Pre-Talbot seminar, lecture 1 Scott Carnahan

This talk is an overview. For simplicity, everything is done over the complex numbers.

What is Langlands duality? It relates two reductive algebraic groups G and $G^\vee$ through their maximal tori. In particular, the weights of G, defined as maps $T \to \mathbb{G}_m$, are identified with the coweights of $G^\vee$, defined as maps $\mathbb{G}_m \to T^\vee$.

$$\begin{tikzcd}
G \arrow[rr, dotted, leftrightarrow] & & G^\vee \\
\cup & & \cup \\
T \arrow[dr] \arrow[rr, phantom, "\leftrightarrows"] & & T^\vee \\
& \mathbb{G}_m \arrow[ur] &
\end{tikzcd}$$

There is also an identification of roots with coroots, which I will not describe. The typical example is GL_n, which happens to be dual to itself. Sp_{2n} is dual to SO_{2n+1}.

A standard strategy in the Langlands program is the use of geometry involving one group to study the representation theory of the Langlands dual group. This is sufficiently vague that I can say it with some confidence.

Representations of G - the finite dimensional representations form an abelian category. Since G is reductive, they are completely reducible, i.e., they can be decomposed as direct sums of irreducibles. The standard strategy for understanding them is looking at the action of the subgroup T. Since T is abelian, any representation decomposes as a sum of one-dimensional irreducibles, known as weights. Irreducible representations of G are parametrized by dominant integral weights.

There is a geometric description of the irreducible representations, provided by Borel-Weil-Bott. It asserts a bijection:

$$\left\{ \begin{array}{c} \text{Irreducible} \\ \text{reps of } G \end{array} \right\} \leftrightarrows \left\{ \begin{array}{c} \text{Equivariant line} \\ \text{bundles on } G/B \end{array} \right\}$$

The right arrow is given by the associated bundle construction: A dominant integral weight is a representation of T, and we can extend it uniquely to a representation of the Borel B (this is a maximal solvable subgroup of G containing T - for $G = GL_n$, it is conjugate to the group of upper triangular matrices). The projection $G \to G/B$ has fiber B, and we replace B by this one-dimensional representation to get a line bundle on G/B.

The left arrow is given by taking global sections.

This is a nice picture, but it does not capture the full category. In particular, the tensor structure is non-obvious. Also, if G is a torus, then G/B is a point, equivariant line bundles are tautologically representations, and we have no new information.

There is an alternative geometric description of representations of the torus T, namely as sheaves of vector spaces on the weight lattice Λ. The lattice is a discrete topological space, so a sheaf is just an assignment of vector spaces to points. The torus acts on any particular vector space by the corresponding weight. Irreducibles are one-dimensional spaces supported at a single point, direct sums are obvious, and tensor products are given by convolution:

$$(V \otimes W)_\lambda = \bigoplus_{\mu \in \Lambda} V_\mu \otimes W_{\lambda - \mu}$$

We can also bring the Langlands dual into this picture:

$$\Lambda = \mathrm{Hom}(T, \mathbb{G}_m) = \mathrm{Hom}(\mathbb{G}_m, T^\vee)$$

The object on the right is called the space of "polynomial loops in $T^\vee$." There is another algebraic notion of loops, known as Laurent loops: $T^\vee((t)) := \underline{\mathrm{Hom}}_{\mathrm{Spec}\,\mathbb{C}}(\mathrm{Spec}\,\mathbb{C}((t)), T^\vee)$. This space is rather large, but if we quotient out by $T^\vee[[t]]$, the space of "jets in $T^\vee$," we have a natural isomorphism:

$$\mathrm{Hom}(\mathbb{G}_m, T^\vee) = \left(T^\vee((t))/T^\vee[[t]]\right)(\mathbb{C})$$

If T is one dimensional we can see this in the following way. An invertible Laurent series has the form $a_N t^N + a_{N+1} t^{N+1} + \dots$ where N is some integer, and $a_N \neq 0$. For each such series, there is a unique invertible Taylor series $b_0 + b_1 t + b_2 t^2 + \dots$ which yields a monic monomial upon multiplication. Thus, points of the quotient are identified with integers by taking the exponent (in this case, N).

$T^\vee((t))/T^\vee[[t]]$ is called $Gr_{T^\vee}$, the affine Grassmannian of $T^\vee$. There is another way to look at $Gr_{T^\vee}$, using algebraic geometry.

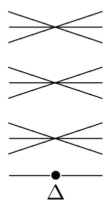
Let X be a complete curve, and let $x \in X$ be a point. Make a trivial $T^\vee$ bundle on the punctured curve $X \setminus \{x\}$. To extend this to a $T^\vee$ bundle on X, we glue a trivial bundle over a small disc, along a small punctured disc. Our disc will be Spec $\mathbb{C}[[t]]$, and the punctured disc will be its generic point Spec $\mathbb{C}((t))$. There is a transitive action of $T^\vee((t))$ on the space of gluing data, with stabilizer $T^\vee[[t]]$ (since it just changes the trivialization of the bundle on the disc). The quotient can then be written:

$$Gr_{T^\vee,x} := \left\{ \begin{array}{c} T^\vee\text{-bundles on } X \text{ with} \\ \text{trivialization away from } x \end{array} \right\}$$

Similarly, we can remove more than one point, so we have a family over X^n for any n. In particular, for any pair $(x_1, x_2) \in X^2$, we have

$$Gr_{T^\vee, x_1, x_2} := \left\{ \begin{array}{c} T^\vee\text{-bundles on } X \text{ with} \\ \text{trivialization away from } x_1, x_2 \end{array} \right\}$$

When x_1 is far away from x_2, the fiber is $Gr_{T^\vee, x_1} \times Gr_{T^\vee, x_2}$. However, the fiber only depends on the punctured curve $X \setminus \{x_1, x_2\}$, so when $x_1 = x_2$, we just have $Gr_{T_\vee}$. In fact, when x_1 and x_2 are infinitesimally nearby, we get an identification of fibers, and this endows the family with an integrable connection. This property of the Grassmannian, where we can degenerate $Gr_{T^\vee} \times Gr_{T^\vee}$ to $Gr_{T^\vee}$, is called factorization, and we say that the Grassmannian is a factorization space. Here is a picture of the fibers near the diagonal:



Each point above the diagonal is given by joining infinitely many points away from it, following the convolution law mentioned above. I should point out that the Grassmannian is not just a countable collection of points. If you have a flat family in which multiple points combine into a single point as above, that single point is non-reduced, i.e., there is some nilpotent fuzz (more precisely, maps from Spec R to $Gr_{T^\vee}$ are not in natural bijection with the weight lattice when a ring R has nilpotent elements). In our case, this non-reduced behavior is quite extreme, since those multiple points in question are themselves fuzzy, each being isomorphic to the single point.

Recall that representations of T are sheaves on $Gr_{T^\vee}$. Because of this factorization property, we can define the tensor product of modules geometrically, by mashing two points on the curve X together, fusing the T-representations over them. This idea is due to Beilinson and Drinfeld.

We would like to make this picture work for any reductive group G. As before, we can choose a curve X and a point $x \in X$.

$$Gr_{G^\vee, x} := \left\{ \begin{array}{c} G^\vee\text{-bundles on } X \text{ with} \\ \text{trivialization away from } x \end{array} \right\}$$

This also gives us a factorization space, but for nonabelian G, it has highly nontrivial geometry. However, $Gr_{G^\vee}$ can be described as a union of finite-dimensional $G^\vee[[t]]$ orbits, where $G^\vee[[t]]$ naturally acts on the left through its inclusion into $G^\vee((t))$. These orbits are locally closed, and parametrized by dominant integral weights of G (or more canonically, W-orbits of weights).

We'd like to associate some kind of sheaf data to these orbits, so that the sheaves are in bijection with irreducible representations of G. A first approximation would be taking the constant sheaf $\underline{\mathbb{C}}_\lambda$ on the orbit Gr_λ corresponding to the dominant integral weight λ, and pushing it somehow into $Gr_{G^\vee}$ along the inclusion map. We should choose a good category of sheaves on $Gr_{G^\vee}$, so that our pushforward is well-behaved. The Grassmannian is a stratified space (with strata given by $G^\vee[[t]]$-orbits), and perverse sheaves are more or less designed to work on stratified spaces. In particular, $G^\vee[[t]]$-equivariant perverse sheaves are locally constant on the strata of $Gr_{G^\vee}$. The following theorem (known as the geometric Satake equivalence) indicates that our guess is more or less correct:

Theorem (Mirkovič, Vilonen) There is an equivalence of symmetric tensor categories:

$$\left\{ \begin{array}{c} \text{Finite dimensional} \\ \text{representations of } G \end{array} \right\} \cong \left\{ \begin{array}{c} G^\vee[[t]]\text{-equivariant perverse} \\ \text{sheaves on } Gr_{G^\vee} \end{array} \right\}$$

The irreducible representation with highest weight λ is taken to the perverse sheaf IC_λ, which is supported on the closure of Gr_λ in $Gr_{G^\vee}$, and its restriction to Gr_λ is constant in cohomological degree $-2\langle\rho, \lambda\rangle$.

Any equivariant perverse sheaf $\mathcal{F}$ is taken to its global cohomology space $\bigoplus H^i(Gr_{G^\vee}, \mathcal{F})$.

Quantum groups

The category of representations of group G has a tensor structure, induced by the diagonal inclusion $\Delta : G \to G \times G$, which is a group homomorphism. We'd like to deform it somehow, but groups are too rigid, so we need to pass to an equivalent category with a floppier deformation theory. If G is simple and simply connected, we have an equivalence Rep $G \cong$ Rep $\mathfrak{g} \cong$ Rep $U(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G, and $U(\mathfrak{g})$ is its universal enveloping algebra.

$U(\mathfrak{g})$ has a Hopf algebra structure, which means in particular that we have an algebra homomorphism $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ satisfying some identities. This gives the representation category a monoidal structure (the same as for G), but since we are now working with an algebra structure over vector spaces, there is some additional freedom.

Theorem (Drinfeld) If G is simple, then the tensor category Rep $U(\mathfrak{g})$ has a one dimensional space of deformations in braided tensor categories.

Drinfeld proved it for infinitesimal deformations, but it was later shown that these could be integrated to a one-parameter family $U_q(\mathfrak{g})$ for $q \in \mathbb{C}^\times$ not a root of unity. When q is a root of unity, one can still construct Hopf algebras, but it is more complicated.

Is there a geometric way to see representations of $U_q(\mathfrak{g})$? One answer comes from Bezrukavnikov, Finkelberg, and Schechtman, who constructed a category $FS^c(G)$ of factorizable sheaves, and showed that it is equivalent to representations of the small quantum group for $q = e^{-2\pi i c}$. The word "small" here refers to a distinction that only matters at roots of unity.

We'd like to use $Gr_{G^\vee}$ here somehow. One idea is to use the Riemann-Hilbert correspondence, which gives an equivalence between perverse sheaves and D-modules, to construct a category of $G^\vee[[t]]$-equivariant twisted D-modules on $Gr_{G^\vee}$. Unfortunately, Gaitsgory found that when the twisting corresponds to q not a root of unity, the resulting category is equivalent to representations of the trivial group.

Lurie's idea was to use an additional equivalence of categories proved by Frenkel, Gaitsgory, and Vilonen, between $G^\vee[[t]]$-equivariant sheaves on $Gr_{G^\vee}$ and the Whittaker category of $N((t))$-equivariant sheaves, and twist the Whittaker category. Gaitsgory then showed that this works:
Theorem (Gaitsgory) For $c \notin \mathbb{Q}$ (i.e., q not a root of unity),

$$Whit(Gr_{G^\vee}) \cong FS^c(G).$$