18.784 Homework Set 5
Due Friday, March 5, 2010.

Part I (AH, 2/26/10)

1. Let L be the lattice described by the Gram matrix $\binom{21}{12}$. Describe the dual lattice $L^\vee$.

Part II (PE, 3/1/10)

1. Let f be a rapidly decreasing (aka Schwartz) function on $\mathbb{R}^n$, and let $\check{f}$ denote its Fourier transform (which is automatically Schwartz). Show that:
$$\sum_{x\in\mathbb{Z}^n} f(x) = \sum_{y\in\mathbb{Z}^n} \check{f}(y).$$
Hint: Consider the function $\tilde{f}(z) = \sum_{x\in\mathbb{Z}^n} f(x+z)$. $\tilde{f}$ is periodic, so it has a Fourier series. Do the same for $\check{f}$.

2. Show that if Γ is a unimodular lattice in a Euclidean space V, then $\mathrm{Vol}(V/\Gamma) = 1$. Hint: Choose a basis and consider a Gram matrix.

Part III (CK, 3/3/10)

1. Prove that $[SL_2(\mathbb{Z}) : \Gamma_+] = 3$, where
$$\Gamma_+ = \langle S, T^2\rangle = \{M \in SL_2(\mathbb{Z}) | M \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mod 2\}.$$

2. (a) Prove that the function s_2 is modular of weight 1 for Γ_+ (i.e., that $s_2(z) = \theta_{\mathbb{Z}}(z)^2$), where:
$$\begin{aligned} s_2(z) &= \sum_{m=-\infty}^{\infty} \operatorname{sech}(\pi miz) \\ &= \sum_{m=-\infty}^{\infty} \sec(m\pi z) \\ &= 1 + 4\sum_{m=1}^{\infty} \frac{q^{m/2}}{1-q^m} \end{aligned}$$
Hint: Use the fact that the Fourier transform of $f(x) = \operatorname{sech}(\pi tx)$ is $\check{f}(y) = t^{-1}\operatorname{sech}(\pi t^{-1}y)$ for $t \in \mathbb{R}^\times$.

   (b) Find an expression for the number of ways to represent a positive integer n as a sum of two squares of integers.