18.784 Homework Set 4
Due Friday, February 26, 2010.

Part I (AF, 2/19/10)

1. Recall that for a function f that is meromorphic in a neighborhood of a point $p \in \mathbb{C}$, $v_p(f)$ is the unique integer n such that $\frac{f(z)}{(z-p)^n}$ has a limit at p that is a nonzero complex number. Show that for any weakly modular function f, any $g \in PSL_2(\mathbb{Z})$, and any $p \in \mathbb{H}$, we have $v_p(f) = v_{g(p)}(f)$.

2. Recall that for $k \geq 2$, $E_{2k}(\tau)$ is the normalized weight $2k$ Eisenstein series, with q-expansion $1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n$. Prove that $E_6 E_8 = E_4 E_{10}$.

Part II (BW, 2/22/10)

1. Let Γ be a lattice in $\mathbb{C}$ generated by ω_1 and ω_2, and let Γ' be a sublattice of index n. For any choice of basis $\{\omega_1', \omega_2'\}$ of Γ', there exist unique $a, b, c, d \in \mathbb{Z}$ such that $\omega_1' = a\omega_1 + b\omega_2$ and $\omega_2' = c\omega_1 + d\omega_2$, i.e., $\binom{\omega_1'}{\omega_2'} = \binom{ab}{cd}\binom{\omega_1}{\omega_2}$. Show that:
   (a) The determinant of the matrix $\binom{ab}{cd}$ is n.
   (b) The set of lattices of index n is represented by the following set of integer matrices $\{\binom{ab}{0d} \mid ad = n, 0 \leq b < d\}$.

2. Let $f(z) = \sum_{n \geq 0} c(n) q^n$ be a modular form that is an eigenfunction of all Hecke operators T_m, so there exist complex numbers λ_m such that $T_m f(z) = \lambda_m f(z)$ for all $m > 0$. Show that if $c(1) = 1$, then $c(n) = \lambda_n$ for all $n > 0$.

Part III (SK, 2/24/10)

1. Define the coefficients $c(n)$ by $j(\tau) - 744 = \sum_{n \geq -1} c(n) q^n$, where $j = 1728 g_2^3/\Delta$ is Dedekind's modular invariant. Use the Koike-Norton-Zagier identity
$$j(\sigma) - j(\tau) = p^{-1} \prod_{m=1}^{\infty} \prod_{n=-1}^{\infty} (1 - p^m q^n)^{c(mn)}$$
to express $c(6)$ in terms of $c(1)$, $c(2)$, and $c(3)$.