FACTORIZABLE SHEAVES AND QUANTUM GROUPS

LECTURES BY DENNIS GAITSGORY, 1/24/08 AND 1/28/08

The idea is to formulate a kind of Langlands duality for quantum groups (and later, a quantum geometric Langlands conjecture). To this end, we consider the following diagram of equivalences:

$$\begin{array}{ccc} \mathrm{Whit}(\mathrm{Gr}_{\check{G}}) & & \mathrm{Rep}(U_q(G)) \\ & \searrow \quad \swarrow & \\ & \mathrm{FS}_q & \end{array}$$

Here $q \in \mathbb{C}^\times$ is not a root of unity, $\mathrm{Rep}(U_q(G))$ is a certain category of representations of a quantum group $U_q(G)$, and $\mathrm{Whit}(\mathrm{Gr}_{\check{G}})$ is the category of twisted Whittaker sheaves on the affine Grassmannian of the dual group $\check{G}$. The intermediate category FS_q is the category of factorizable sheaves of Finkelberg and Schechtman. The goal of this talk is to give a conceptual understanding of the equivalence between $\mathrm{Rep}(U_q(G))$ and FS_q using Koszul duality.

1. Quantum groups

1.1. Recall that $U_q(G)$ is the Hopf algebra generated by E_i, F_i, and $t \in T$. Let Λ and $\check{\Lambda}$ denote the lattices of weights and coweights, respectively. Given $\check{\lambda} \in \check{\Lambda}$, let $t_{\check{\lambda}} = \check{\lambda}(q) \in T$. As usual, for any $t \in T$, we have the relation

$$tE_i t^{-1} = E_i \alpha_i(t)$$

where α_i is the simple root corresponding to E_i. We also have

$$E_i F_i = F_i E_i = \frac{t_{d_i\check{\alpha}_i} - t^{-1}_{d_i\check{\alpha}_i}}{q^{d_i} - q^{-d_i}}$$

where $(\alpha_i, \alpha_i) = 2d_i$. These generators also satisfy the rest of the quantum Serre relations.

The co-multiplication is given by

$$\Delta t = t \otimes t$$

$$\Delta E_i = E_i \otimes 1 + t_{d_i\check{\alpha}_i} \otimes E_i$$

$$\Delta F_i = 1 \otimes F_i + F_i \otimes t_{d_i\check{\alpha}_i}$$

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1.2. A representation of $U_q(G)$ is a Λ-graded vector space (not nec. finite dim.) with an action of this algebra. An element $t \in T$ acts via

$$tv^\lambda = \lambda(t)v^\lambda$$

Let $U_q(\mathfrak{n}^+)$ denote the sub-algebra generated by the $\{E_i\}$. Define the subcategory $\mathcal{O}$ to be the representations on which $U_q(\mathfrak{n}^+)$ acts locally nilpotently. $\mathcal{O}$ is a braided monoidal category.

2. Factorizable Sheaves

Let X be a smooth complex curve and $x_0 \in X$ (e.g. $X = \mathbb{A}^1, x_0 = 0$). Let $\Lambda^{pos} \subset \Lambda$ denote the positive span of simple roots.

2.1. Given $\lambda \in -\Lambda^{pos}$, let X^λ be the variety which classifies $-\Lambda^{pos}$-valued divisors of total weight λ, i.e. divisors of the form $\sum \lambda_i x_i$, such that $\sum \lambda_i = \lambda$. If $\lambda = -\sum n_i \alpha_i$, then

$$X^\lambda = \prod_i X^{(n_i)}$$

where $X^{(n_i)} = \mathrm{Sym}^{n_i}(X)$ denotes the n_i-th symmetric power of the curve.

2.2. If $\lambda \in \Lambda$, let $X^\lambda_{\infty\cdot x_0}$ denote the ind-scheme which classifies Λ-valued divisors of the form $\sum \lambda_i x_i$, where $\sum \lambda_i = \lambda$, and $-\lambda_i \in \Lambda^{pos}$ for $x_i \neq x_0$.

2.3. If $\mu \in \Lambda$, then $X^\lambda_{\leq \mu x_0} \subset X^\lambda_{\infty\cdot x_0}$ classifies divisors of the form $\lambda_0 x_0 + \sum_{x_i \neq x_0} \lambda_i x_i$ with $\lambda_0 \leq \mu$. Note that if $\mu = 0$, then $X^\lambda_{\leq \mu x_0} = X^\lambda$.

2.4. Next we define a line bundle $\mathcal{P}^\lambda$ on $X^\lambda_{\infty\cdot x_0}$. The fiber of $\mathcal{P}^\lambda$ at $\sum \lambda_i x_i$

$$\bigotimes_i \omega_{x_i}^{(\lambda_i, \lambda_i + 2\rho)}.$$

(This was followed by a discussion of why this glues to a line bundle.)

2.5. By adding divisors, we get a map

$$\begin{array}{c} X^{\lambda_1} \times X^{\lambda_2}_{\infty\cdot x_0} \\ \big\downarrow \\ X^{\lambda_1+\lambda_2}_{\infty\cdot x_0} \end{array}$$

Let $(X^{\lambda_1} \times X^{\lambda_2}_{\infty\cdot x_0})_{disj} \subset X^{\lambda_1} \times X^{\lambda_2}$ denote the open subscheme consisting of disjoint divisors. Then we have the *factorization property*:

$$\mathcal{P}^{\lambda_1+\lambda_2}\big|_{(X^{\lambda_1} \times X^{\lambda_2}_{\infty\cdot x_0})_{disj}} = \mathcal{P}^{\lambda_1} \boxtimes \mathcal{P}^{\lambda_2}$$

Let $\overset{\circ}{X}{}^\lambda \subset X^\lambda$ denote the divisors of the form $\sum \lambda_i x_i$ where each λ_i is the negative of a simple root. Then $\mathcal{P}^\lambda\big|_{\overset{\circ}{X}{}^\lambda}$ is trivial.

2.6. Next we define a basic q-twisted perverse sheaf Ω^λ on X^λ. Let $\overset{\circ}{\Omega}{}^\lambda = \Omega^\lambda\big|_{\overset{\circ}{X}{}^\lambda}$ be the *sign* local system. Then we set

$$\Omega^\lambda = j_{!*}\overset{\circ}{\Omega}{}^\lambda$$

where $j : \overset{\circ}{X}{}^\lambda \to X^\lambda$ is the inclusion map. These sheaves have the factorization property:

$$\Omega^{\lambda_1+\lambda_2}\big|_{(X^{\lambda_1}\times X^{\lambda_2}_{\infty\cdot x_0})_{disj}} = \Omega^{\lambda_1}\boxtimes\Omega^{\lambda_2}$$

2.7. The fibers of Ω^λ have the following property:

$$(\Omega^\lambda)_{\sum\lambda_i x_i} = \bigotimes_i (\Omega^{\lambda_i})_{\lambda_i x_i}$$

Moreover,

$$(\Omega^\lambda)_{\lambda x} = \begin{cases} 0 & \text{unless } \lambda = w(\rho)-\rho, w\in W \\ \mathbb{C} & \text{else} \end{cases}$$

2.8. A *factorizable sheaf* (at x_0) is a collection of q-twisted perverse sheaves $\mathcal{F}^\lambda$ on $X^\lambda_{\infty\cdot x_0}$ such that

$$\mathcal{F}^{\lambda_1+\lambda_2}\big|_{(X^{\lambda_1}\times X^{\lambda_2}_{\infty\cdot x_0})_{disj}} = \Omega^{\lambda_1}\boxtimes\mathcal{F}^{\lambda_2}$$

(plus associativity conditions).

2.9. Let FS denote the category of factorizable sheaves at x_0. Then FS $\simeq \mathcal{O}$ as abelian categories. For example, given the following diagram

$$(X^\lambda_{=\mu x_0})_{disj} \xrightarrow{j_2} X^\lambda_{=\mu x_0} \xrightarrow{j_1} X^\lambda_{\leq\mu x_0}$$

we define

$$\nabla_\mu = (j_1)_*(j_2)_{!*}(sign)$$
$$\Delta_\mu = (j_1)_!(j_2)_{!*}(sign)$$
$$L_\mu = (j_1)_{!*}(j_2)_{!*}(sign)$$

There are all examples of factorizable sheaves at x_0, corresponding to the Verma, co-Verma, and irreducible representations, respectively.

2.10. Next we repeat this construction for n points. We define

$$\begin{array}{c} X^\lambda_n \\ \downarrow \\ X^n \end{array}$$

as the ind-scheme which classifies $(x_0^1,\ldots,x_0^n,\sum\lambda_i x_i)$ where $\sum\lambda_i=\lambda$, and λ_i is negative away from $x_0^1,\ldots,x_0^n$. Therefore, $X^\lambda_{\infty\cdot x_0}$ is the fiber over x_0 of X^λ_1.

2.11. Let FS_n denote the category of factorizable sheaves on X^λ_n. For example, FS_1 is the category of local systems on S with coefficients in FS. Also, $\mathrm{FS}_2\,/\,\mathrm{FS}_1$ is the category local systems on $X\times X-\Delta(X)$ with coefficients in FS $\times$ FS.

3. Koszul duality

In this section we state the main theorem/construction and explain how it relates to Koszul duality. From now on $X = \mathbb{C}$.

3.1. Let $\Lambda \supset \Lambda^{pos}$ be a lattice containing a semi-group of positive elements. Let A be a Λ-graded Hopf algebra. Suppose that $A_0 = k$ and A_μ is finite dimensional.

Note that $U_q(\mathfrak{n}^+)$ is not a Hopf algebra in the usual category of Λ-graded vector spaces. However, it is a Hopf algebra in the category of Λ-graded vector spaces equipped with a *different braiding*:

$$\mathbb{C}^\mu \otimes \mathbb{C}^\nu \xrightarrow{q^{(\mu,\nu)}} \mathbb{C}^\nu \otimes \mathbb{C}^\mu \ .$$

3.2. **Theorem.** To a Hopf algebra A one attaches canonically a system of (not twisted!) perverse sheaves Ω_A^λ on X^λ with the factorization property. Moreover:

(1) $i^*_{\lambda x}(\Omega^\lambda_A) = (\mathrm{Tor}_A(k,k))^\lambda$
(2) This construction yields an equivalence of categories between these Hopf algebras and systems of perverse sheaves on X^λ with the factorization property.
(3) There is a canonical equivalence of categories between

$$(A\sharp A^{*op})\text{-modules on which } A^{>0} \text{ acts locally nilpotently}$$

and

$$\text{factorizable sheaves with respect to } \Omega_A$$

3.3. The dual sheaf $\mathbb{D}(\Omega_A) = \Omega_{A^*}$ is also factorizable. Therefore

$$i^!_{\lambda x}(\Omega_A) = (\mathrm{Ext}_{A^*}(k,k))^\lambda$$

Moreover, the Ω^λ from above corresponds to $\Omega_{U_q(\mathfrak{n}^-)}$.

3.4. Let A be an augmented Λ^{pos}-graded associative algebra. Let $B = k \otimes_A k$ thought of as a DG co-algebra via the bar construction. Koszul duality yields an equivalence of categories

$$D(A\text{-modules on which } A^{>0} \text{ acts locally nilpotently }) \simeq D(B\text{-comodules})$$

$$M \mapsto \mathrm{Tor}_A(k, M)$$

The quasi-inverse to this functor is given by

$$N \mapsto \mathrm{Ext}_B(k, N)$$

3.5. Let us now discuss factorizable sheaves in dimension 1. Let B be a DG co-algebra. Let $\mathcal{R}\mathrm{an}(\mathbb{R})$ denote Ran space of $\mathbb{R}$. It is a topological space whose points are finite non-empty collections of points of $\mathbb{R}$. We define a complex of sheaves Ω_B on $\mathcal{R}\mathrm{an}(\mathbb{R})$.

$$(\Omega_B)_{\{x_1,\dots,x_n\}} = B \otimes \dots \otimes B$$

Since $\mathbb{R}$ is one-dimensional and oriented (S^1 would work too), it suffices to define

$$(\Omega_B)_{\{x\}} \to (\Omega_B)_{\{x_1,x_2\}}$$

We take this map to be the co-multiplication $B \to B \otimes B$.

We have the following:

$$H^*(\mathcal{R}\mathrm{an}(S^1), \Omega_B) = \mathbb{H}_*(B) = \text{ the Hochschild homology of } B$$

(Beilinson made a comment that one could guess the S^1-equivariant cohomology...)

$$H^*_{S^1}(\mathcal{R}\mathrm{an}(S^1), \Omega_B) = \text{ the cyclic homology of } B \text{ ??}$$

3.6. We have a map

$$\mathcal{R}\mathrm{an}(\mathbb{R}) \times \mathcal{R}\mathrm{an}(\mathbb{R}) \to \mathcal{R}\mathrm{an}(\mathbb{R})$$

given by taking the union of finite subsets. Let $(\overset{\circ}{\mathcal{R}\mathrm{an}})_{disj} \subset \mathcal{R}\mathrm{an}(\mathbb{R}) \times \mathcal{R}\mathrm{an}(\mathbb{R})$ denote the open subset of pairs of disjoint points. Then Ω_B has a factorization property on $\mathcal{R}\mathrm{an}(\mathbb{R})$.

3.7. Let $x_0 \in \mathbb{R}$. Then $\mathcal{R}\mathrm{an}_{x_0}(\mathbb{R})$ is the space of finite subsets that contain x_0. Let M be a bi-comodule over B. We define a sheaf $\Omega_{B,M}$ on $\mathcal{R}\mathrm{an}_{x_0}(\mathbb{R})$. We let

$$(\Omega_{B,M})_{\{x_0,\ldots,x_n\}} = M \otimes B \otimes \ldots \otimes B$$

as before, the structure maps are sufficient to define a sheaf:

$$M \to B \otimes M$$

$$M \to M \otimes B$$

Moreover, we have

$$H^*(\mathcal{R}\mathrm{an}_{x_0}(S^1), \Omega_{B,M}) = \mathbb{H}_*(B, M)$$

3.8. Suppose B is augmented. Then

$$H^*_c(\mathcal{R}\mathrm{an}(\mathbb{R}), \Omega_B) = \mathrm{Ext}_B(k,k) \simeq A$$

If M is a left B-comodule, then $\Omega_{B,M}$ is a sheaf on $\mathcal{R}\mathrm{an}_{x_0}(\mathbb{R}^{\leq x_0})$. Furthermore,

$$H^*(\mathcal{R}\mathrm{an}_{x_0}(\mathbb{R}^{\leq x_0}), \Omega_{B,M}) = \mathrm{Ext}_B(k, M)$$

3.9. For each n, we have a diagram of DG co-algebras:

$$\begin{array}{ccc} B = k \otimes_A k & \longrightarrow & k \otimes_{A^n} k \\ & & \uparrow \sim \\ & & (k \otimes_A k)^{\otimes n} \end{array}$$

where the vertical arrow is a quasi-isomorphism. Such a structure is called an E_2 *co-algebra.*

3.10. If B is an E_2 co-algebra, then Ω_B is a factorizable complex on $\mathcal{R}\mathrm{an}(\mathbb{R}^2)$.

3.11. On the other hand, suppose we have such an Ω_B. Then

$$H_c^*(\Omega_B\big|_{\mathcal{R}\mathrm{an}(\mathbb{R})}) = A = H^*_{\mathcal{R}\mathrm{an}(i\mathbb{R})}(\Omega_B)$$

which implies that Ω_B is a perverse sheaf. Now let I_1, I_2 be two disjoint open intervals in $\mathbb{R}$. We have a map

$$\mathcal{R}\mathrm{an}(I_1) \times \mathcal{R}\mathrm{an}(I_2) \to \mathcal{R}\mathrm{an}(\mathbb{R})$$

which gives

$$H_c^*(\mathcal{R}\mathrm{an}(I_1), \Omega_B) \otimes H_c^*(\mathcal{R}\mathrm{an}(I_2), \Omega_B) \to H_c(\mathcal{R}\mathrm{an}(\mathbb{R}), \Omega_B)$$

Since each open interval is homeomorphic to $\mathbb{R}$, this yield a multiplication map $A \otimes A \to A$. Similarly, using $H^*_{\mathcal{R}\mathrm{an}(i\mathbb{R})}$, we get a co-multiplication $A \to A \otimes A$.

(Here Drinfeld made a comment that this picture is what originally led him to define quantum groups).