} \right)^b \\
&= \sum_{m=0}^{\infty} \sum_{\substack{i_1, \ldots, i_a, j_1, \ldots, j_b \geq 0 \\ i_1 + \cdots + i_a + j_1 + \cdots + j_b = m}} t(\xi_1, i_1) \cdots t(\xi_1, i_a) t(\xi_2, j_1) \cdots t(\xi_2, j_b) \\
&=: \sum_{m=0}^{\infty} \alpha^{-m} \rho(\mathbf{k}, m). \qquad (2.2)
\end{aligned}$$

Observe that $\rho(\mathbf{k}, m) > 0$ if and only if $m \in aS(\xi_1) + bS(\xi_2)$. In the proof of Theorem 1.1, we used the relation

$$0 \in S(\xi) \subset 2S(\xi) \subset \cdots$$

in order to find nonzero islands. Indeed, (1.8) implies that $(k-1)S(\xi) \subset kS(\xi)$ (see also (1.13)). On the other hand, let $(a_1, a_2), (a_1', a_2') \in \Lambda$. Then, in general, neither $a_1 S(\xi_1) + a_2 S(\xi_2) \subset a_1' S(\xi_1) + a_2' S(\xi_2)$ nor $a_1' S(\xi_1) + a_2' S(\xi_2) \subset a_1 S(\xi_1) + a_2 S(\xi_2)$ holds. This is the most different point between the proofs of Theorems 1.1 and 2.1. Let $\succ$ be the lexicographical order in $\mathbb{N}^2$. Namely, $(a_1, a_2) \succ (a_1', a_2')$, if $a_1 > a_1'$, or if $a_1 = a_1'$ and $a_2 > a_2'$. Let $\mathbf{g} = (g_1, g_2) \in \Lambda$ be the greatest element of Λ with respect to $\succ$. Without loss of generality, we may assume that $A_{\mathbf{g}} > 0$. For any $(a_1, a_2) \in \Lambda$, if $a_1 = g_1$, then $a_2 \leq g_2$. Thus, we have $a_1 S(\xi_1) + a_2 S(\xi_2) \subset g_1 S(\xi_1) + g_2 S(\xi_2)$. If $a_1 < g_1$, then the relation above does not hold generally. However, by the third assumption of Theorem 2.1, the set $a_1 S(\xi_1) + a_2 S(\xi_2)$ is approximated by $(1 + a_1) S(\xi_1)$ because $\kappa(a_1) = a_1 + 1$. Moreover, we have $(1 + a_1) S(\xi_1) \subset g_1 S(\xi_1)$.

Based on the observation above, we give nonzero islands in Section 4.4. Let N be a sufficiently large integer. We construct an interval $J = [T_1, T_2) \subset [0, N)$ satisfying the following three conditions:

(1) $T_1 \in p_1 S(\xi_1) + p_2 S(\xi_2)$ for some $(p_1, p_2) \in \Lambda$ with $p_1 < g_1$.
(2) If $T_2 < N$, then $T_2 \in q_1 S(\xi_1) + q_2 S(\xi_2)$ for some $(q_1, q_2) \in \Lambda$ with $q_1 < g_1$.
(3) Let m be any integer with $T_1 < m < T_2$ and let $(a_1, a_2) \in \Lambda$ with $a_1 < g_1$. Then $m \notin a_1 S(\xi_1) + a_2 S(\xi_2)$.

Since $p_1 S(\xi_1) + p_2 S(\xi_2)$ and $q_1 S(\xi_1) + q_2 S(\xi_2)$ are approximated by $g_1 S(\xi_1)$, we get a subinterval $I = [R_1, R_2)$ of J satisfying the following three conditions:

(1) $R_1 \in g_1 S(\xi_1) + (g_2 - 1) S(\xi_2)$.
(2) $R_2 \in g_1 S(\xi_1) + (g_2 - 1) S(\xi_2)$.

(3) Let m be any integer with $R_1 < m < R_2$ and let $\mathbf{k} = (a_1, a_2) \in \Lambda$ with $\mathbf{g} \succ \mathbf{k}$. Then

$$m \notin a_1 S(\xi_1) + a_2 S(\xi_2). \tag{2.3}$$

Denote the length of I by $|I| = R_2 - R_1$. Using the second assumption of Theorem 2.1, we deduce that there is an $m_0 \in \mathbb{N}$ satisfying $m_0 \in S(\xi_2)$ and

$$\frac{C_1}{1+C_1}|I| \le m_0 \le \frac{1}{1+C_1}|I|.$$

Putting $U := R_1 + m_0$, we obtain $U \in g_1 S(\xi_1) + g_2 S(\xi_2)$ and

$$R_1 + \frac{C_1}{1+C_1}|I| \le U \le R_1 + \frac{1}{1+C_1}|I|. \tag{2.4}$$

In particular,

$$\rho(\mathbf{g}, U) > 0. \tag{2.5}$$

Now we observe the α-ary expansion of the left-hand side of (2.1), using (2.2). Recall that $A_{\mathbf{g}} > 0$ and that $\alpha^{-m}\rho(\mathbf{k}, m)$ causes carry, $O(\log(\mathbf{k}, m))$ to the higher digits. Hence, combining (2.3), (2.4), and (2.5), we conclude that positive digits left in the α-ary expansion of (2.1), which is a contradiction. To explain the details of remaining positive digits, we introduce BBP tails Y_R in the last of Section 4.2.

3. Proof of main results

Proof of Theorem 1.3. Let $\{\eta_{l_1}, \eta_{l_2}, \dots, \eta_{l_r}\}$ be any finite subset of $\{\eta_l \mid l \ge 1\}$. Without loss of generality, we may assume that $l_1 < l_2 < \cdots < l_r$. Let

$$\xi_i = \eta_{l_i} - [\eta_{l_i}] + 1 \in (1, 2)$$

for $i = 1, \dots, r$. Then $S(\xi_i) \ni 0$ for $i = 1, \dots, r$. We check that $\xi_1, \dots, \xi_r$ satisfy the assumptions of Theorem 2.1. Let l be a positive number. In Section 1 we proved that, for any sufficiently large x,

$$f_l(x) \le f_l(x+1) \le 2 f_l(x). \tag{3.1}$$

Therefore, we verified the second assumption with $C_1 = 1/2$. For any positive numbers l and x with $x \ge 1$, put

$$g_l(x) = \exp\left((\log x)^{1/(1+l)}\right). \tag{3.2}$$

Note that g_l is the inverse function of f_l. Namely, for any $x \ge 1$, we have $f_l(g_l(x)) = x$. Let i and j be integers with $1 \le i, j \le r$. As we mentioned in Section 1, for any $\varepsilon > 0$,

$$\lambda(\alpha, \xi_i, R) \sim g_{l_i}(R) = o\left(R^{\varepsilon}\right) \tag{3.3}$$

as R tends to infinity. If $i < j$, then, for any $\varepsilon > 0$,

$$g_{l_j}(R) = o\left(\exp\left(\varepsilon(\log R)^{1/(1+l_i)}\right)\right) = o\left(g_{l_i}(R)^{\varepsilon}\right). \tag{3.4}$$

Thus the first assumption is fulfilled by (3.3) and (3.4). Finally we check the third assumption. We introduce the results of Daniel [6]. Let $(\mu_n)_{n=1}^{\infty}$ be the strictly increasing sequence of those positive integers that can be represented as the sum of three cubes of positive integers. Then Daniel showed that

$$\mu_{n+1} - \mu_n = O\left(\mu_n^{8/27}\right).$$

In the same way as the proof of the result above, we get the following:

Lemma 3.1. *Let* $\mathbf{k} = (a_1, \ldots, a_r) \in \mathbb{N}^r \backslash \{(0, \ldots, 0)\}$. *Then, for* $R \geq 2$,

$$R - \theta\left(R; \sum_{i=1}^{r} a_i S(\xi_i)\right) \ll R(\log R)^{a_1+\cdots+a_r} \mathbf{g}(R)^{-\mathbf{k}}, \tag{3.5}$$

where

$$\mathbf{g}(R)^{-\mathbf{k}} = \prod_{i=1}^{r} g_{l_i}(R)^{-a_i}.$$

Proof. We prove Lemma 3.1 by induction on the value $a_1 + \cdots + a_r$. Assume that $a_1 + \cdots + a_r = 1$. Then there exists an integer h with $1 \leq h \leq r$ and $a_h = 1$. We have

$$f'_{l_h}(x) = \frac{(1+l_h) f_{l_h}(x) (\log f_{l_h}(x))^{l_h/(1+l_h)}}{g_{l_h}(f_{l_h}(x))}.$$

Let x be a sufficiently large real number. Then, by the mean value theorem, there exists $\rho = \rho(x) \in (0,1)$ such that

$$\begin{aligned} f_{l_h}(x+1) - f_{l_h}(x) &= \frac{(1+l_h) f_{l_h}(x+\rho)(\log f_{l_h}(x+\rho))^{l_h/(1+l_h)}}{g_{l_h}(f_{l_h}(x+\rho))} \\ &\leq \frac{(1+l_h) f_{l_h}(x+1)(\log f_{l_h}(x+1))^{l_h/(1+l_h)}}{g_{l_h}(f_{l_h}(x))} \\ &\ll \frac{f_{l_h}(x) \log f_{l_h}(x)}{g_{l_h}(f_{l_h}(x))}, \end{aligned}$$

where for the last inequality we use (3.1). For $R > 1$, let

$$F(R) = \frac{R \log R}{g_{l_h}(R)}.$$

Taking the logarithm of $F(R)$, we deduce that $F(R)$ is monotone increasing for sufficiently large R. If R is sufficiently large, then there exists $m \in \mathbb{N}$ such that

$$[f_{l_h}(m)] < R \leq [f_{l_h}(1+m)].$$

Thus, we get $\theta(R; S(\xi_h)) = [f_{l_h}(m)]$. Since

$$F(f_{l_h}(m)) \ll F([f_{l_h}(m)]) \leq F(R),$$

we obtain

$$\begin{aligned} 0 < R - \theta(R; S(\xi_h)) &\leq f_{l_h}(m+1) - f_{l_h}(m) + 1 \\ &\ll F(f_{l_h}(m)) \ll F(R), \end{aligned}$$

which implies (3.5) in the case of $a_1 + \cdots + a_r = 1$. Next, assume that $a_1 + \cdots + a_r \geq 2$. Let

$$d = \max\{i \geq 1 | a_i \geq 1\}.$$

Put

$$\mathbf{k}' = (a'_1, \ldots, a'_r) := (a_1, \ldots, a_{d-1}, -1 + a_d, 0, \ldots, 0).$$

Then we deduce, using the case of $a_1 + \cdots + a_r = 1$, that there exists a positive constant C satisfying

$$R' := R - \theta(R; S(\xi_d)) \leq C \frac{R \log R}{g_{l_d}(R)}.$$

Note that

$$R - \theta\left(R; \sum_{i=1}^{r} a_i S(\xi_i)\right) \leq R'$$

because

$$\sum_{i=1}^{r} a_i S(\xi_i) \supset S(\xi_d).$$

Thus, we may assume that $R' \geq 2$. By the induction hypothesis, we get

$$R' - \theta\left(R'; \sum_{i=1}^{r} a'_i S(\xi_i)\right) \ll R' (\log R')^{a'_1 + \cdots + a'_r} \mathbf{g}(R')^{-\mathbf{k}'} =: G(R').$$

Let

$$\gamma = \theta(R; S(\xi_d)) + \theta\left(R'; \sum_{i=1}^{r} a'_i S(\xi_i)\right).$$

Then since

$$\gamma \in \sum_{i=1}^{r} a_i S(\xi_i),$$

we get

$$\begin{aligned} 0 &< R - \theta\left(R; \sum_{i=1}^{r} a_i S(\xi_i)\right) \leq R - \gamma \\ &= R' - \theta\left(R'; \sum_{i=1}^{r} a'_i S(\xi_i)\right) \ll G(R'). \end{aligned} \tag{3.6}$$

Taking the logarithm of $G(R)$, we deduce that the function $G(R)$ is monotone increasing for sufficiently large R. Thus, we obtain

$$\begin{aligned}
G(R') &\ll G\left(C\frac{R\log R}{g_{l_d}(R)}\right) \ll G\left(\frac{R\log R}{g_{l_d}(R)}\right)\\
&= \frac{R\log R}{g_{l_d}(R)}\left(\log\frac{R\log R}{g_{l_d}(R)}\right)^{a_1'+\cdots+a_r'}\mathbf{g}\left(\frac{R\log R}{g_{l_d}(R)}\right)^{-\mathbf{k}'}\\
&\ll \frac{R}{g_{l_d}(R)}(\log R)^{a_1+\cdots+a_r}\mathbf{g}\left(\frac{R}{g_{l_d}(R)}\right)^{-\mathbf{k}'}. \qquad (3.7)
\end{aligned}$$

Let $i \in \mathbb{N}$ with $1 \leq i \leq d$. Since $l_i \geq 1$, we observe that, for any sufficiently large R,

$$\begin{aligned}
\left(\log\left(\frac{R}{g_{l_i}(R)}\right)\right)^{1/(1+l_i)} &= \left(\log R - (\log R)^{1/(1+l_i)}\right)^{1/(1+l_i)}\\
&= (\log R)^{1/(1+l_i)}\left(1-(\log R)^{-l_i/(1+l_i)}\right)^{1/(1+l_i)}\\
&\geq (\log R)^{1/(1+l_i)}\left(1-\frac{2}{1+l_i}(\log R)^{-l_i/(1+l_i)}\right)\\
&\geq (\log R)^{1/(1+l_i)} - 1
\end{aligned}$$

and hence

$$g_{l_i}\left(\frac{R}{g_{l_d}(R)}\right) \gg g_{l_i}\left(\frac{R}{g_{l_i}(R)}\right) \gg g_{l_i}(R).$$

Therefore, we obtain

$$\mathbf{g}\left(\frac{R}{g_{l_d}(R)}\right)^{-\mathbf{k}'} \ll \mathbf{g}(R)^{-\mathbf{k}'}. \qquad (3.8)$$

Combining the inequalities (3.6), (3.7) and (3.8), we conclude that

$$0 < R - \theta\left(R; \sum_{i=1}^{r} a_i S(\xi_i)\right) \ll R(\log R)^{a_1+\cdots+a_r}\mathbf{g}(R)^{-\mathbf{k}},$$

which implies (3.5). □

Let $\mathbf{k} = (a_1, \ldots, a_r) \in \mathbb{N}^r$. Then, by (3.3), (3.4) and Lemma 3.1,

$$\begin{aligned}
&R - \theta\left(R; \sum_{i=1}^{r-2} a_i S(\xi_i) + (1+a_{r-1})S(\xi_{r-1})\right)\\
&\leq R g_{l_{r-1}}(R)^{-1/2}\prod_{i=1}^{r-1} g_{l_i}(R)^{-a_i} = o\left(R\prod_{i=1}^{r}\lambda(\alpha,\xi_i,R)^{-a_i}\right)
\end{aligned}$$

as R tends to infinity. Hence, the third assumption of Theorem 2.1 is satisfied with

$$\kappa = \kappa(a_1, \ldots, a_{r-1}) = 1 + a_{r-1}.$$

Therefore we proved Theorem 1.3. □

Proof of Theorem 1.4. Without loss of generality, we may assume that $h < l$. Let $\xi_1 = \eta_h - [\eta_h] + 1$ and $\xi_2 = \eta_l - [\eta_l] + 1$. Note that $1 < \xi_1, \xi_2 < 2$ and that $S(\xi_1), S(\xi_2) \ni 0$. In the same way as in the proof of Theorem 1.3, we can verify that ξ_1 and ξ_2 satisfy the first and second assumptions of Theorem 2.1 with $C_1 = 1/2$. In what follows, we prove that the third assumption is satisfied. Let $g_l(x)$ be defined by (3.2).

Lemma 3.2. *Let b be a positive integer. Then, for any positive number ε, we have*

$$R - \theta(R; bS(\xi_1)) \ll R g_h(R)^{-b+\varepsilon} \tag{3.9}$$

for $R \geq 2$.

Proof. We show (3.9) by induction on b. Assume that $b = 1$. In the same way as in the proof of Lemma 3.1, we deduce that there exists a positive constant C satisfying

$$R' := R - \theta(R; S(\xi_1)) \leq C \frac{R \log R}{g_h(R)}, \tag{3.10}$$

which implies (3.9) because, for any positive ε,

$$\log R = o(g_h(R)^{\varepsilon})$$

as R tends to infinity. Suppose that $b \geq 2$. Without loss of generality, we may assume that $R' \geq 2$ and that $\varepsilon < 1$. In particular, we have

$$-b + 1 + \varepsilon < 0.$$

By the induction hypothesis,

$$R' - \theta(R'; (b-1)S(\xi_1)) \ll R' g_h(R')^{-b+1+\varepsilon/3} =: H(R').$$

We obtain, taking the logarithm of $H(R)$, that the function $H(R)$ is monotone increasing for sufficiently large R. Hence,

$$\begin{aligned}
0 &< R - \theta(R; bS(\xi_1)) \\
&\leq R - \theta(R; S(\xi_1)) - \theta(R'; (b-1)S(\xi_1)) \\
&= R' - \theta(R'; (b-1)S(\xi_1)) \ll H(R') \\
&\ll H\left(C \frac{R \log R}{g_h(R)}\right) \ll H\left(\frac{R \log R}{g_h(R)}\right) \\
&= \frac{R \log R}{g_h(R)} g_h\left(\frac{R \log R}{g_h(R)}\right)^{-b+1+\varepsilon/3} \\
&\ll R g_h(R)^{-1+\varepsilon/3} g_h\left(\frac{R}{g_h(R)}\right)^{-b+1+\varepsilon/3}. \qquad (3.11)
\end{aligned}$$

Let

$$\varepsilon' := \frac{\varepsilon}{3b-3-\varepsilon} \in (0,1).$$

Then

$$(1-\varepsilon')\left(-b+1+\frac{\varepsilon}{3}\right) = -b+1+\frac{2}{3}\varepsilon.$$

For all sufficiently large R, we obtain

$$\begin{aligned}\left(\log\left(\frac{R}{g_h(R)}\right)\right)^{1/(1+h)} &= \left(\log R - (\log R)^{1/(1+h)}\right)^{1/(1+h)} \\ &\geq (1-\varepsilon')(\log R)^{1/(1+h)}\end{aligned}$$

and hence

$$\begin{aligned} g_h\left(\frac{R}{g_h(R)}\right)^{-b+1+\varepsilon/3} &\leq g_h(R)^{(1-\varepsilon')(-b+1+\varepsilon/3)} \\ &= g_h(R)^{-b+1+2\varepsilon/3}.\end{aligned}$$

Combining (3.11) and the inequality above, we proved (3.9). $\square$

Let $(a_1,a_2)\in\mathbb{N}^2$. Then, applying Lemma 3.2 with $b=a_1+1$ and $\varepsilon=1/2$, we get

$$\begin{aligned} R-\theta(R;(a_1+1)S(\xi_1)) &\ll Rg_h(R)^{-a_1-1/2} \\ &= o\left(R\prod_{i=1}^{2}\lambda(\alpha,\xi_i,R)^{-a_i}\right)\end{aligned}$$

as R tends to infinity. Therefore we checked the third assumption of Theorem 2.1 with $\kappa=\kappa(a_1)=a_1+1$ and hence verified Theorem 1.4. $\square$

4. Proof of Theorem 2.1

4.1. **Base-α expansions of powers of real numbers.** We prove Theorem 2.1 by induction on r. Using Corollary 1.2, we deduce the case of $r=1$. In what follows, suppose that $r\geq 2$. Without loss of generality we may assume that $1\leq \xi_1,\ldots,\xi_r<2$. In fact, $\xi_1,\ldots,\xi_r$ are algebraically independent if and only if $\xi_1',\ldots,\xi_r'$ are algebraically independent, where

$$\xi_i' = \xi_i - [\xi_i] + 1 \text{ for } i=1,\ldots,r.$$

For simplicity, let

$$\lambda_i(R) = \lambda(\alpha,\xi_i,R) \text{ for } i=1,\ldots,r \text{ and } R>0.$$

For convenience, put $\mathbb{N}^0 = \{0\}$. Let $\xi > 0, b \in \mathbb{N}, \mathbf{x} = (x_1, \dots, x_b) \in \mathbb{N}^b$, and $\mathbf{s} = (s_1, \dots, s_b) \in \mathbb{Z}^b$. Put

$$\begin{aligned} |\mathbf{x}| &= \begin{cases} 0 & (b = 0), \\ x_1 + \cdots + x_b & (b \geq 1), \end{cases} \\ t(\xi, \mathbf{x}) &= \begin{cases} 1 & (b = 0), \\ t(\xi, x_1) \cdots t(\xi, x_b) & (b \geq 1), \end{cases} \\ \mathbf{x}^{\mathbf{s}} &= \begin{cases} 1 & (b = 0), \\ x_1^{s_1} \cdots x_b^{s_b} & (b \geq 1). \end{cases} \end{aligned}$$

In what follows, we denote $(\xi_1, \dots, \xi_r)$ and $(\lambda_1(R), \dots, \lambda_r(R))$ by $\underline{\xi}$ and $\underline{\lambda}(R)$, respectively. Then, for each $\mathbf{k} = (a_1, \dots, a_r) \in \mathbb{N}^r \backslash \{(0, \dots, 0)\}$, we have

$$\begin{aligned} \underline{\xi}^{\mathbf{k}} &= \prod_{i=1}^{r} \left(\sum_{x=0}^{\infty} t(\xi_i, x) \alpha^{-x} \right)^{a_i} \\ &= \prod_{i=1}^{r} \left(\sum_{\mathbf{x} \in \mathbb{N}^{a_i}} t(\xi_i, \mathbf{x}) \alpha^{-|\mathbf{x}|} \right) \\ &= \sum_{\mathbf{x}_1 \in \mathbb{N}^{a_1}, \dots, \mathbf{x}_r \in \mathbb{N}^{a_r}} t(\xi_1, \mathbf{x}_1) \cdots t(\xi_r, \mathbf{x}_r) \alpha^{-|\mathbf{x}_1| - \cdots - |\mathbf{x}_r|} \\ &= \sum_{m=0}^{\infty} \rho(\mathbf{k}, m) \alpha^{-m}, \end{aligned} \tag{4.1}$$

where

$$\rho(\mathbf{k}, m) := \sum_{\substack{\mathbf{x}_1 \in \mathbb{N}^{a_1}, \dots, \mathbf{x}_r \in \mathbb{N}^{a_r} \\ |\mathbf{x}_1| + \cdots + |\mathbf{x}_r| = m}} t(\xi_1, \mathbf{x}_1) \cdots t(\xi_r, \mathbf{x}_r) \in \mathbb{N}.$$

Note that, for each $m \in \mathbb{N}$, $\rho(\mathbf{k}, m) > 0$ if and only if $m \in \sum_{i=1}^{r} a_i S(\xi_i)$. It is easily seen that

$$\begin{aligned} \rho(\mathbf{k}, m) &\leq \sum_{\substack{\mathbf{x}_1 \in \mathbb{N}^{a_1}, \dots, \mathbf{x}_r \in \mathbb{N}^{a_r} \\ |\mathbf{x}_1| + \cdots + |\mathbf{x}_r| = m}} (\alpha - 1)^{|\mathbf{k}|} \\ &= (\alpha - 1)^{|\mathbf{k}|} \binom{m + |\mathbf{k}| - 1}{|\mathbf{k}| - 1}. \end{aligned} \tag{4.2}$$

We now check the following:

Lemma 4.1. *Let* $\mathbf{k} = (a_1, \dots, a_r) \in \mathbb{N}^r \backslash \{(0, \dots, 0)\}$ *and* $N \in \mathbb{N}$.

(1)

$$\sum_{m=0}^{N} \rho(\mathbf{k}, m) \leq (\alpha - 1)^{|\mathbf{k}|} \underline{\lambda}(N)^{\mathbf{k}}.$$

(2)

$$\mathit{Card}\{m \in \mathbb{N} \mid m \leq N, \rho(\mathbf{k}, m) > 0\} \leq (\alpha - 1)^{|\mathbf{k}|} \underline{\lambda}(N)^{\mathbf{k}}.$$

Proof. Put, for $i = 1, \ldots, r$,

$$S_i = \{m \in S(\xi_i) \mid m \leq N\},\ S_i^0 = \{0\}.$$

Then

$$\begin{aligned}\sum_{m=0}^{N} \rho(\mathbf{k}, m) &= \sum_{\substack{\mathbf{x}_1 \in \mathbb{N}^{a_1}, \ldots, \mathbf{x}_r \in \mathbb{N}^{a_r} \\ |\mathbf{x}_1| + \cdots + |\mathbf{x}_r| \leq N}} t(\xi_1, \mathbf{x}_1) \cdots t(\xi_r, \mathbf{x}_r) \\ &\leq \sum_{\mathbf{x}_1 \in S_1^{a_1}, \ldots, \mathbf{x}_r \in S_r^{a_r}} (\alpha - 1)^{|\mathbf{k}|} = (\alpha - 1)^{|\mathbf{k}|} \underline{\lambda}(N)^{\mathbf{k}},\end{aligned}$$

which implies the first statement of Lemma 4.1. The second statement follows from the first one because $\rho(\mathbf{k}, m) \in \mathbb{N}$ for each $m \in \mathbb{N}$. $\square$

4.2. **Auxiliary functions.** We define the lexicographical order $\succ$ on $\mathbb{N}^r$ as follows. For any $\mathbf{k} = (a_1, \ldots, a_r)$, $\mathbf{k}' = (a_1', \ldots, a_r')$ with $\mathbf{k} \neq \mathbf{k}'$, there exists a positive l such that the first $(l-1)$ symbols in $\mathbf{k} = (a_1, \ldots, a_r)$ and $\mathbf{k}' = (a_1', \ldots, a_r')$ coincide, but their lth symbols are different. Then $\mathbf{k} = (a_1, \ldots, a_r) \succ \mathbf{k}' = (a_1', \ldots, a_r')$ if and only if $a_l > a_l'$.

Each nonzero polynomial $Q(\underline{X}) \in \mathbb{Z}[X_1, \ldots, X_r]$ is uniquely written as

$$Q(\underline{X}) = \sum_{\mathbf{k} \in \Lambda(Q)} B_{\mathbf{k}} \underline{X}^{\mathbf{k}},$$

where $\Lambda(Q)$ is a finite subset of $\mathbb{N}^r$ determined by Q, $B_{\mathbf{k}}$ a nonzero integer and $\underline{X} = (X_1, \ldots, X_r)$. Recall for $\mathbf{k} = (a_1, \ldots, a_r)$ that $\underline{X}^{\mathbf{k}} = X_1^{a_1} \cdots X_r^{a_r}$. Let $\mathbf{g}(Q) = (g_1(Q), \ldots, g_r(Q))$ be the greatest element of $\Lambda(Q)$ with respect to $\succ$. Moreover, put

$$\begin{aligned}\Lambda_1(Q) &= \{\mathbf{k} \in \Lambda(Q) \mid a_1 = g_1(Q), \ldots, a_{r-1} = g_{r-1}(Q), a_r < g_r(Q)\}, \\ \Lambda_2(Q) &= \Lambda(Q) \backslash \big(\Lambda_1(Q) \cup \{\mathbf{g}(Q)\}\big), \\ \Lambda_3(Q) &= \{\mathbf{k} \in \Lambda(Q) \mid a_1 = g_1(Q), \ldots, a_{r-2} = g_{r-2}(Q), a_{r-1} < g_{r-1}(Q)\},\end{aligned}$$

where $\mathbf{k} = (a_1, \ldots, a_r)$. We define the number $e(Q)$ as follows. If $\Lambda_3(Q)$ is empty, then put $e(Q) = 0$. Otherwise, let

$$e(Q) = \max\{a_{r-1} \mid (a_1, \ldots, a_{r-1}, a_r) \in \Lambda_3(Q)\}.$$

Now assume that $\xi_1, \ldots, \xi_r$ are algebraically dependent. Then there exists a nonzero polynomial $P(\underline{X}) \in \mathbb{Z}[X_1, \ldots, X_r]$ such that

$$P(\underline{\xi}) = 0.$$

By the induction hypothesis $\xi_2, \ldots, \xi_r$ are algebraically independent. Thus, the degree of $P(\underline{X})$ in X_1 is positive. Namely, $g_1(P) \geq 1$. Without loss of generality, we may assume that

$$X_r(X_r - 1) | P(\underline{X}). \tag{4.3}$$

In what follows, let

$$\kappa(n) := \kappa(g_1(P), g_2(P), \ldots, g_{r-2}(P), n),$$

where n is a nonnegative integer and the right-hand side of the equality above is defined in the third assumption of Theorem 2.1. Let m and n be integers with $0 \leq m \leq n$. Then, for any positive number R, we have

$$\begin{aligned} R - \theta\left(R; \sum_{i=1}^{r-2} g_i(P)S(\xi_i) + nS(\xi_{r-1})\right) \\ \leq R - \theta\left(R; \sum_{i=1}^{r-2} g_i(P)S(\xi_i) + mS(\xi_{r-1})\right) \end{aligned}$$

because

$$\sum_{i=1}^{r-2} g_i(P)S(\xi_i) + nS(\xi_{r-1}) \supset \sum_{i=1}^{r-2} g_i(P)S(\xi_i) + mS(\xi_{r-1}).$$

So, if necessary, by increasing $\kappa(n)$, we may assume that $\kappa(n) \geq 1$ for any $n \in \mathbb{N}$ and that the sequence $(\kappa(n))_{n=0}^{\infty}$ is monotone increasing.

Lemma 4.2. *There is a nonzero polynomial $F(X_{r-1}, X_r) \in \mathbb{Z}[X_{r-1}, X_r]$ such that*

$$g_{r-1}(FP) \geq \kappa(e(FP)).$$

Proof. We define the nonzero polynomial $\sigma(X_{r-1}, X_r) \in \mathbb{Z}[X_{r-1}, X_r]$ as follows. If $r = 2$, then put

$$\sigma(X_1, X_2) := P(X_1, X_2). \tag{4.4}$$

If $r \geq 3$, then $P(\underline{X})$ is uniquely written as

$$P(\underline{X}) = \sum_{\mathbf{k}=(a_1,\ldots,a_{r-2})\in\Gamma} \varphi_{\mathbf{k}}(X_{r-1}, X_r) X_1^{a_1} \cdots X_{r-2}^{a_{r-2}}, \tag{4.5}$$

where Γ is a finite subset of $\mathbb{N}^{r-2}$ and $\varphi_{\mathbf{k}}(X_{r-1}, X_r) \in \mathbb{Z}[X_{r-1}, X_r]$ a nonzero polynomial. Note that $\mathbf{l} := (g_1(P), \ldots, g_{r-2}(P)) \in \Gamma$. Now put

$$\sigma(X_{r-1}, X_r) := \varphi_{\mathbf{l}}(X_{r-1}, X_r).$$

Let

$$\sigma(X_{r-1}, X_r) =: \sum_{i=0}^{b} \sigma_i(X_r) X_{r-1}^i,$$

where $\sigma_i(X_r) \in \mathbb{Z}[X_r]$ with $\sigma_b(X_r) \neq 0$. We show for any integer n with $n \geq b$ that there is a nonzero polynomial $\psi^{(n)}(X_{r-1}, X_r) \in \mathbb{Z}[X_{r-1}, X_r]$ satisfying the following: $\sigma(X_{r-1}, X_r)\psi^{(n)}(X_{r-1}, X_r)$ is written as

$$\text{(4.6)}\quad \sigma(X_{r-1}, X_r)\psi^{(n)}(X_{r-1}, X_r) = \psi_b^{(n)}(X_r)X_{r-1}^n + \sum_{i=0}^{b-1} \psi_i^{(n)}(X_r)X_{r-1}^i,$$

where $\psi_i^{(n)}(X_r) \in \mathbb{Z}[X_r]$ for $i = 0, 1, \ldots, b$ with $\psi_b^{(n)}(X_r) \neq 0$. In the case of $b = 0$, it is clear that $\psi^{(n)}(X_{r-1}, X_r) = X_{r-1}^n$ satisfies (4.6). Suppose that $b \geq 1$. We check (4.6) by induction on n. If $n = b$, then putting $\psi^{(b)}(X_{r-1}, X_r) = 1$, we get (4.6). Assume that $n \geq b+1$. Then the induction hypothesis implies that

$$\psi^{(n)}(X_{r-1}, X_r) = \sigma_b(X_r)X_{r-1}\psi^{(n-1)}(X_{r-1}, X_r) - \psi_{-1+b}^{(n-1)}(X_r)$$

fulfills (4.6). $\psi^{(n)}(X_{r-1}, X_r) \neq 0$ because $\sigma_b(X_r)\psi^{(n-1)}(X_{r-1}, X_r) \neq 0$. Let $w = \max\{0, b-1\}$. In what follows, we verify that

$$F(X_{r-1}, X_r) := \psi^{(\kappa(w))}(X_{r-1}, X_r)$$

satisfies the statement of Lemma 4.2. Using (4.5) and (4.6), we deduce that the first $(r-2)$ symbols of $\mathbf{g}(P)$ and $\mathbf{g}(FP)$ coincide in the case of $r \geq 3$. Moreover, we obtain

$$\text{(4.7)}\qquad g_{r-1}(FP) = \kappa(w)$$

and

$$\text{(4.8)}\qquad e(FP) \leq w.$$

In fact, if $\Lambda_3(FP)$ is not empty, then by (4.4), (4.5), and (4.6), we get $e(FP) \leq b-1$. Hence, combining (4.7) and (4.8), we conclude that

$$g_{r-1}(FP) \geq \kappa(e(FP))$$

because the sequence $(\kappa(n))_{n=0}^{\infty}$ is monotone increasing. □

For simplicity, put

$$\begin{aligned} \Lambda &= \Lambda(FP), \\ \Lambda_h &= \Lambda_h(FP) \text{ for } 1 \leq h \leq 3, \\ \mathbf{k}_0 &= (g_1, \ldots, g_r) := \mathbf{g}(FP). \end{aligned}$$

Recall that, for $i = 1, 2, \ldots, r-2$,

$$g_i = g_i(FP) = g_i(P),$$

so

$$\kappa(n) = \kappa(g_1, g_2, \ldots, g_{r-2}, n)$$

for each $n \in \mathbb{N}$. Let

$$F(X_{r-1}, X_r)P(\underline{X}) = \sum_{\mathbf{k}\in\Lambda} A_{\mathbf{k}}\underline{X}^{\mathbf{k}},$$

where $A_{\mathbf{k}}$ is a nonzero integer. Then

$$\sum_{\mathbf{k}\in\Lambda} A_{\mathbf{k}}\underline{\xi}^{\mathbf{k}} = 0. \tag{4.9}$$

Note that, for each $\mathbf{k} \in \Lambda$, we have $|\mathbf{k}| \geq 1$ because X_r divides $P(\underline{X})$. Without loss of generality, we may assume that $A_{\mathbf{k}_0} \geq 1$.

Lemma 4.3. Λ_1 *and* Λ_2 *are not empty.*

Proof. First suppose that Λ_2 is empty. Then, for each $\mathbf{k} = (a_1, \ldots, a_r)$, we have $a_1 = g_1, \ldots, a_{r-1} = g_{r-1}$. Thus, (4.9) implies that ξ_r is an algebraic number, which contradicts to the induction hypothesis.

Next, assume that Λ_1 is empty. Then we get

$$\sum_{\substack{\mathbf{k}=(a_1,\ldots,a_r)\in\Lambda \\ a_1=g_1,\ldots,a_{r-1}=g_{r-1}}} A_{\mathbf{k}}\underline{X}^{\mathbf{k}} = A_{\mathbf{k}_0}\underline{X}^{\mathbf{k}_0}. \tag{4.10}$$

Let $\Phi : \mathbb{Z}[X_1, \ldots, X_r] \to \mathbb{Z}[X_1, \ldots, X_{r-1}]$ be defined by

$$\Phi(Q(X_1, \ldots, X_r)) = Q(X_1, \ldots, X_{r-1}, 1).$$

By (4.10), the greatest element of $\Phi(F(X_{r-1}, X_r)P(\underline{X}))$ with respect to the lexicographical order on $\mathbb{N}^{r-1}$ is $(g_1, \ldots, g_{r-1})$. So, $\Phi(F(X_{r-1}, X_r)P(\underline{X}))$ is not zero. Namely, $X_r - 1$ does not divide $F(X_{r-1}, X_r)P(\underline{X})$, which contradicts (4.3). $\square$

Let

$$D = 1 + \max\{|\mathbf{k}| \big| \mathbf{k} \in \Lambda\}.$$

Denote the greatest element of Λ_1 and Λ_2 with respect to $\succ$ by $\mathbf{k}_1$ and $\mathbf{k}_2$, respectively. Let

$$\mathbf{e} = (g_1, g_2, \ldots, g_{r-2}, e(FP), D).$$

Then, for each $\mathbf{k} \in \Lambda_2$, we have $\mathbf{k} \prec \mathbf{e}$. In fact, if Λ_3 is empty, then there exists a positive l with $l \leq r-2$ such that the first $(l-1)$ symbols in $\mathbf{e}$ and $\mathbf{k}_2$ coincide, but the l th symbol of $\mathbf{e}$ is greater than that of $\mathbf{k}_2$. Otherwise, $\mathbf{k}_2$ is written as

$$\mathbf{k}_2 = (g_1, g_2, \ldots, g_{r-2}, e(FP), a)$$

with $a < D$, and so $\mathbf{k}_2 \prec \mathbf{e}$. By the first assumption of Theorem 2.1, for any $\mathbf{k} \in \Lambda_2$,

$$\underline{\lambda}(n)^{\mathbf{k}} = o\left(\underline{\lambda}(n)^{\mathbf{e}}\right). \tag{4.11}$$

Lemma 4.2 implies

$$g_{r-1} \geq \kappa(e(FP)). \tag{4.12}$$

Let Ξ be the set of nonnegative integers N such that, for every integer n with $0 \leq n \leq N$,

$$n\underline{\lambda}(n)^{-\mathbf{e}} \leq N\underline{\lambda}(N)^{-\mathbf{e}}. \tag{4.13}$$

Note that Ξ is an infinite set. In fact, by the first assumption of Theorem 2.1, we have

$$\lim_{N\to\infty} N\underline{\lambda}(N)^{-\mathbf{e}} = \infty.$$

If necessary, by increasing $C_2(\mathbf{e})$, we may assume that $\lambda_r(n) \geq 5$ for every $n \in \mathbb{N}$ with $n \geq C_2(\mathbf{e})$, where $C_2(\mathbf{e})$ is defined in the third assumption of Theorem 2.1. For simplicity, let

$$\theta(R) = \theta\left(R; \sum_{i=1}^{r-1} g_i S(\xi_i)\right).$$

Lemma 4.4. *Let M and E be any positive real numbers with*

$$M \geq C_2(\mathbf{e})$$

and

$$E \geq 4M\underline{\lambda}(M)^{-\mathbf{e}}.$$

Then

$$M + \frac{1}{2}E < \theta(M+E).$$

Proof. Using

$$\begin{aligned} \frac{E}{4} &\geq M\underline{\lambda}(M)^{-\mathbf{e}}, \\ \frac{E}{4} &> E\underline{\lambda}(M)^{-\mathbf{e}}, \end{aligned}$$

we get

$$\frac{E}{2} > (M+E)\underline{\lambda}(M)^{-\mathbf{e}} \geq (M+E)\underline{\lambda}(M+E)^{-\mathbf{e}}.$$

Note that $M + E \geq C_2(\mathbf{e})$. Thus, using (4.12) and the third assumption of Theorem 2.1 with

$$(a_1, \ldots, a_r) = \mathbf{e},\ R = M + E,$$

we deduce that

$$M + E - \theta(M+E) < (M+E)\underline{\lambda}(M+E)^{-\mathbf{e}} < \frac{E}{2},$$

which implies Lemma 4.4. □

Using (4.1) and (4.9), we get, for each $R \in \mathbb{N}$,

$$0 = \alpha^R \sum_{\mathbf{k}\in\Lambda} A_{\mathbf{k}} \underline{\xi}^{\mathbf{k}} = \sum_{\mathbf{k}\in\Lambda} A_{\mathbf{k}} \sum_{m=-R}^{\infty} \rho(\mathbf{k}, m+R)\alpha^{-m},$$

so

$$Y_R := \sum_{\mathbf{k}\in\Lambda} A_{\mathbf{k}} \sum_{m=1}^{\infty} \rho(\mathbf{k}, m+R)\alpha^{-m} \in \mathbb{Z}.$$

Let $N \in \mathbb{N}$. In what follows, we estimate the number $y(N)$ of $R \in \mathbb{N}$ satisfying $R \le N$ and $Y_R > 0$, namely,

$$y(N) = \mathrm{Card}\{R \in \mathbb{N} \mid R \le N, Y_R > 0\}.$$

4.3. **Bounds for $y(N)$.** First, we consider upper bounds for $y(N)$.

Lemma 4.5.

$$y(N) = o(N)$$

as N tends to infinity.

Proof. For $\mathbf{k} \in \Lambda$ and $R \in \mathbb{N}$, let

$$Y(\mathbf{k}, R) = \sum_{m=1}^{\infty} \rho(\mathbf{k}, m+R)\alpha^{-m} \ge 0.$$

Then by (4.2)

$$\begin{aligned} Y(\mathbf{k}, R) &\le \sum_{m=1}^{\infty} (\alpha-1)^{|\mathbf{k}|} \binom{m+R+|\mathbf{k}|-1}{|\mathbf{k}|-1} \alpha^{-m} \\ &\le (\alpha-1)^{|\mathbf{k}|} \sum_{m=1}^{\infty} \binom{m+R+|\mathbf{k}|-1}{|\mathbf{k}|-1} 2^{-m}. \end{aligned}$$

In the proof of Theorem 2.1 of [1], Bailey, Borwein, Crandall, and Pomerance showed for $R \ge 0$ and $l \ge 1$ that

$$\sum_{m=1}^{\infty} \binom{m+R+l-1}{l-1} 2^{-m} < \frac{(R+l)^l}{(l-1)!(R+1)}.$$

Since $|\mathbf{k}| \ge 1$, we get

$$Y(\mathbf{k}, R) < \frac{(\alpha-1)^{|\mathbf{k}|}(R+|\mathbf{k}|)^{|\mathbf{k}|}}{(|\mathbf{k}|-1)!(R+1)}. \tag{4.14}$$

In particular,

$$\begin{aligned} \sum_{R=0}^{N} Y(\mathbf{k}, R) &< \sum_{R=0}^{N} \frac{(\alpha-1)^{|\mathbf{k}|}(N+|\mathbf{k}|)^{|\mathbf{k}|}}{(|\mathbf{k}|-1)!} \\ &\le \frac{(\alpha-1)^{|\mathbf{k}|}(N+|\mathbf{k}|)^{|\mathbf{k}|+1}}{(|\mathbf{k}|-1)!}. \end{aligned} \tag{4.15}$$

By the first assumption of Theorem 2.1, we have

$$\underline{\lambda}(N)^{\mathbf{k}} = o(N). \tag{4.16}$$

Let $K = \lceil D \log_\alpha N \rceil$, where $\log_\alpha N = (\log N)/(\log \alpha)$ and $\lceil x \rceil$ is the smallest integer greater than or equal to a real number x. Then by (4.15), (4.16) and the first statement of Lemma 4.1, we get

$$\begin{aligned}
\sum_{R=0}^{N-K} Y(\mathbf{k}, R) &= \sum_{m=1}^{\infty} \alpha^{-m} \sum_{R=0}^{N-K} \rho(\mathbf{k}, m+R) \\
&\le \sum_{m=1}^{K} \alpha^{-m} \sum_{R=0}^{N} \rho(\mathbf{k}, R) + \alpha^{-K} \sum_{m=1+K}^{\infty} \alpha^{K-m} \sum_{R=0}^{N-K} \rho(\mathbf{k}, m+R) \\
&\le (\alpha-1)^{|\mathbf{k}|} \underline{\lambda}(N)^{\mathbf{k}} + \alpha^{-K} \sum_{R=0}^{N-K} \sum_{m=1}^{\infty} \rho(\mathbf{k}, m+R+K) \alpha^{-m} \\
&= o(N) + \alpha^{-K} \sum_{R=0}^{N-K} Y(\mathbf{k}, R+K) \\
&\le o(N) + N^{-D} (\alpha-1)^{D} (N+D)^{D} = o(N).
\end{aligned}$$

Since $Y(\mathbf{k}, R) \ge 0$,

$$\sum_{R=0}^{N-K} |Y_R| \le \sum_{\mathbf{k} \in \Lambda} |A_{\mathbf{k}}| \sum_{R=0}^{N-K} Y(\mathbf{k}, R) = o(N).$$

Using $Y_R \in \mathbb{Z}$, we obtain

$$y(N) \le K + \sum_{R=0}^{N-K} |Y_R| = o(N).$$

□

Next, we estimate lower bounds for $y(N)$.

Lemma 4.6. *Let $N \in \mathbb{N}$ be sufficiently large and $I = [U_1, U_2)$ an interval with $I \subset [0, N)$. Suppose that $\rho(\mathbf{k}, x) = 0$ for any integer $x \in (U_1, U_2)$ and $\mathbf{k} \in \Lambda \backslash \{\mathbf{k}_0\}$. Moreover, assume that there exists $U \in \mathbb{N}$ satisfying*

$$U_1 < U \le U_2 - D \log_\alpha N$$

and

$$\rho(\mathbf{k}_0, U) > 0.$$

Then $Y_n > 0$ for any $n \in [U_1, U)$.

Proof. We prove Lemma 4.6 by induction on n. First we consider the case of $n = U - 1$. Using (4.14), $A_{\mathbf{k}_0} \geq 1$, and the assumptions on I and U, we obtain

$$\begin{aligned}
Y_{U-1} &= \sum_{\mathbf{k}\in\Lambda} A_{\mathbf{k}} \sum_{m=1}^{\infty} \rho(\mathbf{k}, m+U-1)\alpha^{-m} \\
&\geq \frac{1}{\alpha} - \sum_{\mathbf{k}\in\Lambda\setminus\{\mathbf{k}_0\}} |A_{\mathbf{k}}| \sum_{m=1+U_2-U}^{\infty} \rho(\mathbf{k}, m+U-1)\alpha^{-m} \\
&= \frac{1}{\alpha} - \sum_{\mathbf{k}\in\Lambda\setminus\{\mathbf{k}_0\}} |A_{\mathbf{k}}| \alpha^{U-U_2} Y(\mathbf{k}, U_2 - 1) \\
&\geq \frac{1}{\alpha} - \sum_{\mathbf{k}\in\Lambda\setminus\{\mathbf{k}_0\}} |A_{\mathbf{k}}| N^{-D}(\alpha-1)^{D-1}(N+D-1)^{D-1} > 0
\end{aligned}$$

for all sufficiently large N.

Next, suppose that $Y_n > 0$ for some $n \in \mathbb{N}$ with $1 + U_1 \leq n \leq U - 1$. Then by $A_{\mathbf{k}_0}\rho(\mathbf{k}_0, n) \geq 0$ we get

$$\begin{aligned}
Y_{n-1} &= \frac{1}{\alpha}\sum_{\mathbf{k}\in\Lambda} A_{\mathbf{k}}\rho(\mathbf{k}, n) + \frac{1}{\alpha}\sum_{\mathbf{k}\in\Lambda} A_{\mathbf{k}} \sum_{m=2}^{\infty} \rho(\mathbf{k}, m+n-1)\alpha^{-m+1} \\
&= \frac{1}{\alpha} A_{\mathbf{k}_0}\rho(\mathbf{k}_0, n) + \frac{1}{\alpha} Y_n > 0.
\end{aligned}$$

Hence we verified Lemma 4.6. □

4.4. **Completion of the proof of Theorem 2.1.** We construct intervals $I = [U_1, U_2)$ satisfying the assumptions of Lemma 4.6. Using (4.11) and the second statement of Lemma 4.1, we deduce the following: Let $N \in \Xi$ be sufficiently large. Then the number of nonnegative integers T with $T \leq N$ such that there exists a $\mathbf{k} \in \Lambda_2$ with $\rho(\mathbf{k}, T) > 0$ is at most

$$\sum_{\mathbf{k}\in\Lambda_2} (\alpha-1)^{|\mathbf{k}|} \underline{\lambda}(N)^{\mathbf{k}} \leq \frac{1}{32} \underline{\lambda}(N)^{\mathbf{e}}.$$

Say these T's are $0 = T_1 < T_2 < \cdots < T_\tau$, where

$$\tau \leq \frac{1}{32} \underline{\lambda}(N)^{\mathbf{e}}. \tag{4.17}$$

Set $T_{1+\tau} = N$ and

$$\mathcal{J} = \{J = J(j) = [T_j, T_{1+j}) \mid 1 \leq j \leq \tau\}.$$

For any interval $I \subset \mathbb{R}$, let $|I|$ denote its length. Then we have

$$\sum_{J\in\mathcal{J}} |J| = N. \tag{4.18}$$

Moreover, put

$$\begin{aligned}\mathcal{J}_1 &= \left\{J \in \mathcal{J} \,\middle|\, |J| \geq 16N\underline{\lambda}(N)^{-\mathbf{e}}\right\},\\ \mathcal{J}_2 &= \{J \in \mathcal{J}_1 \mid J \subset [C_2(\mathbf{e}), N)\}.\end{aligned}$$

Lemma 4.7. *Let $N \in \Xi$ be sufficiently large.*
(1)

$$\sum_{J\in\mathcal{J}_1} |J| \geq \frac{N}{2}.$$

(2)

$$\sum_{J\in\mathcal{J}_2} |J| \geq \frac{N}{3}.$$

Proof. By (4.17) and (4.18)

$$\begin{aligned}\sum_{J\in\mathcal{J}_1} |J| &= \sum_{J\in\mathcal{J}} |J| - \sum_{J\in\mathcal{J}\backslash\mathcal{J}_1} |J|\\ &\geq N - \tau\cdot 16N\underline{\lambda}(N)^{-\mathbf{e}} \geq \frac{N}{2},\end{aligned}$$

which implies the first statement of Lemma 4.7. We now check the second statement. Take positive integers $N_0 < N_1$ satisfying $N_i > C_2(\mathbf{e})$ and $\rho(\mathbf{k}_2, N_i) > 0$ for $i = 0, 1$. If $N > N_1$, then there exists $j_0 = j_0(N)$ with

$$T_{j_0} = N_0,\ T_{1+j_0} \leq N_1$$

by the definition of $T_1, T_2, \ldots, T_{1+\tau}$. Let $J(j) \in \mathcal{J}_1\backslash\mathcal{J}_2$. Then $j \leq j_0$. Hence, for any $N \in \Xi$ with $N \geq 6N_1$,

$$\begin{aligned}\sum_{J\in\mathcal{J}_2} |J| &\geq \sum_{J\in\mathcal{J}_1} |J| - \sum_{j=1}^{j_0} |J(j)|\\ &\geq \frac{1}{2}N - N_1 \geq \frac{1}{3}N.\end{aligned}$$

□

By Lemma 4.1 the number of nonnegative integers R with $R \leq N$ such that there exists a $\mathbf{k} \in \Lambda_1$ with $\rho(\mathbf{k}, R) > 0$ is at most

$$\sum_{\mathbf{k}\in\Lambda_1} (\alpha-1)^{|\mathbf{k}|}\underline{\lambda}(N)^{\mathbf{k}} \leq C_3\underline{\lambda}(N)^{\mathbf{k}_1},$$

where C_3 is a positive constant. Say these R's are $0 = R_1 < R_2 < \cdots < R_\mu$, where

$$\mu \leq C_3\underline{\lambda}(N)^{\mathbf{k}_1}. \tag{4.19}$$

Let $R_{1+\mu} = N$ and

$$\mathcal{I} = \{I = [R_i, R_{1+i}) \mid 1 \leq i \leq \mu\}.$$

Then

$$\sum_{I \in \mathcal{I}} |I| = N.$$

Put

$$\begin{aligned}
\mathcal{I}_1 &= \{I \in \mathcal{I} \mid I \subset J \text{ for some } J \in \mathcal{J}\}, \\
\mathcal{I}_2 &= \left\{ I \in \mathcal{I}_1 \,\middle|\, |I| \geq \frac{1}{12C_3} N \underline{\lambda}(N)^{-\mathbf{k}_1} \right\}.
\end{aligned}$$

Lemma 4.8. *Let $N \in \Xi$ be sufficiently large.*
(1)

$$\sum_{I \in \mathcal{I}_1} |I| \geq \frac{N}{6}.$$

(2)

$$\sum_{I \in \mathcal{I}_2} |I| \geq \frac{N}{12}.$$

Proof. We check the first statement. For any $J = [T_j, T_{1+j}) \in \mathcal{J}_2$, we have $C_2(\mathbf{e}) \leq T_j < T_{1+j} \leq N$. If $N \in \Xi$ is sufficiently large, then by (4.13)

$$\frac{|J|}{4} \geq 4N \underline{\lambda}(N)^{-\mathbf{e}} \geq T_{1+j} \underline{\lambda}(T_{1+j})^{-\mathbf{e}}.$$

So, using (4.12) and the third assumption of Theorem 2.1 with

$$(a_1, \ldots, a_r) = \mathbf{e},\ R = T_{1+j},$$

we obtain

$$\begin{aligned}
T_{1+j} > \theta(T_{1+j}) &> T_{1+j} - T_{1+j} \underline{\lambda}(T_{1+j})^{-\mathbf{e}} \\
&\geq T_{1+j} - \frac{|J|}{4}.
\end{aligned}$$

$\mathbf{k}_1$ is written as $\mathbf{k}_1 = (g_1, \ldots, g_{r-1}, u)$. Since

$$\theta(T_{1+j}) \in \sum_{h=1}^{r-1} g_h S(\xi_h) \subset \sum_{h=1}^{r-1} g_h S(\xi_h) + u S(\xi_r),$$

we get

$$\rho(\mathbf{k}_1, \theta(T_{1+j})) > 0.$$

Thus, by the definition of $R_1, R_2, \ldots, R_{1+\mu}$,

$$\theta(T_{1+j}) = R_i \tag{4.20}$$

for some $i \in \mathbb{N}$. Consequently, we put

$$\begin{aligned}\beta(J) &= \min\{n \in \mathbb{N} \mid n > T_j, n = R_i \text{ for some } i \in \mathbb{N}\},\\ \gamma(J) &= \max\{n \in \mathbb{N} \mid n < T_{1+j}, n = R_i \text{ for some } i \in \mathbb{N}\}.\end{aligned}$$

Then it is clear that

$$\sum_{I \in \mathcal{I}, I \subset J} |I| = \gamma(J) - \beta(J) \tag{4.21}$$

and that

$$\gamma(J) \geq \theta(T_{1+j}) > T_{1+j} - \frac{|J|}{4}. \tag{4.22}$$

Similarly, we have

$$\frac{|J|}{4} \geq 4N\underline{\lambda}(N)^{-\mathbf{e}} \geq 4T_j\underline{\lambda}(T_j)^{-\mathbf{e}}.$$

Applying Lemma 4.4 with

$$M = T_j,\ E = \frac{|J|}{4},$$

we get

$$T_j + \frac{|J|}{8} < \theta\left(T_j + \frac{|J|}{4}\right) < T_j + \frac{|J|}{4}.$$

In the same way as in the proof of (4.20), we deduce that

$$\theta\left(T_j + \frac{|J|}{4}\right) = R_i$$

for some $i \in \mathbb{N}$. Hence

$$\beta(J) \leq \theta\left(T_j + \frac{|J|}{4}\right) < T_j + \frac{|J|}{4}. \tag{4.23}$$

Therefore, combining (4.21), (4.22) and (4.23), we obtain

$$\sum_{I \in \mathcal{I}, I \subset J} |I| \geq \frac{1}{2}|J|.$$

Consequently, using Lemma 4.7, we conclude that

$$\sum_{I \in \mathcal{I}_1} |I| \geq \sum_{J \in \mathcal{J}_2} \sum_{I \in \mathcal{I}, I \subset J} |I| \geq \frac{1}{2} \sum_{J \in \mathcal{J}_2} |J| \geq \frac{1}{6}N,$$

which implies the first statement.

Using (4.19) and the first statement of Lemma 4.8, we get

$$\begin{aligned}\sum_{I \in \mathcal{I}_2} |I| &= \sum_{I \in \mathcal{I}_1} |I| - \sum_{I \in \mathcal{I}_1 \setminus \mathcal{I}_2} |I|\\ &\geq \frac{1}{6}N - \mu\frac{1}{12C_3}N\underline{\lambda}(N)^{-\mathbf{k}_1} \geq \frac{1}{12}N.\end{aligned}$$

Thus we verified the second statement. □

In what follows, we show that each interval $I \in \mathcal{I}_2$ satisfies the assumptions of Lemma 4.6. The first assumption of Theorem 2.1 implies that, for any $\mathbf{k} \in \Lambda$,

$$\log_\alpha N = o\left(N\underline{\lambda}(N)^{-\mathbf{k}}\right). \tag{4.24}$$

By the second assumption of Theorem 2.1, there exists a positive constant C_4 such that, for any real number R with $R \geq C_4$,

$$S(\xi_r) \cap [C_1 R, R] \neq \emptyset.$$

Moreover, by (4.24) there is a positive constant C_5 such that, for each natural number N with $N \geq C_5$,

$$\frac{1}{12C_3} N\underline{\lambda}(N)^{-\mathbf{k}_1} - D\log_\alpha N \geq C_4. \tag{4.25}$$

Let $N \in \Xi$ and $I = [R_i, R_{1+i}) \in \mathcal{I}_2$. Suppose that N is sufficiently large. Then we have

$$|I| \geq \frac{1}{12C_3} N\underline{\lambda}(N)^{-\mathbf{k}_1}. \tag{4.26}$$

If $N \geq C_5$, then by (4.25) and (4.26), there exists $V \in S(\xi_r)$ with

$$C_1(|I| - D\log_\alpha N) \leq V \leq |I| - D\log_\alpha N. \tag{4.27}$$

Using (4.24) and (4.26), we get

$$C_1(|I| - D\log_\alpha N) \geq 1 + \left[\frac{1}{2}C_1|I|\right] \tag{4.28}$$

because N is sufficiently large. Let

$$U = R_i + V.$$

Then there exists $\mathbf{k} = (g_1, \ldots, g_{r-1}, b) \in \Lambda_1$ $(b < g_r)$ such that

$$U \in \sum_{i=1}^{r-1} g_i S(\xi_i) + (1+b)S(\xi_r) \subset \sum_{i=1}^{r} g_i S(\xi_i),$$

so

$$\rho(\mathbf{k}_0, U) > 0.$$

Moreover, by (4.27) and (4.28)

$$R_i + 1 + \left[\frac{1}{2}C_1|I|\right] \leq U \leq R_{i+1} - D\log_\alpha N.$$

By the definition of $\mathcal{I}_2$, there exists a positive integer j such that

$$I = [R_i, R_{i+1}) \subset [T_j, T_{j+1}).$$

Hence, for any integer x with $x \in (R_i, R_{i+1})$ and $\mathbf{k} \in \Lambda \backslash \{\mathbf{k}_0\}$, we have

$$\rho(\mathbf{k}, x) = 0$$

because $\Lambda\backslash\{\mathbf{k}_0\} = \Lambda_1 \cup \Lambda_2$. Thus, by Lemma 4.6, $Y_n > 0$ for any $n \in \mathbb{N}$ with

$$R_i \leq n \leq R_i + \left[\frac{1}{2}C_1|I|\right].$$

Hence, using Lemma 4.8, we conclude that

$$y(N) \geq \sum_{I \in \mathcal{I}_2} \left(1 + \left[\frac{1}{2}C_1|I|\right]\right) \geq \frac{1}{24}C_1 N,$$

which contradicts the statement of Lemma 4.5. Therefore we proved Theorem 2.1.

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