18.086 Problem Set 2, Solutions

Written part for # 1:

(1) Compute the dispersion relation for the equation $u_{tt} = c^2 u_{xx} - d^2 u_{xxxx}$, i.e., find how the speed of wave propagation depends on the wavenumber k of the initial conditions $u(x, 0) = e^{ikx}$.
Solution: We consider solutions of the form $u(x, t) = e^{i\omega t} e^{ikx}$, and solve for ω. Taking derivatives, we find that $u_{tt} = -\omega^2 u$, $u_{xx} = -k^2 u$, and $u_{xxxx} = k^4 u$. The differential equation implies $-\omega^2 = -c^2 k^2 - d^2 k^4$, so the dispersion relation is $\omega = \pm k\sqrt{c^2 + d^2 k^2}$. The speed of propagation is $|\omega/k| = \sqrt{c^2 + d^2 k^2}$.

(2) Let $r = \frac{c\Delta t}{\Delta x}$ and let $R = \frac{d\Delta t}{(\Delta x)^2}$. Evaluate a leapfrog method for accuracy and stability.
Solution: Leapfrog is determined by the difference approximations:

$$U_{tt}(j\Delta x, n\Delta t) = \frac{U_{j,n+1} - 2U_{j,n} + U_{j,n-1}}{(\Delta t)^2}$$
$$U_{xx}(j\Delta x, n\Delta t) = \frac{U_{j+1,n} - 2U_{j,n} + U_{j-1,n}}{(\Delta x)^2}$$
$$U_{xxxx}(j\Delta x, n\Delta t) = \frac{U_{j+2,n} - 4U_{j+1,n} + 6U_{j,n} - 4U_{j-1,n} + U_{j+2,n}}{(\Delta x)^4}$$

Using a periodic solution $U(x, n\Delta t) = G^n e^{ikx}$, we substite:

$$e^{ikj\Delta x}(G^{n+1} - 2G^n + G^{n-1})$$
$$= r^2 G^n e^{ikj\Delta x}(2\cos k\Delta x - 2) - R^2 G^n e^{ikj\Delta x}(2\cos 2k\Delta x - 8\cos k\Delta x + 6)$$

We divide by $G^{n-1}e^{ikj\Delta x}$ and let $a = 1 - r^2 - 3R^2 + (r^2 + 4R^2)\cos k\Delta x - r^2 \cos 2k\Delta x$. Then the equation becomes

$$G^2 - 2aG + 1 = 0$$

The roots of this polynomial in G have magnitude 1 if and only if $|a| \le 1$. If $|a| > 1$, then there is a root larger than one, and the numerical approximation becomes unstable. It remains to find the parameters for which $|a| < 1$. Using a double angle formula, we find that:

$$a = 1 - r^2(1 - \cos k\Delta x) - 2R^2(1 - \cos k\Delta x)^2$$

Both r^2 and R^2 are non-negative quantities, and so are the quantities inside the parentheses, so the worst possible behavior occurs when $\cos k\Delta x = -1$. In this case, we need r and R to satisfy $1 - 2r^2 - 8R^2 \ge -1$, or $r^2 + 4R^2 \le 1$. The admissible values of r and R therefore fall in an ellipse, where the r axis ranges from -1 to 1, and the R axis ranges from $-1/2$ to $1/2$.

Computational part:

(1) Suppose our constants are $c = 10$ and $d = 1$, the initial conditions are $U(x, 0) = 0$, $U_t(x, 0) = 10^6 x^5 (1 - x)^{20}$ on the interval $[0, 1]$, and boundary conditions are fixed at zero. Choose an approximation method with stable parameters (e.g., explicit u_{xx}, implicit u_{xxxx}), and plot the end state of the string at time $t = 0.05$.
Solution: Code is be on the webpage. There should be a hump that rises on the left and migrates to the right.

Written part for # 2:

(1) Note that by removing the dispersion term u_{xxx} from the KdV equation $u_t + 6uu_x + u_{xxx} = 0$, we get the inviscid Burger's equation (rescaled by a factor of 6). Explain how this difference should affect the qualitative behavior of solutions of the two equations.

Solution: The inviscid Burger's equation can form shocks from initial conditions with negative slope. KdV does not form shocks, because the dispersion term damps any rapid slope formation.

Computational part:

(1) Consider the following family of exact solutions to KdV:

$$f_c(x,t) = \frac{c}{2\cosh^2\left(\frac{\sqrt{c}}{2}(x-ct)\right)}$$

Write a program to approximate time evolution for this equation by finite differences on the interval $[-1,1]$, with periodic boundary conditions. Choose your grid and method so that the initial condition $f_{400}(x,0)$ retains its shape after several cycles (last years recommendation: at least 300 grid points, and an implicit step for the dispersion term). Run your program with the initial condition $f_{400}(x+0.7)+f_{200}(x)$. Plot the result at time $t=0.015$, and explain briefly how the solitons interact.

Solution: The taller soliton overtakes the shorter one, they merge into a shorter, smaller hill with two small humps, then the second hump grows and separates until the peaks reach the original amplitudes. Code is on the website.