Pre-Talbot seminar, lecture 4

David Jordan - Quantum groups
Notes by Scott Carnahan

I hope you're familiar with tensor categories. A tensor category is braided if it is equipped with natural isomorphisms $c_{V,W} : V \otimes W \to W \otimes V$ for all objects V, W, and they satisfy $c_{12}c_{23}c_{12} = c_{23}c_{12}c_{23}$ for every triple. This is the structure you need to have a representation of the braid group. If $c_{V,W} \circ c_{W,V} = id$, then we just have a symmetric tensor category.

Universal R-matrix

Let H be a Hopf algebra. We want H-mod to form a braided tensor category. For motivation, suppose we already have a braided tensor structure on H-mod, and consider the regular representation. Then $c_{H,H} : H \otimes H \to H \otimes H$ takes $1 \otimes 1$ to some element $c_{H,H}(1 \otimes 1)$, and we define $R = \tau_{12} c_{H,H}(1 \otimes 1) \in H \otimes H$, where τ_{12} is the plain switching map.

Proposition We can express the braiding using R, via $c_{V,W} = \tau_{V,W} \circ \mu_R$, where μ_R is just multiplication by the R-matrix.

The upshot is that H-mod admits a braided tensor category structure if and only if there exists $R \in H \otimes H$ satisfying some conditions. The conditions can be reconstructed by looking at the axioms for braiding.

Bi-cross products of Hopf algebras

This concerns Drinfeld's construction of U_q using the quantum double.

A pair (X, A) of Hopf algebras is called matched if we have

(1) a left action $\triangleright : A \otimes X \to X$
(2) a right action $\triangleleft : A \otimes X \to A$
(3) a condition: $\triangleright$ and $\triangleleft$ make X and A into module-coalgebras over each other.
(4) some more technical conditions involving coproducts and composition (enough to put a Hopf algebra structure on $X \otimes A$).

Proposition We can construct a Hopf algebra structure $X \bowtie A$ on $X \otimes A$ by

- specifying that the inclusions $i_X : X \to X \bowtie A$ given by $x \mapsto x \otimes 1$, and $i_A : A \to X \bowtie A$ given by $a \mapsto 1 \otimes a$ are Hopf algebra homomorphisms
- specifying cross relations: $a \cdot x = (a_1 \triangleright x_1)(a_2 \triangleleft x_2)$ - note that the first factor is in X, and the second in A.

Here, the subscripts denote Sweedler notation with tacit summation: $\Delta(a) = a_1 \otimes a_2$. The cross relations allow us to switch the order of factors.

Example: Group factorizations. Let $G = H \cdot K$ (so any $g \in G$ can be uniquely written as hk for some $h \in H$, $k \in K$), where H and K are not necessarily normal subgroups of G (standard examples: Sylow subgroups of $A_4 \times S_3$, and $A_4 \cdot C_5 = A_5$). Suppose someone gives you kh, and you want to put it into the other order. Let $g' = kh$, and use the uniqueness of decomposition to get $g' = h'k'$. We can define the action of H and K on each other by $k \triangleright h := h_1$ and $k \triangleleft h := k_1$. Then $\mathbb{C}[G] \cong \mathbb{C}[H] \bowtie \mathbb{C}[K]$.

Quantum Double

Let H be a Hopf algebra. Then H has an adjoint action on itself, by $(\mathrm{ad}\, a) \triangleright h := a_1 h S(a_2)$ and $h \triangleleft (\mathrm{ad}\, a) := S(a_1) h a_2$. In group land, we have $\mathrm{ad} g \triangleright h = ghg^{-1}$.

Proposition This is a module-algebra action: $(\mathrm{ad}\, a) \triangleright xy = (\mathrm{ad}\, a_1) \triangleright x \cdot (\mathrm{ad}\, a_2) \triangleright y$. (For example, with groups we have $gxyg^{-1} = gxg^{-1}gyg^{-1}$.

This implies that H acts on H^* as a module coalgebra (if H is infinite dimensional, we just ask for H^* to have a nondegenerate pairing). Then H acts on $(H^{op})^*$ by module-coalgebra actions. By symmetry, since $(((H^{op})^*)^{op})^* \cong H$, we get a module-coalgebra action of $(H^{op})^*$ on H. This gives us a matched pair.

Definition: The quantum double of H is $D(H) := (H^{op})^* \bowtie H$.

Why do we do this construction? We'll see that the universal R-matrix gives us an isomorphism $R : H \to H^{co-op}$, where H^{co-op} is just H with the same multiplication but opposite comultiplication. The quantum double construction makes this self-dual.

To construct the R-matrix, take $id \in \mathrm{End}(H) \underset{\lambda_{H,H}}{\overset{\simeq}{\to}} (H^{op})^* \otimes H$, and let $\rho := \lambda_{H,H}(id)$. If H is infinite-dimensional, then $\lambda_{H,H}$ is not necessarily a well-defined map, but ρ can still be defined formally by its action on finite dimensional representations as a power series of nilpotents. We choose a basis such that $\rho = e^i \otimes e_i$, and let $R = (i_{(H^{op})^*} \otimes i_H)(\rho)$. Then we can write $R = \sum_i (1 \otimes e_i) \otimes (e^i \otimes 1)$. If you've ever seen an R-matrix for $U_q(\mathfrak{g})$ written explicitly, you might wonder what happened to the huge mess. I'll get to that soon.

We can recast this construction categorically. The existence of the R-matrix for the double follows from the universal property of the Drinfeld center of the category Rep H, i.e., Rep $D(H)$ is the universal braided tensor category with a monoidal functor to Rep H.

Claim: $U_q(\mathfrak{g}) \cong D(U_q(\mathfrak{b}^{\pm}))$/some stuff.

We can decompose $U_q(\mathfrak{g})$ as a vector space into $U_q(\mathfrak{n}^-) \otimes U_q(\mathfrak{h}) \otimes U_q(\mathfrak{n}^+)$, with $U_q(\mathfrak{b}^-) = U_q(\mathfrak{n}^-) \otimes U_q(\mathfrak{h})$ and $U_q(\mathfrak{b}^+) = U_q(\mathfrak{n}^+) \otimes U_q(\mathfrak{h})$. For $\mathfrak{g} = \mathfrak{sl}_n$, these are generated by the lower and upper triangular matrices with trace zero, and they intersect on the torus of diagonals. The explicit Hopf algebra structure can be found on line or in any textbook on quantum groups.

The Killing form on the Lie algebra gives a nondegenerate pairing $\langle \mathfrak{b}^+, \mathfrak{b}^- \rangle \to \mathbb{C}$, so we can write $\mathfrak{b}^- \cong (\mathfrak{b}^{+,op})^*$, where the ^{op} comes from the transpose anti-involution.

Identifying the tori gives us a surjection $D(U_q(\mathfrak{b}^+)) \twoheadrightarrow U_q(\mathfrak{g})$, and the rather nice R-matrix we saw above gets taken to something rather nasty. You might see some red flags involving finite-dimensionality with respect to this map, but it can be made rigorous.

Three contexts for quantum groups

(1) Formal power series $U_\hbar(\mathfrak{g})$ - defined by Drinfeld.
(2) Rational case $U_q(\mathfrak{g})$ - Lusztig and Jimbo noticed that setting $q = e^\hbar$ gives rational functions. The Hopf algebra is not quasitriangular, but if we restrict to finite dimensional modules, we get a braiding, because the formal R-matrix looks like $\sum_k E^k \otimes F^k$, and the pieces act nilpotently.
(3) Roots of unity - this has a large center, so if we work in a relative setting, we only need to worry about finite dimensional behaviors.

Duality and covariance

In classical groups, we have a commutative algebra $\mathcal{O}(G)$ of functions with a nondegenerate pairing $\langle U(\mathfrak{g}), \mathcal{O}(G) \rangle \to \mathbb{C}$, given by differential operators acting on germs of functions at the identity. A natural question is, if we replace $U(\mathfrak{g})$ by $U_q(\mathfrak{g})$, what should replace $\mathcal{O}(G)$?

There are two answers:

(1) (Majid's first braided reconstruction theorem) We can get $\mathcal{O}(G)$ from Tannaka-Krein. It is the coordinate ring of the automorphism group of the fiber functor $U(\mathfrak{g}) \to Vect$. We can do something similar to get $\mathcal{O}_q(G)$, where our fiber functor $U_q(\mathfrak{g}) \to Vect$ is adjusted so that the essential image has a nontrivial braiding pushed forward. Since $Vect$ has no nontrivial braidings, this braiding cannot be extended to all of $Vect$. We get a noncommutative Hopf algebra.
(2) (Algebra of matrix coefficients) Let V be a $U_q(\mathfrak{g})$-representation, and let $v \in V$ and $f \in V^*$. Then we have functionals $c_{f,v} \in U_q(\mathfrak{g})^*$, defined by $c_{f,v}(u) = f(u \cdot v)$. Here, the star just means vector space dual. We write $\mathcal{O}_q(G)$ for the span of all $c_{f,v}$ inside

$U_q(\mathfrak{g})$ as V ranges over all finite dimensional $U_q(\mathfrak{g})$-modules V. There is a multiplication: $c_{f,v} \cdot c_{g,w} = c_{f\otimes g, v\otimes w}$, which makes this space a subalgebra. Actually, we get n or 2^n copies of the algebra we want, but there is a preferred one, called $\mathcal{O}_q^+(G)$.

Proposition: $U_q(\mathfrak{g})$-mod has a projective element, i.e., every finite dimensional (projective?) module lives in some $V^{\otimes n}$.

A consequence of this is that $\mathcal{O}_q(G)$ is a quotient of the tensor algebra $T(V^* \otimes V)$ by some relations encoding the embeddings of representations W in the tensor powers of V. For $G = SL_n$ you get a relation, $det_q = 1$.

We can decompose: $\mathcal{O}_q(G) \cong \bigoplus_{V \text{ simple}} V^* \otimes V$, and $U_q(\mathfrak{g})$ acts on both sides, generalizing the actions of left and right vector fields. $\mathcal{O}_q(G)$ is a module-algebra for these actions, and when $q = 1$, this implies $\mathcal{O}(G)$ is covariant with respect to the adjoint action. However, when $q \neq 1$, $\mathcal{O}_q(G)$ is not commutative, and the left and right actions do not combine to give an adjoint action.

Majid's solution: $BO_q(G)$, the braided covariantized algebra.

Again, we have two definitions.

(1) (Fancy version) For any faithful exact braided tensor functor U_q-mod $\to \mathcal{C}$, we get a Tannakian reconstruction of a Hopf algebra in $\mathcal{C}$. One obvious choice is the identity functor on U_q-mod, and its automorphisms form a Hopf algebra BO_q in U_q-mod. Majid's second reconstruction theorem asserts that this is unique. Note that this is not a Hopf algebra in vector spaces - you use the braided tensor structure to get compatibility of multiplication and comultiplication.
(2) (Potatoes version) $BO_q \cong \{c_{f,v}\}$ as a coalgebra. We change the multiplication to be compatible with the braiding, since multiplying an element of $V^* \otimes V$ by an element of $W^* \otimes W$ involves switching them. Writing $R = r^+ \otimes r^-$, we have $c_{f,v} \cdot c_{g,w} = c_{f\otimes r^- g, r^+ v \otimes w}$. This is still generated by the defining module.

What is this good for? Well, this is the punchline of the talk.

Proposition

(1) Both $\mathcal{O}_q$ and BO_q have the property that all U_q-modules are comodules for $\mathcal{O}_q$ and BO_q. For BO_q-comodules, we get a well-behaved braided diagram calculus.
(2) BO_q is covariant for the adjoint action of U_q.
(3) BO_q is braided commutative, unlike $\mathcal{O}_q$.

Theorem There is a unique isomorphism of algebras $K : BO_q \to U_q^{lf}$, called the "Fourier transform," where the lf means the locally finite part.

As $q \to 1$, this degenerates, and is no longer an isomorphism. The fact that it exists is a strange thing, since we constructed BO_q as a dual to U_q. One consequence of this theorem is that an analogue of harmonic analysis works in the quantum context, and it is similar to the case of abelian groups.

This theory was worked out by Majid (and independently by several others), and you can find it in this hot pink textbook that he wrote.