18.086 Problem Set 1 Solutions

(1) 6.2 # 8. Solve $u' = -Ku$ starting from the delta vector

$$u(0) = [\texttt{zeros}(N,1); 1; \texttt{zeros}(N,1)].$$

For large sizes $n = 2N + 1 = 201$ and 2001, compare as many methods as possible for accuracy and time step Δt:

Backward Euler BDF2 Runge-Kutta `ode45` `ode15s`

Solution: The equation in this question is a model of heat flow with delta initial conditions and a conductive boundary. I have placed sample code on the web site. You should be careful to run the numerical approximation for the full length of time, since an off-by-one error in your loop parameter will yield only first order accuracy.

Results: Unless you chose your time interval to be extremely large, the errors for $N = 201$ and $N = 2001$ should be almost identical, since there is negligible heat leakage at the boundary. Additional distortions to analysis of accuracy can come from setting Δt too large (stability problems) or too small (for high-order methods like Runge-Kutta, you can run afoul of floating-point precision limits). The following error results are for $T = 10$ - other time intervals may yield different constants, but they should yield the same order.

Method	$\Delta t = 0.1$	$\Delta t = 0.05$	Accuracy
Backward Euler	$8.17 \cdot 10^{-4}$	$4.08 \cdot 10^{-4}$	$8 \cdot 10^{-3}(\Delta t)$
BDF2	$5.97 \cdot 10^{-6}$	$1.47 \cdot 10^{-6}$	$6 \cdot 10^{-4}(\Delta t)^2$
RK2	$6.99 \cdot 10^{-6}$	$1.72 \cdot 10^{-6}$	$7 \cdot 10^{-4}(\Delta t)^2$
RK4	$5.66 \cdot 10^{-10}$	$3.46 \cdot 10^{-11}$	$6 \cdot 10^{-6}(\Delta t)^4$
ode45	$8.38 \cdot 10^{-6}$	$8.38 \cdot 10^{-6}$	N/A
ode15s	$1.50 \cdot 10^{-5}$	$1.50 \cdot 10^{-5}$	N/A

The errors for `ode45` and `ode15s` are constant, because the solvers automatically adjust step size. There may be a way to override this behavior, but I haven't found it.

(2) 6.2 # 10. The semidiscrete form of $\partial u/\partial t = \partial^2 u/\partial x^2$ is a system of ordinary differential equations. Periodic boundary conditions produce the -1, 2, -1 circulant matrix C in $u' = -n^2 Cu$. Starting from $u(0) = (1 : n)/n$ test these methods for their stability limits with $n = 11$ and $n = 101$, and find the steady state $u(\infty)$ for large t:

Forward Euler Runge-Kutta Trapezoidal (15) Adams-Bashforth (17)

Solution: This is a model of heat flow with sawtooth initial conditions and periodic boundary conditions. The stability limits on Δt are as follows:

Method	$n = 11$	$n = 101$
Forward Euler	0.00421	$4.90 \cdot 10^{-5}$
RK2	0.00421	$4.90 \cdot 10^{-5}$
RK4	0.00587	$6.82 \cdot 10^{-5}$
Trapezoidal	∞	∞
Adams-Bashforth	0.00210	$2.45 \cdot 10^{-5}$

The steady state is given by the average value $u(\infty) = \frac{n+1}{2n}$. Code is on the web site.

(3) 6.4 # 3. An odd 2π-periodic sawtooth function $ST(x)$ is the integral of an even square wave $SW(x)$. Solve $u_{tt} = u_{xx}$ starting from SW and also ST with $u_t(x,0) = 0$, by a double Fourier series in x and t:

$$SW(x) = \frac{\cos x}{1} - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} \dots \qquad ST(x) = \frac{\sin x}{1} - \frac{\sin 3x}{9} + \frac{\sin 5x}{25} \dots$$

Solution: Any exponential input e^{ikx} to the wave equation $u_{tt} = u_{xx}$ evolves with time as a linear combination of $e^{ik(x+t)}$ and $e^{ik(x-t)}$, with coefficients determined by the initial time derivative. In particular, with an interval of length 2π and periodic boundary conditions, we can decompose our solution $u(x,t)$ into a double Fourier sum

$$\sum_{m,n\in\mathbb{Z}} c_{m,n} e^{imx} e^{int}$$

where $c_{m,n} = 0$ if $m \neq \pm n$. The initial condition $u_t(x,0) = 0$ implies

$$\sum_{m\in\mathbb{Z}} e^{imx}(imc_{m,m} + (-im)c_{m,-m}) = 0$$

By orthogonality of exponentials, $c_{m,m} = c_{m,-m}$ for all m. In particular, our initial signal is divided into two equal waves travelling in opposite directions. If $u(x,0) = SW(x)$, then $u(x,t) = \frac{SW(x+t)+SW(x-t)}{2}$. If $u(x,0) = ST(x)$, then $u(x,t) = \frac{ST(x+t)+ST(x-t)}{2}$.

(4) 6.4 #4. Draw the graphs of $SW(x)$ and $ST(x)$ for $|x| \leq \pi$. If they are extended to be 2π-periodic for all x, what is the d'Alembert solution $u_{SW} = \frac{1}{2}SW(x+t) + \frac{1}{2}SW(x-t)$? Draw its graph at $t = 1$ and $t = \pi$, and similarly for ST.

Solution: I have given MATLAB code for plotting. The implementation looks a bit odd, because the textbook and MATLAB definitions of square wave and sawtooth function differ. Even after the obvious adjustments, we need to scale amplitudes by constant multiples:

$$\pi/4 = 1 - 1/3 + 1/5 - \dots$$

and

$$\pi^2/8 = 1 + 1/9 + 1/25 + \dots$$

(5) 6.4 #5. Solve the wave equation $u_{tt} = u_{xx}$ by the leapfrog method (14) starting from rest with $u(x,0) = SW(x)$. Periodic boundary conditions replace u_{xx} by the second difference circulant $-CU/(\Delta x)^2$. Compare with the exact solution in Problem 4 at $t = \pi$, for CFL numbers $\Delta t/\Delta x = 0.8, 0.9, 1.0, 1.1$.

Solution: CFL numbers 0.8 and 0.9 yield some limited high-frequency oscillation arising from the discontinuity. CFL number 1.0 will in general produce high-frequency oscillation with amplitude comparable to the input size. However, when the time interval has length a multiple of π, it is possible to arrange parameters so that all of the oscillations cancel out at the end (You should not think of this as a good general strategy). CFL number 1.1 explodes. Code is on the website.