Pre-Talbot seminar, lecture 5

Chris Dodd, D-modules and Riemann-Hilbert correspondence

We have the following setup:

- X is a smooth complex algebraic variety.
- $\mathcal{O}_X$ is the sheaf of regular functions on X.
- Θ_X is the sheaf of vector fields. It is a locally free $\mathcal{O}_X$ module whose rank is the dimension of X.
- D_X is the sheaf (of algebras) of differential operators. It is the subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathcal{O}_X)$ generated by $\mathcal{O}_X$ and Θ_X.

A left D-module on X is a sheaf of (left) modules for D_X.

Examples

(1) If $X = \mathbb{A}^1$, then $D_X(X) = \mathbb{C}[x, \frac{d}{dx}]/([\frac{d}{dx}, x] = 1)$.

(2) What is the sigmificance of a D_X-module here? Take a differential equation, e.g., $(x\frac{d}{dx} - \alpha)f = 0$ for $\alpha \in \mathbb{C}$. Let $M = D/D(x\frac{d}{dx} - \alpha)$, and let F be a space of functions on $\mathbb{C}$ that admits an action of D. Then $\mathrm{Hom}_D(M, F) = \{u \in F | (x\frac{d}{dx} - \alpha)u = 0\}$ We get x^α away from zero.

(3) Let X be any complex manifold, and M a sheaf with an integrable connection on X. This can be viewed as a map $\nabla : M \to \Omega^1_X \otimes M$ or a Lie algebra homomorphism $\nabla : \Theta_X \to \mathrm{End}_{\mathbb{C}} M$. We can define an action of Θ_X on M by $s \cdot m = \nabla(s)(m)$. The action of D_X comes from the integrability. In fact, any D_X-module which is $\mathcal{O}_X$-coherent has a connection.

(4) We can also consider right D_X-modules. Left and right D_X-modules form equivalent categories, but they are convenient for different things. The key example is the canonical sheaf $\omega_X = \bigwedge^{\dim X} \Omega^1_X$. This is a right D-module via $\omega \cdot \theta = -(\mathrm{lie}\ \theta)(\omega)$, for $\theta \in \Theta_X$. In fact $M \mapsto \omega_X \otimes_{\mathcal{O}_X} M$ realizes this equivalence of categories.

Operations

Pullback: Let $f : X \to Y$ be a morphism of smooth complex varieties. Given a differential equation on Y, can we find a differential equation on X such that solutions are pullbacks of solutions on Y?

Let M be a left D_Y-module. The $\mathcal{O}_X$-module pullback is

$$f^*M = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}M.$$

Let $\theta \in \Theta_X$. $df_*\theta$ gives us a map $\Theta_X \to \mathrm{End}_{\mathbb{C}}(f^{-1}M)$, denoted by $\tilde{\theta}$. We write $\theta(\psi \otimes s) = \theta(\psi) \otimes s + \psi \cdot \tilde{\theta}(s)$. This makes f^*M into a left D_X-module.

As a special case, consider $f^*D_Y = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}D_Y$. It is a $(D_X, f^{-1}D_Y)$-bimodule, called the transfer bimodule, and we will denote it by $D_{X\to Y}$. We can now define pullback using just D-modules:

$$f^*M = D_{X\to Y} \otimes_{f^{-1}D_Y} f^{-1}M$$

For reasons that will become clear soon, we really want to work in the derived category, and the correct functor is $Lf^*M^\cdot = D_{X\to Y} \otimes^L_{f^{-1}D_Y} f^{-1}M^\cdot$. Note that f^{-1} is an exact functor.

Here's an example. Let $\iota : X \hookrightarrow Y$ be a closed immersion. We can take local coordinates on Y and X such that we get $\{y_i, \frac{\partial}{\partial y}\}_{i=1}^n$ on Y and locally $X = \{y_{r+1} = \cdots = y_n = 0\}$. We need smoothness here, since proving it requires some statement about regular local rings. In this case, $D_{X\to Y} = D_x \otimes \mathbb{C}[\frac{\partial}{\partial y_{r+1}}, \ldots, \frac{\partial}{\partial y_n}]$. Open immersions give restriction.

Pushforward: We are given a map $f : X \to Y$. In general, we cannot push functions forward canonically. However, we can push distributions, essentially by integrating over fibers. Let M be a (complex of) right D-modules. The push-forward is

$$Rf_*(M^\cdot \otimes^L_{D_X} D_{X\to Y}),$$

written $f_+M^\cdot$ or $\int_f M$. Note that this involves a combination of left and right exact functors. This is *not* the derived functor of an ordinary functor.

Here are some examples: First, let $\iota : U \hookrightarrow X$ be an open embedding, so $D_{U\to X} = \iota^{-1}D_X = D_U$ and $\iota_+M = R\iota_*M$. Note that if the embedding isn't affine, then we get derived stuff.

If $\iota : X \to Y$ is a closed embedding, then

$$\iota_+M = \mathbb{C}[\frac{\partial}{\partial y_{r+1}}, \ldots, \frac{\partial}{\partial y_n}] \otimes_{\mathbb{C}} \iota_*M.$$

Theorem (Kashiwara) Let $\iota : X \to Y$ be a closed embedding. Then $\iota_+ : Mod_{coh}(D_X) \to Mod^X_{coh}(D_Y)$ is an equivalence of categories, where $Mod^X_{coh}(D_Y)$ is the category of coherent (i.e., finitely generated) D_Y-modules that are set-theoretically supported on X.

Holonomic modules

For any coherent D-module M, we can consider the singular support $SS(M) \subset T^*X$. D_X has a filtration by order, and taking the associated graded kills the Lie bracket on Θ_X, so $grD_X \cong \mathcal{O}_{T^*X}$. There exists a "good" filtration on M, compatible with the filtration on D_X, and it makes grM into a coherent $\mathcal{O}_{T^*X}$-module. Its support is called $SS(M)$, and is independent of our choice of good filtration.

$SS(M)$ is a conical subset of T^*X, meaning it is preserved by fiber dilations. Bernstein's inequality asserts that if M is nonzero, then the dimension of each component of $SS(M)$ is at least the dimension of X. A D-module is *holonomic* if $M = 0$ or $\dim SS(M) = \dim X$. We say that holonomic modules come from maximally overdetermined systems of linear partial differential equations.

Examples:

(1) If (M, ∇) is a vector bundle with a flat connection, then
$$SS(M) = X \subset T^*X.$$

(2) Choose a point $x \in X$, and let $\iota : * \hookrightarrow X$ be the inclusion at x. Then $i_+(\mathbb{C}) = \delta_x = D_x/\mathfrak{m}_x$. This is called the δ-module (after Dirac's δ-distribution, which generates it), and $SS(\delta_x) = T^*_x(X)$.

T^*X has a natural symplectic structure, and Gabber showed that if M is holonomic, then $SS(M)$ is Lagrangian. If $M \in Mod_{hol}(D_X)$, then M is generically $\mathcal{O}$-coherent. We will use this fact to connect D-modules to constructible sheaves.

Given M, the DeRham functor produces the complex
$$DR_X(M)[-dimX] = [\Omega^0_{X^{an}} \otimes_{\mathcal{O}_{X^{an}}} M \to \Omega^1_{X^{an}} \otimes_{\mathcal{O}_{X^{an}}} M \to \ldots].$$
The differential is $d^p : \Omega^p_{X^{an}} \otimes_{\mathcal{O}_{X^{an}}} M \to \Omega^{p+1}_{X^{an}} \otimes_{\mathcal{O}_{X^{an}}} M$ given by $d(\omega \otimes s) = d\omega \otimes s + \sum_i dx_i \wedge (\omega \otimes \frac{\partial}{\partial x_i} s)$, where $\{x_i\}$ are local coordinates.

For example, if (M, ∇) is a module with integrable connection, then $H^0(DR_X(M)[-dimX] = ker(\nabla)$ is the sheaf of horizonal sections. This is a local system with rank equal to the rank of M. $H^i = 0$ for $i > 0$ by the analytic Poincaré lemma.

If we take any holonomic D-module, it is generically a vector bundle with connection. We can restrict to the complement and do the same. This suggests that we should get a constructible complex.

Theorem (Kashiwara)

If $M^{\cdot} \in D^b_{hol}(D_X)$, then $DR_X(M^{\cdot}) \in D^b_{const}(\mathbb{C}_{X^{an}})$.

There is a problem, which is that we can have two holonomic modules which yield the same constructible complex. The solution is to restrict to complexes with regular singularities. For example, $D/D(x^2\frac{d}{dx} + 1)$ is not regular, since the solutions are generated by $e^{1/x}$ which has an essential singularity at zero. We get a category $D^b_{rh}(D_X) \subset D^b_{hol}(D_X)$.

Theorem (Riemann-Hilbert correspondence, proved by Menkhout and Kashiwara)

(1) $DR_X : D^b_{rh}(D_X) \to D^b_{const}(\mathbb{C}_{X^{an}})$ is an equivalence of triangulated categories.

(2) $DR_X : Mod_{rh}(D_X) \to Perv_{X^{an}}$ is an equivalence of abelian categories.