18.086 Problem Set 2 due Friday, March 6, 2009.

Please hand in your solutions to the written problems in class together with plots from the computational problems, and send the code for the computational portions to the grader by email.

(1) The vibrations of a string in a piano do not precisely obey the wave equation, because the nonzero diameter of the wire introduces some resistance to bending, and this adds to the restoring force contribution from the string tension. The resulting dispersion yields inharmonicities for overtones, i.e., higher normal modes have frequencies that are non-integer multiples of the fundamental. An exact treatment would use a nonlinear formula like $u_{tt} = Ku_{xx} + \kappa u_{xx}(1 + u_x^2)^{-3/2}$, but we will consider the traditional model for this phenomenon, given by the equation

$$u_{tt} = c^2 u_{xx} - d^2 u_{xxxx}.$$

κ and d are determined by the Young's modulus of the string material and the area moment of inertia of the string cross section.

Written part:

(a) Compute the dispersion relation, i.e., find how the speed of wave propagation depends on the wavenumber k of the initial conditions $u(x, 0) = e^{ikx}$.

(b) Let $r = \frac{c\Delta t}{\Delta x}$ and let $R = \frac{d\Delta t}{(\Delta x)^2}$. Evaluate a leapfrog method for accuracy and stability.

Computational part:

(a) Suppose our constants are $c = 10$ and $d = 1$, the initial conditions are $U(x, 0) = 0$, $U_t(x, 0) = 10^6 x^5(1 - x)^{20}$ on the interval $[0, 1]$, and boundary conditions are fixed at zero. Choose an approximation method with stable parameters (e.g., explicit u_{xx}, implicit u_{xxxx}), and plot the end state of the string at time $t = 0.05$.

(2) This problem (shamelessly stolen from last year) concerns the Korteweg-de Vries equation $u_t + 6uu_x + u_{xxx} = 0$. It was originally used to study water wave phenomena in shallow channels (you can look up stories of Sir Russell riding his horse along a canal). However, its applicability has grown substantially since the late 1800s, even making an appearance in two-dimensional conformal field theory. One key fact about this equation is that the soliton

$$f_c(x, t) = \frac{c}{2\cosh^2\left(\frac{\sqrt{c}}{2}(x - ct)\right)}$$

is an exact solution to this equation, that forms a smooth pulse moving to the right at speed c.

Written part:

(a) Note that by removing the dispersion term u_{xxx}, we get the inviscid Burger's equation (rescaled by a factor of 6). Explain how this difference should affect the qualitative behavior of solutions of the two equations.

Computational part:

(a) Write a program to approximate time evolution for this equation by finite differences on the interval $[-1, 1]$, with periodic boundary conditions. Choose your grid and method so that the initial condition $f_{400}(x, 0)$ retains its shape after several cycles (last years recommendation: at least 300 grid points, and an implicit step for the dispersion term). Run your program with the initial condition $f_{400}(x + 0.7) + f_{200}(x)$. Plot the result at time $t = 0.015$, and explain briefly how the solitons interact.