Pre-Talbot seminar, lecture 2

John Francis - Tannakian Formalism and the Barr-Beck Theorem

Tannakian formalism - gives a criterion for recognizing a tensor category $\mathcal{C}^{\otimes}$ as the representations of an affine algebraic group.

(1) Geometric motivation
(2) Category theory, Barr-Beck
(3) Tannaka-Krein duality

1. Say we have X some scheme, stack, etc. If you're not comfortable with this, think of some equations with coefficients in a field k, and they might have symmetries. For any k-algebra R, you can say

$$X(R) = \left\{ \begin{array}{c} R\text{-valued solutions} \\ \text{to these equations} \end{array} \right\}$$

With symmetries, you don't just have a set. The symmetries give you a groupoid.

We want to study QC_X, the category of quasi-coherent sheaves on X. A quasi-coherent sheaf M on X consists of the data of an A-module $M(x)$ (= a quasi-coherent sheaf on Spec A) for every map $x : \text{Spec } A \to X$.

Unfortunately, geometry is hard - equations can be complicated, and so can symmetries. We'd like to reduce our problems to questions about vector spaces.

Shape of a solution We want to say that the data of $M \in QC_X$ is equivalent to the data of a vector space + some extra structure on it.

Take # 1: Take global sections. We have a canonical map $X \xrightarrow{p} *$, and pushing forward gives us a functor $p_* = \Gamma : QC_X \to QC_* = k\text{-mod}$. We ask for the data of M to be given by the data of $R\Gamma(M)$ together with some extra structure on it.

Problem: Is there a problem? [Nick says, "It's not faithful."] What? What does that have to do with anything? Okay, so Nick's *pessimism* says that $R\Gamma$ can kill things - it's not conservative. Actually, if X is affine, it's just forgetful. It's not a murderer. But if $X = \mathbb{P}^1$, then $\mathcal{O}(-1)$ has vanishing cohomology. We can't put any extra structure on zero, so Nick might be right.

Also, $R\Gamma$ doesn't preserve tensor structure, but we have no need to go there, since we're already dead in the water.

Take # 2: Cover X. We pick an affine cover $f : \text{Spec } A \to X$. Then $f^* : QC_X \to A$-mods **is** conservative. This cover describes X by gluing, so we can form a simpicial object that maps to X:

$$\cdots X_2 \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} X_1 \underset{\pi_2}{\overset{\pi_1}{\begin{smallmatrix}\longrightarrow\\ \xleftarrow{\Delta}\\ \longrightarrow\end{smallmatrix}}} X_0 \longrightarrow X$$

where $X_1 = X_0 \underset{X}{\times} X_0$, $X_0 = \text{Spec } A$, and Δ is the relative diagonal. $X = \text{colim } X_i$. This is called a geometric realization, and it is good enough to describe QC_X in terms of A-modules.

Unfortunately, this is more data than just k-modules. [Kobi mentions that A-modules are just k-modules with extra structure.] Well, the extra structure of an A-module is monadic, while the above is comonadic. We don't want to mix them. Bad idea. [Mixing makes it difficult to understand the tensor structure.]

Geometrically, we are asking for X to be covered by Spec k. Asking to be covered by a point is asking for equations to have a single solution over k. We can try to build the simplicial object

$$\cdots \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} \text{Spec } k \underset{X}{\times} \text{Spec } k \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} \text{Spec } k \longrightarrow X$$

Let's suggestively write $G = \text{Spec } k \underset{X}{\times} \text{Spec } k$, so $X_2 = * \underset{X}{\times} * \underset{X}{\times} * = G \underset{*}{\times} G$. Then our simplicial structure looks like

$$\cdots G \times G \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \xrightarrow{m}\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} G \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} * \longrightarrow X,$$

with m denoting a multiplication map. In other words, covering X by a point gives an isomorphism $X \cong BG$ for G a monoid. In fact, G is a group, because the symmetries in our stacks take values in groupoids. If we took a more general notion of stack, allowing arbitrary categories of symmetries, then we wouldn't have a group. Anyway, describing QC_X in terms of vector spaces with extra data is the same as giving it the structure of representations of a group.

This suggests the following: Given $\mathcal{C}^{\otimes}$, if there exists a functor $F : \mathcal{C}^{\otimes} \to k\text{-mod}^{\otimes}$ that is conservative, and $\mathcal{C}^{\otimes}$ has duals, then $\mathcal{C}^{\otimes} \cong \text{Rep}_k G$ for G an affine group scheme. F is called a fiber functor - the motivation for this terminology comes from a similar idea in fundamental groups using sets instead of vector spaces.

There is an analogous idea in homotopy theory. A space that is covered by a point is just a pointed connected space, which gives an equivalent theory to that of loop spaces, by the functors B (classifying space) and Ω (loop space).

2. Category theory. We'll give a formal setup for describing the "extra structure."

Suppose we have some categories $\mathcal{C} \overset{F}{\underset{G}{\rightleftarrows}} \mathcal{A}$. F and G are adjoint functors if there exists a natural equivalence

$$Hom_{\mathcal{A}}(FX, Y) \cong Hom_{\mathcal{C}}(X, GY).$$

A typical example is $\mathcal{C} = Vect$, $\mathcal{A} = Comm - alg$, F is the free algebra functor, Sym^*, and G forgets the algebra structure.

Objects of $\mathcal{C}$ in the essential image of F have some extra structure, which we'd like to extract. First, we note that composition gives us functors $G \circ F : \mathcal{C} \to \mathcal{C}$ and $F \circ G : \mathcal{A} \to \mathcal{A}$. By using the above natural equivalence on identity maps, we get natural transformations $id_{\mathcal{C}} \to G \circ F$, called the unit, and $F \circ G \to id_{\mathcal{A}}$, called the counit. Let $C = F \circ G$. Then there is a natural map $C \to C \circ C$ given by $FG = F \circ id_{\mathcal{C}} \circ G \xrightarrow{unit} FGFG$. This is a coassociative coalgebra structure on C, called a comonad (or cotriple). We get a diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\tilde{F}} & \mathrm{Comod}_C(\mathcal{A}) \\ {\scriptstyle F}\downarrow\uparrow{\scriptstyle G} & & \\ \mathcal{A} & & \end{array}$$

where $\mathrm{Comod}_C(\mathcal{A})$ is the category of comodules over the comonad C. $\tilde{F}$ is given by the unit map $F(X) \to F(GF)(X) = C \circ F(X)$. It gives us an approximation of $\mathcal{C}$ as "$\mathcal{A}$ + extra structure." We'd like to know how good this approximation is.

Barr-Beck Theorem If F is conservative + a modest additional hypothesis, then $\tilde{F}$ is an equivalence.

The hypothesis is that F preserves F-split equalizers, i.e., that it preserves a few limits in addition to all colimits. There is an opposite version, with modules over a monad, but we won't use it.

3. Tannakian formalism - says that if $\mathcal{C}^{\otimes} \overset{F}{\rightleftarrows} k$-mod is a conservative tensor functor, and if $\mathcal{C}$ has duals (i.e., is rigid), then $\mathcal{C}^{\otimes} \cong \mathrm{Rep}_k(G)$ for some G.

Asking for a conservative tensor functor to k-mod is a strong thing to ask, just like asking a stack to be covered by a point.

Sketch of a proof

Our candidate G is given by tensor automorphisms of the fiber functor F. $\mathrm{Aut}^{\otimes}(F)$ is a group. In fact, for any k-algebra R, we can define R-points by base change. There is a natural functor $F \otimes R : \mathcal{C}^{\otimes} \to R$-mod, and we define $\mathrm{Aut}^{\otimes}(F)(R) := \mathrm{Aut}^{\otimes}(F \otimes R)$, so $\mathrm{Aut}^{\otimes}(F)$ is a group scheme over k.

Suppose that $\mathcal{C}^{\otimes} = \mathrm{Rep}(G)$. Let's check that we can recover G through this precedure. There exists a homomorphism $G \to \mathrm{Aut}^{\otimes}(F)$, where each $g \in G$ gives us a commutative diagram

$$\begin{CD} V @>{g}>> V \\ @V{\phi}VV @VV{\phi}V \\ V' @>{g}>> V' \end{CD}$$

We can check that it is an equivalence by looking at the subcategory in $\mathcal{C}$ generated by V. The two maps $\mathrm{Aut}^{\otimes}(F) \hookrightarrow GL(V) \leftarrow G_V$ have the same image. We take a limit over all $V \in \mathcal{C}^{\otimes}$ and we are done.

That was the Tannaka part. This is the Krein part.

Given general $\mathcal{C}^{\otimes}$ with adjunction as above, $\mathcal{C}^{\otimes} \cong \mathrm{Comod}_C(k\text{-mod})$ by Barr-Beck. C is colimit-preserving, so $C(1)$ gets a coalgebra structure. We have an equivalence:

$$\mathrm{Comod}_C(k\text{-mod}) \cong \{\text{comodules over the coalgebra } C(1)\}$$

$C(1)$ as a coalgebra gets an algebra structure via the marked arrow:

$$\begin{array}{ccc} X \otimes X' & \longrightarrow & X \otimes X' \otimes C(1)^{\otimes 2} \\ & \searrow \quad \swarrow^{*} & \\ & (X \otimes X') \otimes C(1) & \end{array}$$

Using this algebra structure (which is commutative), we get $G = \mathrm{Spec}\, C(1)$. Then, $\mathrm{Comod}_C(k\text{-mod}) = \mathrm{Comod}_{\mathcal{O}_G} = \mathrm{Rep}_k(G)$.