Pre-Talbot seminar, lecture 3

Sheel Ganatra - The Geometric Satake Correspondence

Let G be a reductive algebraic group, and let $G^\vee$ be its Langlands dual - it has the dual root datum. The theorem for today is that there is an equivalence of categories

$$\mathrm{Rep}(G^\vee) \cong P_{G[[t]]}(Gr),$$

where we write Gr for the affine Grassmannian for G. For the first half of the talk, I will explain the right side, and for the second half, I will explain the proof.

Affine Grassmannian

There is an object called the "loop group of G," written LG or $G((t))$. It is an ind-scheme, whose complex points are in natural bijection with $G(\mathbb{C}((t)))$. There is also a subscheme of "positive loops," written L^+G or $G[[t]]$, whose complex points are $G(\mathbb{C}[[t]])$. L^+G acts on LG on the left and right, by restricting the multiplication maps. We define the affine Grassmannian $Gr = LG/L^+G$. This is an ind-scheme.

Example 1: Let $G = GL_1 = C^\times$. Then $G(\mathbb{C}((t))) = \{f : \widehat{\mathbb{C}^\times} \to \mathbb{C}^\times\}$ This is the group of invertible formal Laurent series, and elements can be written $f(z) = \sum_{-\infty}^{\infty} a_i z^i$. $G(\mathbb{C}[[t]])$ is the group of invertible formal Taylor series, namely those series supported on non-negative exponents with nonzero constant term. Then $Gr(\mathbb{C}) \cong \mathbb{Z}$, because we can uniquely get a monomial representative z^n by multiplication.

Example 2: $G = T$ a torus. Then $Gr = X_*(T) = \mathrm{Hom}(\mathbb{C}^\times, T)$, i.e., the coweight lattice.

An interesting exercise is the affine Grassmannian of SL_2.

Structure of Gr: Fix a triangular decomposition $T \subset B \subset G$. Pick $\lambda \in X_*(T)$. We can associate to λ an element $t^\lambda \in Gr$ in the following way. λ is a map $\mathbb{C}^\times \to T$. We postcompose with $T \hookrightarrow G$ and precompose with the completion at zero Spec $\mathbb{C}((t)) \hookrightarrow \mathbb{C}^\times$. We get a map Spec $\mathbb{C}((t)) \to G$, i.e., a point $\widetilde{t^\lambda} \in G((t))$, and we let $t^\lambda := \widetilde{t^\lambda} \cdot L^+G$.

L^+G acts on Gr on the left, and we define $Gr^\lambda := L^+G \cdot t^\lambda$ to be the orbit. This has nice properties:

(1) $dim_{\mathbb{C}} Gr^\lambda = 2\rho(\lambda)$, for λ dominant integral.
(2) $\overline{Gr^\lambda} = \bigcup_{\mu \leq \lambda} Gr^\mu$.
(3) Any point in Gr is in some Gr^λ.

This gives us a stratification of Gr by L^+G-orbits.

Definition: Let $\mathcal{S}$ be a poset. A Whitney stratification of a space X is a collection of locally finite disjoint subspaces $S_\alpha, \alpha \in \mathcal{S}$ such that

(1) $\bigcup_\alpha S_\alpha = X$.
(2) All S_α are smooth.
(3) $S_\alpha \cap \overline{S_\beta} \neq 0 \Leftrightarrow \alpha \leq b \Leftrightarrow S_\alpha \subset \overline{S_\beta}$.
(4) technical conditions for containments.

We want to look at sheaves that behave nicely with respect to a stratification.

- A sheaf is locally constant with respect to $\mathcal{S}$ if we can associate to any path on a stratum an isomorphism on the stalks, such that the isomorphism depends only on the homotopy class of the path.
- A complex of sheaves is constructible if
  - The cohomology is locally constant with respect to $\mathcal{S}$.
  - The cohomology has finitely generated stalks.
- $D_c^b(X)$ is the bounded derived category of constructible sheaves. Objects are constructible complexes, and morphisms are given by certain zig-zags.
- The category $P_{\mathcal{S}}(X)$ of perverse sheaves is the full subcategory whose objects are $A^\bullet \in D_c^b(X)$ satisfying:
  - (support) $H^k(j_\alpha^* A) = 0$ for $k > -dim_{\mathbb{C}} S_\alpha$.
  - (cosupport) $H^k(j_\alpha^! A) = 0$ for $k < dim_{\mathbb{C}} S_\alpha$.

The conditions on perverse sheaves control the failure of transversality of cycles with substrata. Note that the two conditions are Verdier dual, so perverse sheaves are self-dual. In fact, the extension of Poincaré duality to singular spaces was the initial motivation for perversity.

We have the following properties:

(1) $P_{\mathcal{S}}(X)$ is abelian.
(2) There is a "truncation" functor ${}^pH^0 : D_c^b(X) \to P_{\mathcal{S}}(X)$
(3) Given a stratified map $j : (X, \mathcal{S}) \to (Y, \mathcal{T})$, there is a functor ${}^pj_* : P_{\mathcal{S}}(X) \to P_{\mathcal{T}}(X)$.
(4) ${}^pj_* := {}^pH^0 Rj_*$.

In $P_{\mathcal{S}}(X)$, there is a unique "simple object" $IC_X = {}^pH^0(\mathbb{C}[dim_{\mathbb{C}} X])$. It is called the intersection cohomology sheaf. Since each $\overline{Gr^\lambda}$ is stratified, we have $IC_{\overline{Gr^\lambda}} \in P_{\mathcal{S}}(\overline{Gr^\lambda})$. For $j : \overline{Gr^\lambda} \hookrightarrow Gr$, we define $IC_\lambda = {}^pj_*(IC_{Gr^\lambda})$.

Part 2: the proof

We now have a bijection between irreducible representations V_λ of $G^\vee$ and simple perverse sheaves IC_λ on Gr. We'd like to promote this to an equivalence of tensor categories. To do this, we use the Tannakian formalism so nicely developed by John last week. [John says, "I think Deligne might have played a bigger role."] If we can construct a faithful exact tensor functor $P_{\mathcal{S}}(Gr) \to Vect^{\otimes}$, then $P_{\mathcal{S}}(Gr) \cong \text{Rep } \widetilde{G}$ for some $\widetilde{G}$. Then we show that $\widetilde{G}$ is reductive, and identify its root datum with that of $G^{\vee}$.

We construct the tensor product in two ways. The first is manifestly associative, the second is manifestly commutative, and it turns out that they coincide.

The convolution tensor structure is given by the following diagram:

$$Gr \times Gr \xleftarrow{p} G((t)) \times Gr \to G((t)) \overset{G[[t]]}{\times} Gr \xrightarrow{m} Gr$$

Given a sheaf $\mathcal{F} \boxtimes \mathcal{G}$ on $Gr \times Gr$, we pull it back to $p^*(\mathcal{F} \boxtimes \mathcal{G})$ on $G((t)) \times Gr$. This sheaf is $G[[t]] \times G[[t]]$-equivariant, where the first copy of $G[[t]]$ acts on the first factor on the left, and the second copy acts by left multiplication on the second factor and by right-inverse on the first factor. Since the action of the second copy of $G[[t]]$ is free, there is a unique $G[[t]]$-equivariant perverse sheaf $\mathcal{F}\widetilde{\boxtimes}\mathcal{G}$ on $G((t)) \overset{G[[t]]}{\times} Gr$. The last map m gives us $\mathcal{F} * \mathcal{G} := Rm_*(\mathcal{F}\widetilde{\boxtimes}\mathcal{G})$. One can show that this is perverse by using the fact that m is a stratified semi-small map. This is a technical condition that amounts to counting dimensions.

The fusion tensor structure is given by a global construction due to Beilinson and Drinfeld. We fix a point x on a smooth complex curve X. The completed local ring is $\widehat{\mathcal{O}_x}$, and its field of fractions is $\mathcal{K}_x$. Choosing a coordinate at x gives isomorphisms $\widehat{\mathcal{O}_x} \cong \text{Spec } \mathbb{C}[[t]]$ and $\mathcal{K}_x \cong \text{Spec } \mathbb{C}((t))$. We define

$$Gr_x := \{G\text{-bundles on } X, \text{ with a trivialization away from } x\}$$

Beilinson and Drinfeld showed that we can make this into a family $Gr_X \to X$, where Gr_X parametrizes a point $x \in X$, a G-bundle on X, and a trivialization of that G-bundle away from x. In fact, we can make a family over X^2 or X^n by choosing more points, so

$$Gr_{X^2} = \{(x_1, x_2), G\text{-bundle}, \text{trivialization on } X \setminus \{x_1, x_2\}\}$$

When $X = \mathbb{A}^1$, the fiber over a diagonal point is just Gr, and the fiber away from the diagonal is $Gr \times Gr$, and the families are trivial on or away from the diagonal. We have a family of groups

$$\begin{tikzcd} G_{x,\widehat{\mathcal{O}_x}} \arrow[r] \arrow[d] & G_{X,\mathcal{O}} \arrow[d] \\ x \arrow[r] & X \end{tikzcd}$$

and we can consider $P_{G_{x,\mathcal{O}}}(Gr_X) \cong P_{L^+G}(Gr)$. We have a diagram $U \overset{j}{\hookrightarrow} X \times X \overset{i}{\hookleftarrow} X$, where U is the complement of the diagonal. Given a $G_{X,\mathcal{O}} \times G_{X,\mathcal{O}}|_U$-equivariant sheaf $\mathcal{F} \boxtimes \mathcal{G}$ on U, $i^*(j_{*!}(\mathcal{F} \boxtimes \mathcal{G}))$ lives in $P_{\mathcal{S}}(Gr)$. This is the fusion tensor product, and it is isomorphic to the convolution tensor product.

Now, we need a tensor functor to $Vect$. This is given by global cohomology $\mathbb{H}(Gr, -)$. This is a fiber functor, i.e., it is faithful and respects the tensor product. By the Tannakian formalism, we get $P_{\mathcal{S}}(Gr) \cong \text{Rep } \widetilde{G}$ for some affine algebraic group $\widetilde{G}$.

Given a reductive group, we can identify its root data by the weight decomposition of its irreducibles. We would like to decompose IC_λ in a similar way. This is done using MV cycles, which arise from "semi-infinite orbits." For $\mu \in X_*(T)$, we define $S_\mu = N((t))t^\mu \subset Gr$, where N is the unipotent radical of our chosen Borel subgroup B.

Theorem (Mirkovic, Vilonen)

- $S_\mu \cap \overline{Gr^\lambda}$ is nonempty if and only if μ appears in the weight decomposition of $V_\lambda = \bigoplus_\alpha V_\alpha$, and in this case, it has pure dimension $\langle \rho, \lambda - \mu \rangle$.
- $\mathbb{H}(Gr, \mathcal{A}) = \bigoplus_\mu H^{2\rho(\mu)}(S_\mu, \mathcal{A})$ for $\mathcal{A} \in P_{L^+G}(Gr)$. If $\mathcal{A} = IC_\lambda$ for some dominant integral weight λ, then this is $\bigoplus_\alpha H^{2\rho(\mu)}(S_\mu \cap Gr^\alpha, IC_\lambda)$.

Therefore, $\mathbb{H}(Gr, IC_\lambda)$ is a free module generated by the irreducible components of $S_\alpha \cap \overline{Gr^\lambda}$.