Notes from Cheewhye's seminar on Langlands correspondence for GL_r over global function fields

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Preface

Cheewhye Chin gave a one year long seminar at Berkeley, on Langlands Correspondence for GL_r over global function fields, leading up to Lafforgue's proof. Since we were had limited time, he focused on precise definitions and precise statements of theorems, and there are few proofs. These notes are incomplete, and I didn't understand some of what was said during the seminar. If the notes get really sketchy somewhere, it usually indicates that Arthur Ogus started some sort of rapid-fire technical digression.

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Principal references:

1. L. Lafforgue. Chtoucas de Drinfeld et Correspondence de Langlands. Invent. Math. 147 (2002) no. 1 pp1-241.

2. L. Lafforgue. Une compactification des champs classifiant les chtoucas de Drinfeld. J. AMS 11 (1998) no. 4 pp1001-1036.

3. L. Lafforgue. Chtoucas de Drinfeld et conjecture de Ramanujan-Petersson. Asterisque 243 (1997) ii+329 pages.

Note that this is just over 600 pages of rather hard mathematics.

Let F be a function field of characteristic p, i.e., some finite extension of $\mathbb{F}_q(t)$. Let $\mathbb{A}$ be its ring of adèles, and let $|F|$ be the set of its places.

Spectral World

Choose an algebraic closure $\mathbb{C}$ of $\mathbb{R}$. For $r \geq 1$ let

$$\mathcal{A}^r(F,\mathbb{C}) := \left\{ \begin{array}{r} \text{isomorphism classes of cuspidal automorphic} \\ \text{irreducible complex representations of } GL_r(\mathbb{A}) \\ \text{whose central character has finite order} \end{array} \right\}$$

Definition An **automorphic function on** $GL_r(\mathbb{A})$ is a function $f : GL_r(\mathbb{A}) \to \mathbb{C}$ satisfying:

1. $f(\gamma x) = f(x)$ for all $x \in GL_r(\mathbb{A})$, $\gamma \in GL_r(F) \underset{\text{discrete}}{\subset} GL_r(\mathbb{A})$.

2. There exists an open compact $K \subset GL_r(\mathbb{A})$ such that $f(xk) = f(x)$ for all $x \in GL_r(\mathbb{A})$, $k \in K$.

3. The subrepresentation $GL_r(\mathbb{A}) \cdot f \subset C(GL_r(\mathbb{A}), \mathbb{C})$ of the right regular representation is admissible.

Definition f is called **cuspidal** if and only if for any proper parabolic subgroup $P \subset GL_r$ and any $x \in GL_r(\mathbb{A})$,

$$\int_{N_P(F)\backslash N_P(\mathbb{A})} f(nx)dn = 0,$$

for any Haar measure dn, where N_P denotes the unipotent radical of P.

Let $\mathcal{A}_0(GL_r(\mathbb{A}), \mathbb{C})$ be the complex vector space of all cuspidal automorphic functions on $GL_r(\mathbb{A})$. Any irreducible subquotient of $\mathcal{A}_0(GL_r(\mathbb{A}), \mathbb{C})$ as a representation of $GL_r(\mathbb{A})$ is called **cuspidal automorphic irreducible**. The set of isomorphism classes of these is written $\mathcal{A}^r(F, \mathbb{C})$.

For any $\pi \in \mathcal{A}^r(F, \mathbb{C})$, let $S_\pi \subset |F|$ be the finite set of ramified places. For all $x \in |F| - S_\pi$, let $\{z_1(\pi_x), \ldots, z_r(\pi_x)\} \subset \mathbb{C}$ be the **Satake parameters** (also called **Hecke eigenvalues**) of π at x. This is an unordered multiset of r complex numbers.

Algebraic World

Choose a separable algebraic closure $\overline{F}$ of F. We denote by $Gal(\overline{F}/F)$ the group of automorphisms of $\overline{F}$ fixing F. Choose a prime $l \neq p$. Choose an algebraic closure $\overline{\mathbb{Q}_l}$ of $\mathbb{Q}_l$. For $r > 1$, let

$$\mathcal{G}^r(F, \overline{\mathbb{Q}_l}) := \left\{ \begin{array}{r} \text{isomorphism classes of continuous irreducible representations} \\ \text{of } Gal(\overline{F}/F) \text{ on an } r\text{-dimensional } \overline{\mathbb{Q}_l} \text{ vector space almost} \\ \text{everywhere unramified, with determinant of finite order} \end{array} \right\}$$

The determinant is a character, so when we say it has finite order, this means some tensor power is the trivial character. For any $\sigma \in \mathcal{G}^r(F, \overline{\mathbb{Q}_l})$, let $S_\sigma \subset |F|$ be the finite set of ramified places. For any $x \in |F| - S_\sigma$, let $\{z_1(\sigma_x), \ldots, z_r(\sigma_x)\} \subset \overline{\mathbb{Q}_l}$ be the unordered multiset of r **Frobenius eigenvalues** of σ at x.

Let $\iota : \overline{\mathbb{Q}_l} \xrightarrow{\sim} \mathbb{C}$ be an isomorphism of fields. This exists by Zorn's lemma, and is used in e.g. Deligne's proof of the Weil conjectures.

Definition One says that $\pi \in \mathcal{A}^r(F, \mathbb{C})$ and $\sigma \in \mathcal{G}^r(F, \overline{\mathbb{Q}_l})$ are in **Langlands correspondence with respect to** ι if and only if for all $x \in |F| - S_\pi - S_\sigma$, $\{z_1(\pi_x), \ldots, z_r(\pi_x)\} = \iota(\{z_1(\sigma_x), \ldots, z_r(\sigma_x)\})$.

Theorem (L. Lafforgue, Langlands correspondence for GL_r over function fields) For any $\iota : \overline{\mathbb{Q}_l} \xrightarrow{\sim} \mathbb{C}$ and any $r \geq 1$, the relation "Langlands correspondence with respect to ι" defines a bijection $\mathcal{A}^r(F,\mathbb{C}) \overset{\sim}{\longleftrightarrow} \mathcal{G}^r(F,\overline{\mathbb{Q}_l})$. Moreover, when σ and π are in Langlands correspondence with respect to ι, we have $S_\pi = S_\sigma \subset |F|$.

More precisely, for any $\pi \in \mathcal{A}^r(F,\mathbb{C})$, there exists a unique $\sigma \in \mathcal{G}^r(F,\overline{\mathbb{Q}_l})$ such that σ and π are in Langlands correspondence with respect to ι, and for any $\sigma \in \mathcal{G}^r(F,\overline{\mathbb{Q}_l})$ there exists a unique $\pi \in \mathcal{A}^r(F,\mathbb{C})$ such that σ and π are in Langlands correspondence with respect to ι.

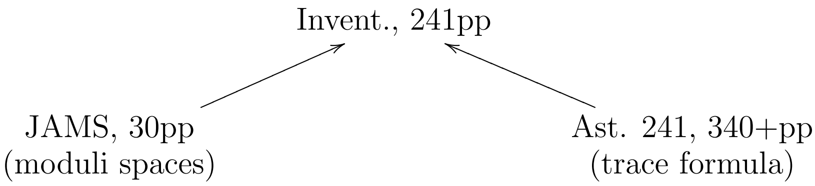
The uniqueness claims have been known for a while. Čebotarev density implies uniqueness in $\mathcal{G}^r(F,\overline{\mathbb{Q}_l})$, and Piatetski-Shapiro's strong multiplicity one theorem implies uniqueness in $\mathcal{A}^r(F,\mathbb{C})$. Existence is proved by induction. The initial case $r = 1$ is just class field theory for F. In fact, Professor Lang made some key contributions to our understanding of this in the early part of the last century. [Everyone laughs. Serge looks completely unperturbed. Vojta asks: why can't we induct from $r = 0$? There is an obvious correspondence between representations of the trivial group and trivial Galois representations.] You can't induct from $r = 0$, because you really need class field theory to power the induction. Beyond $r = 1$, we have:

Deligne's induction machine The goal here is to build a bridge between two worlds, using another two worlds.

Lafforgue's moduli of shtukas, truncations and compactifications, Arthur-Selberg trace formula	(moduli spaces, Drinfeld's shtukas)	Grothendieck's theory of l-adic cohomology, Grothendieck-Lefshetz-Verdier trace formula, Deligne's purity from Pink and Fujiwara
(cuspidal automorphic representations)	Geometric ↗ ↘ Spectral Algebraic ↖ ↙ Analytic	(Galois representations)
Piatetski-Shapiro's converse theorem, Rankin-Selberg convolutions from Jacquet, Piatetski-Shapiro, and Shalika	(L-functions, ϵ-factors)	Deligne's theory of weights (Weil II), product formula, Grothendieck's and Laumon's cohomological interpretation of L-functions and ϵ-factors

Pink's proof used resolution of singularities, but could be extended to stacks. Fujiwara did not use resolution, but his proof required schemes.

Here's a literature diagram:



[Serge asks, "Why isn't the inverse Galois problem solved, then? For function fields, there is no problem at infinity!"]

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Recall the main theorem (Langlands correspondence for GL_r over function fields): For any $r \geq 1$ and any $\iota : \overline{\mathbb{Q}_l} \xrightarrow{\sim} \mathbb{C}$, the relation of Langlands correspondence with respect to ι defines a bijection $\mathcal{A}^r(F, \mathbb{C}) \xleftrightarrow{\sim} \mathcal{G}^r(F, \overline{\mathbb{Q}_l})$.

We can squeeze more juice out of this.

Theorem (Deligne's independence of l conjecture in Weil II for curves over finite fields) Let X/k be a smooth curve over a finite field k of characteristic p. Let $l \neq p$ be a prime, and $\overline{\mathbb{Q}_l}$ an algebraic closure of $\mathbb{Q}_l$. Let $\mathcal{L}$ be a lisse $\overline{\mathbb{Q}_l}$-sheaf on X (meaning an l-adic representation of $Gal(K(X))$) that is irreducible, and has finite order determinant (this is a normalization condition, making the result easier to state). Then:

1. There exists a number field $E \subset \mathbb{Q}_l$ such that $\mathcal{L}$ is E-rational, i.e., for any $x \in |X|$, the polynomial $det(1 - T \cdot Frob_X; \mathcal{L}) \in \overline{\mathbb{Q}_l}[T]$ has coefficients in E.
2. $\mathcal{L}$ is:
   (a) pure of weight 0
   (b) plain of characteristic p
   (c) C-bounded in valuation, where we can take $C \leq \frac{(r-1)^2}{r}$, with r is the rank of $\mathcal{L}$
   (d) D-bounded in denominators for some $D \in \mathbb{Z}_{>0}$
3. There exists a finite extension E' of E in $\overline{\mathbb{Q}_l}$ such that for any $\lambda \in |E'|_{\neq p}$ (giving us a completion $E_\lambda{}'$), there exists an absolutely irreducible lisse $E_\lambda{}'$-sheaf $\mathcal{L}_\lambda$ on X such that $\mathcal{L}$ and $\mathcal{L}'$ are E-**compatible**, i.e., for any $x \in |X|$, $det(1 - T \cdot Frob_x|\mathcal{L}) = det(1 - T \cdot Frob_x|\mathcal{L}_\lambda)$ in $E[T]$. One can say "$\mathcal{L}$ extends to an E-compatible system". This is Deligne's original conjecture.

Definition For any $x \in |X|$ and any eigenvalue $\alpha \in \overline{\mathbb{Q}_l}$ of $Frob_X$ on $\mathcal{L}$,

- (2a) $\Leftrightarrow$ for any archimedean valuation $|\cdot| \in |E(\alpha)|_\infty$, $|\alpha| = 1 \neq q^\alpha$
- (2b) $\Leftrightarrow$ for any $\lambda \notin |E(\alpha)|_p$, $\lambda(\alpha) = 0$, where $\lambda : E(\alpha)^\times \to \mathbb{Z}$ is a nonarchimedean non-p-adic valuation.
- (2c) $\Leftrightarrow$ for any p-adic valuation $v \in |E(\alpha)|_p$, $\left|\frac{v(\alpha)}{v(\#X(\alpha))}\right| \leq C$
- (2d) $\Leftrightarrow$ for any p-adic valuation $v \in |E(\alpha)|_p$, $\frac{v(\alpha)}{v(\#X(\alpha))} \in \frac{1}{D}\mathbb{Z}$

Statements 1 and 2 are known for arbitrary normal varieties over finite fields.

Let $\overline{\eta} \to X$ be a geometric point, so we get $\pi_1(X, \overline{\eta})$. For any l-adic field Λ (i.e., $\mathbb{Q}_l$, some finite extension, $\overline{\mathbb{Q}_l}$), we get an equivalence of categories:

$$\left\{ \begin{array}{c} \text{lisse } \Lambda\text{-sheaves} \\ \text{on } X \end{array} \right\} \overset{\sim}{\to} \left\{ \begin{array}{c} \text{finite dimensional continuous} \\ \Lambda\text{-representations of } \pi_1(X, \overline{\eta}) \end{array} \right\}$$
$$\mathcal{L} \mapsto \mathcal{L}_{\overline{\eta}}, \pi_1(X, \overline{\eta}) \overset{[\mathcal{L}]}{\to} GL(\mathcal{L}_{\overline{\eta}})$$

$[\mathcal{L}]$ is the **monodromy representation** of $\mathcal{L}$. [Check out www.monodromy.com]

Definition The **(arithmetic) monodromy group** of $\mathcal{L}$ is $G_{arith}(\mathcal{L}, \overline{\eta}) :=$ Zariski closure of the image of $[\mathcal{L}]$ in $GL(\mathcal{L}, \overline{\eta})$. This is a Λ-algebraic group.

Conjecture (Independence of l of monodromy groups) Let $L = \{\mathcal{L}_\lambda\}_{\lambda \in |E|_{\neq p}}$ be an E-compatible system. Assume L is

- arithmetically semisimple (i.e., for any $\lambda \in |E|_{\neq p}$, $\mathcal{L}_\lambda$ is semisimple).
- pure of weight $w \in \mathbb{Z}$.

Then there exists an algebraic group G over E such that for any $\lambda \in |E|_{\neq p}$, $G \otimes_E E_\lambda \cong G_{arith}(\mathcal{L}_\lambda, \overline{\eta})$ as an algebraic group over E_λ.

There is also a variant, in which we allow ourselves to replace E by a finite extension.

For each $\lambda \in |E|_{\neq p}$, we get a short exact sequence:

$$1 \to G_{arith}(\mathcal{L}_\lambda)^0 \to G_{arith}(\mathcal{L}_\lambda) \to \Gamma_{arith}(\mathcal{L}_\lambda) \to 1$$

where $G_{arith}(\mathcal{L}_\lambda)^0$ is connected and reductive, $G_{arith}(\mathcal{L}_\lambda)$ is reductive, and $\Gamma_{arith}(\mathcal{L}_\lambda)$ is finite. (We also define $G^0_{geom} := [G^0_{arith}, G^0_{arith}]$.)

Theorem 1 (Serre, letter to Ribet, 1981) With notation and hypotheses as above, $\Gamma_{arith}(\mathcal{L}_\lambda)$ is independent of λ. More precisely, for any λ,

$$ker(\pi_1(X, \overline{\eta}) \overset{[\mathcal{L}_\lambda]}{\to} G_{arith}(\mathcal{L}_\lambda) \twoheadrightarrow \Gamma_{arith}(\mathcal{L}_\lambda))$$

is the same open subgroup of $\pi_1(X, \overline{\eta})$.

Theorem 2 With notation and hypotheses as above, after replacing E by a suitable finite extension, there exists a connected algebraic group G_0 over E such that for any $\lambda \in |E|_{\neq p}$, $G_{arith}(\mathcal{L}_\lambda)^0 \cong G_0 \otimes_E E_\lambda$.

blah blah motives should form a Tannakian category over X. wtf??

Lafforgue gave via Langlands correspondence for GL_r a bijection between automorphic irreducible complex representations π of $GL_r(\mathbb{A}_F)$ and E-compatible systems of l-adic lisse sheaves $L = \{\mathcal{L}_\lambda\}_{\lambda \in |E|_{\neq p}}$ on X, semistable of weight $w \in \mathbb{Z}$. This gives **"the"** monodromy group G_0. [F is the function field of X.]

Conjecture Let H be the F-reductive group whose Langlands dual group is G_0. Then there exists an automorphic representation π_0 of $H(\mathbb{A})$ such that π is a weak functorial lift of π_0.

The disconnected case is an extension problem - we get a split sequence in the case of trivial center. It's hard with a big center.

Proof Steps of theorem 2:

One: Apply Serre's theorem. We may assume all G_{arith}s are connected.

Two: Use Serre's theory of Frobenius tori:

Theorem (Serre, letter to Ribet, 1981) If a lisse l-adic sheaf $\mathcal{L}$ is

1. E-rational for some number field E
2. (a) pure of weight $w \in \mathbb{Z}$
   (b) plain of characteristic p
   (c) C-bounded in valuations for some C
   (d) D-bounded in denominators for some D

Then there exist infinitely many $x \in |X|$ such that $[\mathcal{L}](Frob_x)^{s.s.\mathbb{Z}} \subset G_{arith}(\mathcal{L})$ is a **maximal torus**.

Three Pick $x \in |X|$ as in step 2. Let $T_\lambda := [\mathcal{L}_\lambda](Frob_x^{\mathcal{L}})^{\mathbb{Z}}$ be a maximal torus in $G_{arith}(\mathcal{L}_\lambda)$. Use Lafforgue to show that one has a commutative diagram ["2" and "3" denote **distinct primes**, that are not necessarily 2 and 3]:

$$\begin{array}{ccc}
Irr(G_{arith}(\mathcal{L}_2)) & \xrightarrow[\text{Lafforgue}]{\cong \text{ as sets}} & Irr(G_{arith}(\mathcal{L}_3)) \\
\text{basis} \big\downarrow & & \big\downarrow \text{basis} \\
K(G_{arith}(\mathcal{L}_2)) & \xrightarrow{\cong \text{ as ab. groups}} & K(G_{arith}(\mathcal{L}_3)) \\
\big\downarrow & & \big\downarrow \\
K(T_2) & \xrightarrow{\cong \text{ as rings}} & K(T_3) \\
\big\uparrow & & \big\uparrow \\
X(T_2) & \xrightarrow{\cong \text{ as free ab. gps.}} & X(T_3)
\end{array}$$

This reduces the problem to representation theory of reductive groups. Thus, one can conclude that there exists an isomorphism:

$$f : (G_{arith}(\mathcal{L}_2) \otimes \mathbb{C}, T_2 \otimes \mathbb{C}) \xrightarrow{\sim} (G_{arith}(\mathcal{L}_3) \otimes \mathbb{C}, T_3 \otimes \mathbb{C}).$$

End of pep talk.

Smooth representations and admissible representations

Let G be a topological group. Choose an algebraic closure $\mathbb{C}$ of $\mathbb{R}$.

Definition A **complex representation of** G is a pair (V, π), where V is a complex vector space (of arbitrary dimension), and $\pi : G \to GL(V) = Aut_{\mathbb{C}}(V)$ is a homomorphism (with no assumptions on continuity).

We get a complex-linear abelian category $Rep(G) := Rep(G; \mathbb{C})$. We will ignore the tensor structure.

Definition A complex representation (V, π) of G is **smooth** if and only if for all $v \in V$, $Stab_G(v) := \{g \in G | \pi(g)v = v\}$ is an **open** subgroup of G.

We get $Rep_{smooth}(G)$, a full subcategory of $Rep(G)$ whose objects are smooth. This subcategory is closed under taking subquotients and finite direct sums, so it is abelian, inclusion is exact, and notions of irreducibility, subquotients, $End(V, \pi)$, etc. **make sense**.

However, it is **not** closed under extensions. For example the representation $\mathbb{R} \to GL_2(\mathbb{C})$ given by

$$x \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

is not smooth.

Let $Z \subset G$ denote the center of G.

Definition $(V, \pi) \in Rep_{smooth}(G)$ admits a **central (quasi-)character** if and only if for any $z \in Z$, $\pi(z) = \chi_\pi(z) \cdot id_V \in GL(V)$ for some $\chi_\pi(z) \in \mathbb{C}^\times$.

In this case, the map $\chi_\pi : Z \to \mathbb{C}^\times$ defined by $z \mapsto \chi_\pi(z)$ is a homomorphism, called the **central (quasi-)character**, and it is trivial on some open subgroup of Z.

Note that we get a commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{\pi} & GL(V) \\ \uparrow & & \uparrow \\ Z & \xrightarrow{\chi_\pi} & \mathbb{C}^\times \end{array}$$

Lemma (Jacquet-Schur Lemma) Assume G has a countable basis of open neighborhoods. Let $(V, \pi) \in Rep_{smooth}(G)$ be irreducible. Then the map $C \to End_{Rep(G)}(V, \pi)$ given by $\lambda \mapsto \lambda \cdot id_V$ is **surjective**. In particular, (V, π) admits a central character.

Proof Pick any nonzero $v \in V$. Let $H := Stab_\pi(v) \underset{\text{open}}{\subset} G$. Then V as a complex vector space is spanned by $\{\pi(g) \cdot v | g \in G/H\}$, which is a countable set by hypothesis. Suppose $u \in End_G(V, \pi)$ is not a scalar, i.e., there is no $\lambda \in \mathbb{C}$ such that $u = \lambda \cdot id_V$. Then for any $\lambda \in \mathbb{C}^\times$, $R_\lambda := (u - \lambda \cdot id_V)^{-1}$ exists . It suffices to show what? that $\{R_\lambda(v) \in V, \lambda \in \mathbb{C}\}$ is $\mathbb{C}$-linearly independent (by counting). It suffices to show that for any $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ distinct and any coefficients $a_1, \ldots, a_n \in \mathbb{C}^\times$, the sum $\sum a_i R_{\lambda_i} \in End(V)$ is invertible. From the definition of R_λ, we can write

$$\sum a_i R_{\lambda_i} = P(u) \prod_{i=1}^{n} R_{\lambda_i}, \text{ where } P(T) = \sum_{i=1}^{n} a_i \prod_{i \neq j} (T - \lambda_i) \in \mathbb{C}[T].$$

Now, we factorize: $P(T) = a \prod (T - \alpha_i)$ for some nonzero a, so it doesn't vanish if we plug in u.

Definition A representation (V, π) of G is **admissible** if

1. it is smooth
2. for any compact open subgroup $K \subset G$, $V^K := \{v \in V | k \cdot v = v\}$ is finite dimensional over $\mathbb{C}$.

We get a full abelian subcategory $Rep_{adm}(G) \subset Rep_{smooth}(G)$, and inclusion is exact. $Rep_{adm}(G)$ is closed under subquotients, finite direct sums, and extensions. Furthermore, admissible representations are dualizable via smooth contragradient.

Now, we have three categories, each "more finite" than the next: $Rep_{adm}(G) \subset Rep_{smooth}(G) \subset Rep(G)$. This is analogous to the sequence "coherent and locally free sheaves," "quasi-coherent sheaves," "arbitrary abelian sheaves."

Automorphic representations, cuspidal automorphic representations

Let F be a function field, $\mathbb{A}$ its ring of adèles. Let $r \geq 1$, and consider the group $GL_r(\mathbb{A})$.

Let $C(G, \mathbb{C}) = \{\text{continuous functions } \phi : G \to \mathbb{C}\}$, where $\mathbb{C}$ has its usual topology. Then we have $R_{reg} : G \to GL(C(G, \mathbb{C}))$, the regular representation, given by $g \mapsto (\phi \mapsto (x \mapsto \phi(xg)))$ (You should be familiar with this notation if you've ever seen lambda calculus). Then $(C(G, \mathbb{C}), R_{reg}) \in Rep(G)$.

Definition An **automorphic form** (or **function**) for $GL_r(\mathbb{A})$ is a function $\phi \in C(GL_r(\mathbb{A}), \mathbb{C})$ such that

1. (automorphic) for any $\gamma \in GL_r(F) \underset{\text{discrete}}{\subset} GL_r(\mathbb{A})$ and for any $x \in GL_r(\mathbb{A})$, $\phi(\gamma x) = \phi(x)$.
2. (uniformly locally constant) there is an open compact subgroup $K \subset GL_r(\mathbb{A})$ such that for any $k \in K$ and any $x \in GL_r(\mathbb{A})$, $\phi(xk) = \phi(x)$.
3. (admissible) The subrepresentation $R_{reg}(G)\phi \subset (C(GL_r(\mathbb{A}), \mathbb{C}), R_{reg})$ is admissible.

One can also have a "moderate growth" condition.

Note: The first condition is a **global** condition. It is easy to get a representation, but the $GL_r(F)$ sits diagonally and discretely in $GL_r(\mathbb{A})$, and imposes strong global conditions on ϕ.

Let $\mathcal{A}(GL_r(\mathbb{A}), \mathbb{C}) := \{\phi \in C(GL_r(\mathbb{A}), \mathbb{C}) | \phi \text{ automorphic form}\}$. Let R_{aut} be the restriction of R_{reg} to $\mathcal{A}(GL_r(\mathbb{A}))$. Then $(\mathcal{A}(GL_r(\mathbb{A})), R_{aut}) \in Rep_{smooth}(GL_r(\mathbb{A}))$.

Definition An automorphic irreducible complex representation of $GL_r(\mathbb{A})$ is a complex representation (V, π) of $GL_r(\mathbb{A})$ that is irreducible and isomorphic to a subquotient of $(\mathcal{A}(GL_r(\mathbb{A})), R_{aut})$.

Lemma An automorphic irreducible representation (V, π) of $GL_r(\mathbb{A})$ is automatically admissible.

"Proof" Let $V_1 \subset V_2 \subset \mathcal{A}$ in Rep_{smooth} with V_2/V_1 irreducible, and $\phi \in V_2 \setminus V_1$. Quotients of admissible representations are admissible. fix me

Why are we interested in decomposing function spaces like $(\mathcal{A}(GL_r(\mathbb{A})), R_{aut})$? To answer this, we go back to complex irreducible representations of $\mathbb{R}$. This is an abelian group, so its unitary irreducible representations correspond to points in S^1. Consider the exact sequence:

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$$

Note that $\mathbb{Z}$ is discrete, $\mathbb{R}$ is locally compact, and $\mathbb{R}/\mathbb{Z}$ is compact. By Pontryagin duality, this sequence is self-dual. and?

Definition An automorphic form $\phi \in \mathcal{A}$ is **cuspidal** if and only if for any **proper** parabolic subgroup $P \subset GL_r$ over F and any $x \in GL_r(\mathbb{A})$,

$$\int_{N_P(F)\backslash N_P(\mathbb{A})} \phi(nx)dn = 0$$

where N_P is the unipotent radical of P, and dn is some Haar measure on $N_P(\mathbb{A})$. Ideally one would divide dn by the counting measure on the discrete part, but the vanishing condition works with any choice of Haar measure. It suffices to check vanishing for P running over a set of representatives of F-conjugacy classes of maximal proper parabolics.

(NB: A subgroup is **parabolic** if the quotient is a proper variety over the base field. In particular, for GL_r, the set of parabolic subgroups is the set of all conjugates of block upper triangular matrices. A **unipotent** group is an iterated extension of additive groups, and the **unipotent radical** N_P of a linear algebraic group P is the unique maximal normal unipotent subgroup. The **Levi quotients** P/N_P in this case are the conjugates of block diagonal matrices.)

Let $G = GL_r(\mathbb{A})$, $\mathcal{A}_{cusp}(G) := \{\phi \in \mathcal{A}(G) | \phi \text{ cuspidal}\}$, and R_{cusp} the restriction of R_{aut} to $\mathcal{A}_{cusp}$. We get $(\mathcal{A}_{cusp}(G), R_{cusp}) \in Rep_{smooth}(G)$.

Definition A **cuspidal automorphic irreducible complex representation of** $GL_r(\mathbb{A})$ is an irreducible subquotient of $(\mathcal{A}_{cusp}, R_{cusp})$.

Let $Z \subset GL_r$ be the center. $Z \cong \mathbb{G}_m$ and is given by the diagonal matrices. Let $\chi : Z(\mathbb{A}) \to \mathbb{C}^\times$ be a (quasi-)character (not necessarily unitary) that is trivial on $Z(F)$, i.e., χ factors through $Z(F)\backslash Z(\mathbb{A})$. Let $\mathcal{A}_{cusp}(G;\chi) := \{\phi \in \mathcal{A}_{cusp}(G) | \forall z \in Z(\mathbb{A}), \forall x \in GL_r(\mathbb{A}), \phi(zx) = \chi(z)\phi(x)\}$. Choose a set of representatives of isomorphism classes of cuspidal automorphic irreducible representations of $GL_r(\mathbb{A})$ shose central character is χ. Call this $\mathcal{A}^r(F, \mathbb{C}; \chi)$.

Theorem (Gelfand, Piatetski-Shapiro)

1. The set $\mathcal{A}^r(F, \mathbb{C}; \chi)$ is countable.
2. For each $(V, \pi) \in \mathcal{A}^r(F, \mathbb{C}; \chi)$, the multiplicity
$$M_{cusp}(\pi) := dim_{\mathbb{C}} Hom_{Rep(G)}((V, \pi), \mathcal{A}_{cusp}(GL_r(\mathbb{A}); \chi))$$
is finite and nonzero.
3. One has a noncanonical isomorphism
$$(\mathcal{A}_{cusp}(GL_r(\mathbb{A}); \chi), R_{cusp}) \cong \bigoplus_{\mathcal{A}^r(F,\mathbb{C};\chi)} (V, \pi)^{\oplus M_{cusp}(\pi)}$$

4. $(\mathcal{A}_{cusp}(GL_r(\mathbb{A});\chi), R_{cusp})$ is an admissible representation.

5. For any open compact subgroup $K \subset GL_r(\mathbb{A})$, there exist only finitely many $(V,\pi) \in \mathcal{A}^r(F,\mathbb{C};\chi)$ such that $V^K \neq 0$.

The open compact subgroups are conjugate to those of the form

$$K = ker(GL_r(\mathcal{O}_{\mathbb{A}}) \to GL_r(\mathcal{O}_F/(\text{nontrivial ideal}))).$$

Condition 5 says that for a fixed level of ramification, there are only finitely many representations with that level. Mordell's theorem gives a stronger statement for number fields. ?

Another way to look at condition 3 is that the cuspidal spectrum lies inside the discrete spectrum, which lies inside $\mathcal{A}(G) \cong \mathcal{A}_{disc} \oplus \mathcal{A}_{cts}$. The continuous spectrum $\mathcal{A}_{cts}$ is spanned by Eisenstein series.

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Restricted tensor products of representations

Let $|X|$ be a set (think of the set of places of a function field or closed points on a curve). For each $x \in |X|$, let G_x be a topological group that is locally compact, totally disconnected (every neighborhood of 1 has an open compact subgroup), and unimodular (every left Haar measure is right-invariant).

We want good information on representations of Hecke algebras. Let $K_x \subset G_x$ be an open compact subgroup, e.g. $G_x = GL_r(F_x)$, $K_x = GL_r(\mathcal{O}_x)$. For each finite subset $S \subset |X|$, let $G_S := \prod_{x\in S} G_x$, a locally compact group, and $K^S := \prod_{x\in|X|\setminus s} K_x$ a compact group. The subscripts and superscripts are French notation, describing the support of a group.

Definition The **restricted product** $\prod_{x\in|X|}(G_x,K_x) := \varinjlim_{S\subset|X| \text{ finite}}(G_S \times K^S)$, where the limit is taken in the category of locally compact Hausdorff totally disconnected unimodular topological groups. Transition homomorphisms are induced for all $S \subset S' \underset{\text{finite}}{\subset} |X|$ by inclusions $K^{S'\setminus S} \hookrightarrow G_{S'\setminus S}$.

Lemma $\prod_{x\in|X|}(G_x,K_x)$ exists, and contains $\prod_{x\in|X|} K_x$ as an open compact subgroup.

Construction Choose an algebraic closure $\mathbb{C}$ of $\mathbb{R}$. For each $x \in |X|$, let $(V_x,\pi_x) \in Rep_{smooth}(G_x,\mathbb{C})$. Assume that for almost all $x \in |X|$, one has $dim_{\mathbb{C}}(V_x^{K_x}) = 1$, and choose $S_0 \subset |X|$ such that this condition is satisfied for all $x \in |X| \setminus S_0$. $|X| \setminus S_0$ is called the "unramified set". For each $x \in |X| \setminus S_0$, choose a distinguished vector $\overset{0}{v}_x \in V_x^{K_x} \setminus \{0\}$. For each finite subset $S \subset |X|$ containing S_0, we get

$$(V_S := \bigotimes_{x\in S} V_x, \pi_S := \bigotimes_{x\in S} \pi_x) \in Rep_{smooth}(G_S)$$

Definition The **restricted tensor product** $\bigotimes^{res}_{x\in|X|}(V_x,\pi_x,(\overset{0}{v}_x)_{x\in|X|\setminus S_0})$ is the object $(V,\pi)\in Rep(\prod_{x\in|X|}(G_x,K_x))$, where $V=\varinjlim_{S_0\subset S\underset{\text{finite}}{\subset}|X|}V_S$ as a complex representation, with transitions given for all S,S' satisfying $S_0\subset S\subset S'\underset{\text{finite}}{\subset}|X|$ by $\bigotimes_{x\in S}V_x\to\bigotimes_{x\in S'}V_x$ via $(\bigotimes_{x\in S}V_x)\otimes(\bigotimes_{x\in S'\setminus S}\overset{0}{v}_x)$. fix me

- for any $v_S\in V_S$, denote its image in V by $v_S\otimes\overset{0}{v}^S$
- for any finite $S\subset|X|$, any $g_S\in G_S$, $k^S\in K^S$ (giving $g_S\times k^S\in\prod_{x\in|X|}(G_x,K_x)$), and any $v_S\in V_S$ (giving $v_S\otimes\overset{0}{v}^S\in V$), set $\pi(g_S\times k^S)(v_S\otimes\overset{0}{v}^S):=(\pi_S(g_S)v_S\otimes\overset{0}{v}^S\in V$.

Lemma (V,π) is well defined, and an object in $Rep_{smooth}(\prod_{x\in|X|}(G_x,K_x))$.

Lemma If $S_\infty\subset|X|$ with $(\overset{\infty}{v}_x)_{x\in|X|\setminus S_\infty}$ is another choice, then

$$\bigotimes_{x\in|X|}^{res}(V_x,\pi_x,(\overset{0}{v}_x)_{x\in|X|\setminus S_0})\underset{\text{canonically}}{\cong}\bigotimes_{x\in|X|}^{res}(V_x,\pi_x,(\overset{\infty}{v}_x)_{x\in|X|\setminus S_\infty})$$

We may assume $S_0\subset S_\infty$ or vice versa.

Proposition If every (V_x,π_x) is $\left\{\begin{array}{c}\text{admissible}\\ \text{admissible and irreducible}\end{array}\right\}$, then the same is true for $\bigotimes^{res}_{x\in|X|}(V_x,\pi_x)$.

This is a way to cook up admissible irreducible representations of a restricted product. Unfortunately, the automorphic part is harder. (Someone asks, "Does every admissible irreducible representation come from such a product?" Cheewhye says, "Yes for $GL_r(\mathbb{A})$, given certain hypotheses.")

Theorem (Flath) Suppose for almost all $x\in|X|$, the Hecke algebra $e_{K_x}*\mathcal{H}(G_x)*e_{K_x}$ is commutative (N.B. By Satake's isomorphism, this is true for $GL_r(F_x),GL_r(\mathcal{O}_x)$). Let $(V,\pi)\in Rep_{adm}(G)$ be irreducible. then for all $x\in|X|$, there exists $(V_x,\pi_x)\in Rep_{adm}(G_x)$ irreducible, such that for almost all $x\in|X|$, $dim_{\mathbb{C}}(V_x^{K_x})=1$, and such that $(V,\pi)=\bigotimes^{res}_{x\in|X|}(V_x,\pi_x)\in Rep_{adm}(G)$. If for all $x\in|X|$, $(V'_x,\pi'_x)\in Rep_{adm}(G_x)$ with the same properties, then for all $x\in|X|$, $(V_x,\pi_x)\cong(V'_x,\pi'_x)$ in $Rep_{adm}(G_x)$.

The unramified principal series for GL_r(local field)

Let F be a non-archimedean local field, let k be its residue field, $p=char(k)$, $q=\#k$. By normalizing the valuation, we get an exact sequence

$$1\to\mathcal{O}_F^\times\to F^\times\xrightarrow{v_F}\mathbb{Z}\to 0.$$

Let $r \geq 1$ be an integer. We get a Borel (maximal connected solvable) subgroup $B_r \subset GL_r$ (for groups that aren't GL_r, look for a split or quasi-split form). We have a split exact sequence:

$$1 \to N_r \to B_r \to A_r \to 1,$$

where N_r is the unipotent radical of B_r, and A_r, the Levi component, is the maximal torus. The Levi splitting $A_r \to B_r$ is noncanonical. Let $\delta_{B_r} : B_r(F) \to \mathbb{R}_{>0}$ be the modular (quasi-)character of $B_r(F)$ (i.e., for any left Haar measure dx, then for any $b \in B_r(F)$, $d(bxb^{-1}) = \delta_{B_r(F)}(b)dx$. In fact, this map factors as $B_r(F) \twoheadrightarrow A_r(F) \to q^{\mathbb{Z}} \to \mathbb{R}_{>0}$, and

$$\begin{aligned}
\delta_{B_r(F)}(b) &= \left|detAd_{Lie(N_r)}(b)\right|_F \\
&= \left|\prod_{\alpha \in \Phi(B_r,A_r)} \alpha(b)\right|_F \\
&= \prod_{i<j} \left|\frac{b_{ii}}{b_{jj}}\right|_F \\
&= q^{-((r-1)v_F(b_{11})+(r-3)v_F(b_{22})+\cdots+(1-r)v_F(b_{rr}))},
\end{aligned}$$

where $b_{11}, \ldots, b_{rr}$ are the diagonal entries of the upper triangular matrix b, and $\Phi(B_r, A_r)$ is the set of roots of B_r with respect to A_r.

Choose a square root $p^{1/2} \in \mathbb{C}$. Then we get a $q^{1/2} \in \mathbb{C}$ canonically.

Definition The **parabolic induction functor**

$$\begin{aligned}
ind_{B_r,A_r}^{GL_r} : Rep_{smooth}(A_r(F)) \to \quad & Rep_{smooth}(GL_r(F)) \\
(V_\chi, \chi) \mapsto \quad & (V_{ind(\chi)}, ind(\chi))
\end{aligned}$$

$V_{ind(\chi)}$ is the complex vector space of all uniformly locally constant (i.e., invariant under right translation by some open compact subgroup) functions $GL_r(F) \to V_\chi$ such that for any $a \in A_r(F)$, any $n \in N_r(F)$, and any $x \in GL_r(F)$, $\phi(anx) = \delta_{B_r(F)}^{1/2}(a)\chi(a)\phi(x)$, and $ind(\chi) : GL_r(F) \to GL(V_\chi)$ is given by right translation: $g \mapsto (\phi \mapsto (x \mapsto \phi(xg)))$. The correction term $\delta_{B_r(F)}^{1/2}(a) = \delta_{B_r(F)}^{1/2}(an)$ is to preserve unitarity.

Definition The **Jacquet restriction functor**

$$\begin{aligned}
res_{B_r,A_r}^{GL_r} : Rep_{smooth}(GL_r(F)) \to \quad & Rep_{smooth}(A_r(F)) \\
(V_\pi, \pi) \mapsto \quad & (V_{res(\pi)}, res(\pi))
\end{aligned}$$

$$\begin{aligned}
V_{res(\pi)} &:= N_r(F)\text{-coinvariants of } V_\pi \\
&= V_\pi / \langle \pi(n)v - v \rangle_{n \in N_r(F), v \in V_\pi}
\end{aligned}$$

$res(\pi) : A_r(F) \to GL(V_{res(\pi)})$ takes $a \mapsto \delta_{B_r(F)}^{-1/2}(a)\pi(a)$. The δ term is necessary only for unitarity.

Proposition The functors $Rep_{smooth}(GL_r(F)) \underset{ind}{\overset{res}{\rightleftarrows}} Rep_{smooth}(A_r(F))$

1. are exact.
2. form an adjoint pair (Frobenius reciprocity).
3. send admissibles to admissibles.

The adjunction takes the form

$$\begin{array}{ccc} Hom_{Rep(A_r(F))}(res(\pi), \chi) & \rightleftarrows & Hom_{Rep(GL_r(F))}(\pi, ind(\chi)) \\ \alpha & \mapsto & (v \mapsto (x \mapsto \alpha([\pi(x)v]))) \\ ([v] \mapsto \beta(v)(I \in GL_r(F))) & \leftarrow\!\shortmid & \beta \end{array}$$

By adjunction, res is naturally right exact. To see why it is left exact, use Hecke algebras. A representation of G naturally extends to a representation of $\mathcal{H}(G)$. given an idempotent e_I, make a smaller Hecke algebra $e_I * \mathcal{H}(G) * e_I$. The map $V \mapsto V * e_I$ gives right modules. ?

Definition Let $\chi : A_r(F) \to \mathbb{C}^\times = GL(V_\chi)$ be a smooth (quasi-)character (i.e., the kernel is open), so $(V_\chi, \chi) \in Rep_{smooth}(A_r(F))$. The **principal series representation of $GL_r(F)$ associated to** χ is $(V_{ind(\chi)}, ind(\chi)) \in Rep_{smooth}(GL_r(F))$.

By theorems of Casselman and independently by Bernstein and Zelevinsky, this representation has finite length, that is in fact bounded by $\#Weyl(GL_r, A_r) = r!$.

Definition A representation $(V_\pi, \pi) \in Rep_{smooth}(GL_r(F))$ is **unramified** if and only if it is irreducible, smooth, and $V_\pi^{GL_r(\mathcal{O}_F)} \neq 0$.

It turns out the dimension of the above space is one in this case.

Lemma-Definition Let $\chi : A_r(F) \to \mathbb{C}^\times$ be an unramified quasi-character (i.e., $A_r(\mathcal{O}_F) \subset ker(\chi)$). Consider the principal series representation $(V_{ind(\chi)}, ind(\chi))$.

1. One has $dim_{\mathbb{C}}(V_{ind(\chi)}^{GL_r(\mathcal{O}_f)}) = 1$.
2. There exists a unique irreducible subquotient of $(V_{ind(\chi)}, ind(\chi))$ that is unramified, which we denote as $(V_{\pi(\chi)}, \pi(\chi))$.

Sketch of proof For the first, consider the decompositions

$$\begin{array}{rcl} GL_r(F) \cong B_r(F).GL_r(\mathcal{O}_F) & & \text{(Iwasawa)} \\ A_r(F) \cap GL_r(\mathcal{O}_F) = A_r(\mathcal{O}_F) & & \text{(not Iwasawa).} \end{array}$$

The $\mathbb{C}$-linear map $V_{ind(\chi)}^{GL_r(\mathcal{O}_F)} \to \mathbb{C}$ given by $\phi \mapsto \phi(1 \in GL_r(F))$ is injective and surjective (just need to check the torus). To prove the second claim, use the exactness check of the functor $Mod(\mathcal{H}) \to Mod(\mathcal{H}_{GL_r(\mathcal{O}_F)}$ defined by $V \mapsto V * e_{GL_r(\mathcal{O}_F)}$.

Theorem (Satake) Consider the map

$$\left\{ \begin{array}{c} \text{unramified quasi-} \\ \text{characters of } A_r(F) \end{array} \right\} \to \left\{ \begin{array}{c} \text{unramified represen-} \\ \text{tations of } GL_r(F) \end{array} \right\}_{/\cong}$$

$$\chi \mapsto (V_{\pi(\chi)}, \pi(\chi))$$

1. This map is surjective.
2. χ and χ' on the left are such that $(V_{\pi(\chi)}, \pi(\chi)) \cong V_{\pi(\chi')}, \pi(\chi'))$ in $Rep(GL_r(F))$ if and only if there exists $w \in Weyl(GL_r, A_r) \cong S_r$ such that $\chi' = w(\chi)$.

Proposition Let $\varpi \in \mathfrak{m}_F$ be a uniformizer of $\mathcal{O}_F$. Then we have a bijection:

$$\left\{ \begin{array}{c} \text{unramified quasi-} \\ \text{characters of } A_r(F) \end{array} \right\} \rightleftarrows (\mathbb{C}^\times)^r$$

$$\chi \mapsto \left(\chi\begin{pmatrix} \varpi & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \ldots, \chi\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \varpi \end{pmatrix} \right)$$

$$\left(\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_r \end{pmatrix} \mapsto \prod_{i=1}^r z^{v_F(a_i)} \right) =: \chi_{(z_1,\ldots,z_r)} \leftarrow\!\shortmid (z_1, \ldots, z_r)$$

Let $Weyl(GL_r, A_r) := N_{GL_r}(A_r)/A_r$ act on $(\mathbb{C}^\times)^r$ by transport of structure:

$$S_r \to Weyl(GL_r, A_r)$$
$$\sigma \mapsto w_\sigma := \delta_{\sigma(i),j}$$

Then

$$(w_\sigma)^{-1} \cdot \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_r \end{pmatrix} = w_\sigma^{-1} \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_r \end{pmatrix} w_\sigma = \begin{pmatrix} a_{\sigma(1)} & & \\ & \ddots & \\ & & a_{\sigma(r)} \end{pmatrix}$$

$$w_\sigma(\chi_{(z_1,\dots,z_r)}\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_r \end{pmatrix} \mapsto \prod_{i=1}^r z_i^{v_F(a_i)}) =$$

$$= (\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_r \end{pmatrix} \mapsto \prod_{i=1}^r z_i^{v_F(a_{\sigma(i)})}$$

$$= \prod_{i=1}^r z_{\sigma^{-1}(i)}^{v_F(a_i)}) = \chi_{z_{\sigma^{-1}(1)},\dots,z_{\sigma^{-1}(r)}}$$

Corollary (Satake's parametrization) The following map is a bijection.

$$S_r\backslash(\mathbb{C}^\times)^r \to \left\{ \begin{matrix} \text{unramified irreducible} \\ \text{representations of } GL_r(F) \end{matrix} \right\}_{/\cong}$$

$$(z_1,\dots,z_r) \mapsto (V_{\pi(\chi_{(z_1,\dots,z_r)})}, \pi(\chi_{(z_1,\dots,z_r)}))$$

Definition Let $(V_\pi,\pi) \in Rep_{adm}(GL_r(F))$ be unramified (N.B. Jacquet showed that irreducible and smooth implies admissible for GL_r). The multiset of **Satake parameters** of π is the unique unordered r-tuple of nonzero complex numbers $z_1(\pi),\dots,z_r(\pi)$ such that $(V_\pi,\pi) \cong (V_{\pi(\chi_{(z_1,\dots,z_r)})}, \pi(\chi_{(z_1,\dots,z_r)}))$.

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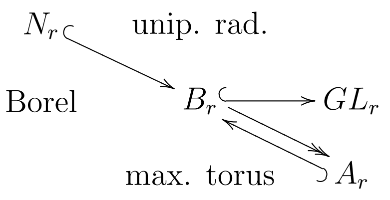
Supplimentary remarks Let F be a nonarchimedean local field, and $r \geq 1$ an integer, so we get a category $Rep_{smooth}(GL_r(F),\mathbb{C})$.

Proposition Let $(V,\pi) \in Rep_{smooth}(GL_r(F),\mathbb{C})$, and assume $dim_{\mathbb{C}} V < \infty$. Then $SL_r(F) \subset ker(\pi)$.

Corollary If in addition π is irreducible, then π factors as $GL_r(F) \xrightarrow{det} \mathbb{G}_m(F) \xrightarrow{\pi} \mathbb{C}^\times = GL(V)$.

Proof Pick a basis $E \underset{\text{finite}}{\subset} V$. Then $ker(\pi) = \bigcap_{v\in E} Stab_{GL_r(F)}(v)$ is an open normal subgroup of $GL_r(F)$. Choose subgroups:



e.g.

$$N_r(F) = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\} \qquad A_r(F) = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}$$

$ker(\pi) \cap N_r(F)$ is an open subgroup of $N_r(F)$ stable under conjugation by $A_r(F) = B_r(F)/N_r(F)$, so it is equal to $N_r(F)$, which is then contained in $ker(\pi)$. Then $ker(\pi)$ contains the subgroup of $GL_r(F)$ generated by all $GL_r(F)$-conjugates of $N_r(F)$, which is equal to $[GL_r(F), GL_r(F)]$ (this equals $SL_r(F)$ when F is infinite).

Question: What do the Satake parameters of one dimensional smooth representations of $GL_r(F)$ look like?

Suppose (V, π) is an unramified, smooth, (irreducible) one dimensional representation of $GL_r(F)$. Then π factors as $GL_r(F) \xrightarrow{det} \mathbb{G}_m(F) \to \mathbb{G}_m(F)/\mathbb{G}_m(\mathcal{O}) \cong \mathbb{Z} \to \mathbb{C}^\times$, i.e., the representations are parametrized by a complex number$z$, with the last map defined by $1 \mapsto z$. Now consider the representation of $A_r(F) \subset GL_r(F) \xrightarrow{\pi} \mathbb{C}^\times$ defined by $\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_r \end{pmatrix} \mapsto z^{\sum v(a_i)}$. $Res^{GL_r}_{B_r, A_r}(V, \pi) = (V_{res(\pi)}, res(\pi))$ given by $V_{res(\pi)} = V$ (since $N_r(F)$ acts trivially on V, making all elements coinvariant), and $res(\pi) : A_r(F) \to \mathbb{C}^\times$ taking $a \mapsto \delta^{-1/2}_{B_r(F)}(a)\pi(a)$. $res(\pi)$ is also written $\chi_{(q^{\frac{r-1}{2}}z, \ldots, q^{\frac{1-r}{2}}z)}$ as it takes $\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_r \end{pmatrix} \mapsto (q^{\frac{r-1}{2}}z)^{v(a_1)} \ldots (q^{\frac{1-r}{2}}z)^{v(a_r)}$

Frobenius reciprocity says that

$$Hom_{Rep(A_r(F))}(res(\pi), \chi_{(q^{\frac{r-1}{2}}z, \ldots, q^{\frac{1-r}{2}}z)}) \cong$$
$$\cong Hom_{Rep(GL_r(F))}(\pi, ind^{GL_r}_{B_r, A_r}(\chi_{(q^{\frac{r-1}{2}}z, \ldots, q^{\frac{1-r}{2}}z)})),$$

and we know these are nonzero, so there is an injection $\pi \hookrightarrow \chi_{(q^{\frac{r-1}{2}}z, \ldots, q^{\frac{1-r}{2}}z)}$. We have just exhibited an unramified subquotient (in fact a subrepresentation), so by uniqueness, the Satake parameters of π **must be** $(q^{\frac{r-1}{2}}z, \ldots, q^{\frac{1-r}{2}}z)$, where q is the size of the residue field of F.

For example, the trivial representation is "as unramified as possible" with Satake parameters $(q^{\frac{r-1}{2}}, \ldots q^{\frac{1-r}{2}})$, and weights $-(r-1), -(r-3), \ldots, r-1$.

Theorem (L. Lafforgue, Ramanujan-Petersson Conjecture) For any function field F, any $r \geq 1$, any $\pi \in \mathcal{A}^r(F, \mathbb{C}) = \left\{ \begin{array}{c} \text{isomorphism classes of } \textbf{cuspidal} \text{ automorphic} \\ \text{irreducible representations of } GL_r(\mathbb{A}_F) \\ \text{with finite order central character} \end{array} \right\}$, and any place $x \in |F|$, the representation $\pi_x \in Rep_{adm}(GL_r(F_x))$ is **tempered**.

N.B. If π_x is unramified, then the tempered condition is equivalent to $|z_i(\pi_x)|_{\mathbb{C}} = 1$ for all i. James Arthur formulated the "correct" generalization of this notion after examining the failure of the Ramanujan-Petersson conjecture in the symplectic case. The induction process requires this theorem to be proved **in concert** with Langlands correspondence.

Theorem (Casselman, Bernstein-Zelevinsky) $ind_{B_r,A_r}^{GL_r}(\chi_{(q^{\frac{r-1}{2}}z,\dots,q^{\frac{1-r}{2}}z)})$ as a representation of $GL_r(F)$ has a **unique** irreducible subobject (in this case trivial), and a unique irreducible quotient object (in this case the Steinberg representation).

Proposition If $(V,\pi) \in Rep_{smooth}(GL_r(\mathbb{A}))$ with $dim_{\mathbb{C}}(V) < \infty$, then $SL_r(\mathbb{A}) \subset ker(\pi)$.

This says something like, "orbital integrals use $\mathcal{O}$." ?

Corollary If (V,π) is also cuspidal, then $r = 1$.

So much for the automorphic side.

Galois side l-adic Galois representations.

Let F be any field. Choose a separable closure $\overline{F}$ of F. We get a profinite topological group $Gal(\overline{F}/F)$. Choose a prime number l. We get $\mathbb{Q}_l$. Choose an algebraic closure $\overline{\mathbb{Q}_l}$ of $\mathbb{Q}_l$.

Definition A (finite dimensional) l-adic Galois representation of $Gal(\overline{F}/F)$ is a pair (V,σ), where V is a finite dimensional $\overline{\mathbb{Q}_l}$ vector space with the l-adic topology (which is not locally compact), and $\sigma : Gal(\overline{F}/F) \to GL(V/\overline{\mathbb{Q}_l})$ is a continuous homomorphism.

We get a category $Rep_{fin}(Gal(\overline{F}/F), \overline{\mathbb{Q}_l})$ whose objects are pairs (V,σ) as above, and whose morphisms are $\overline{\mathbb{Q}_l}$-linear maps of vector spaces intertwining the respective Galois actions. This is a $\overline{\mathbb{Q}_l}$-linear abelian tensor category.

Note This category is **not** semisimple (No Haar measure, and no positive definite inner product).

Counterexample Let $F = \mathbb{Q}$, and let E be an elliptic curve over Spec $\mathbb{Z}[1/N]$ for some suitable integer N. Let S be the zero section, let T be a fiberwise disjoint section, and let $X := E - S - T$. The structure morphism $f : X \to$ Spec $\mathbb{Z}[1/N]$ is smooth, and $R^1 f_! \overline{\mathbb{Q}_l}$ is a lisse $\overline{\mathbb{Q}_l}$ sheaf of rank 3, so it corresponds to a rank 3 representation of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. The cohomology long exact sequence from the diagram $X \overset{j}{\hookrightarrow} E \overset{i}{\hookleftarrow} S \cup T$ is:

$$0 \overset{\text{affine}}{=} H_c^0(X) \to H_c^0(E) \overset{\text{diag}}{\to} H_c^0(S \cup T) \to H_c^1(X) \to H_c^1(E) \to H_c^1(S \cup T) \overset{\text{codim 1}}{=} 0$$

The nonzero terms have rank 1, 2, 3, and 2, respectively. $H_c^1(E)$ is the Tate module, or its dual, depending on who you ask, but it's self-dual for elliptic curves. [I seem to have missed the argument for why the sequence doesn't split.] Kronecker-Weber implies $Gal(\overline{\mathbb{Q}}/\mathbb{Q})^{ab} \cong Gal(\mathbb{Q}^{ab}/\mathbb{Q}) \cong \widehat{\mathbb{Z}}^{\times} \twoheadrightarrow \widehat{Z}$. There is a Poincaré duality pairing $H^1_c(X) \times H^1(X) \to \overline{\mathbb{Q}_l}(-1)$. fix

Why $\overline{\mathbb{Q}_l}$ and not $\mathbb{C}$? The main reason is that any complex Lie group has a neighborhood of the identity that contains no nontrivial subgroups. The preimage of such a neighborhood under the representation map is an open subset of the Galois group, which is compact. This implies the representation factors through a finite quotient, i.e., it is rather trivial. In particular, monodromy groups are all zero dimensional.

Note If $\overline{F}'$ is another separable algebraic closure of F, then we get another group, $Gal(\overline{F}'/F)$, and there exists an isomorphism $\phi : Gal(\overline{F}'/F) \to Gal(\overline{F}/F)$. Then we have an isomorphism

$$\begin{array}{rcl} Rep(Gal(\overline{F}/F)) & \xrightarrow{\sim} & Rep(Gal(\overline{F}'/F)) \\ (V, \sigma) & \mapsto & (V, \sigma \circ \phi) \\ (V, \sigma' \circ \phi^{-1}) & \leftarrow\!\shortmid & (V, \sigma') \end{array}$$

and the sets of isomorphism classes of irreducible objects are in canonical bijection.

Blah blah Katz-Sarnak remark 9.3. fix

$$\begin{array}{ccc} Gal & \xrightarrow{\ \forall\ } & GL_r(\overline{\mathbb{Q}_l}) \\ & \searrow^{\exists} & \uparrow \\ & & GL_r(E_\lambda) \\ & \searrow_{\text{also } \exists} & \uparrow \\ & & GL_r(\mathcal{O}_\lambda) \end{array}$$

For any homomorphism $\sigma : Gal(\overline{F}/F) \to GL_r(\overline{\mathbb{Q}_l})$, there is a finite extension $E_\lambda/\mathbb{Q}_l$ such that σ factors through $GL_r(E_\lambda)$, and given a properly chosen basis, it factors through $GL_r(\mathcal{O}_\lambda)$, where $\mathcal{O}_\lambda$ is the ring of integers in E_λ.

Local field case

Let F be a nonarchimedean local field, k its residue field, and choose $\overline{F}$, l, and $\overline{\mathbb{Q}_l}$ as above. This gives us a representation category. Let F^{ur} be the maximal unramified extension of F in $\overline{F}$. We get an exact sequence of profinite groups:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Gal(\overline{F}/F^{ur}) & \longrightarrow & Gal(\overline{F}/F) & \longrightarrow & Gal(F^{ur}/F) & \longrightarrow & 1 \\ & & \| & & & & \| & & \\ & & I(\overline{F}/F) & & & & Gal(\overline{k}/k) & & \end{array}$$

$I(\overline{F}/F)$ is the **absolute inertia group** of $\overline{F}/F$. $\overline{k}$ is the residue field of F^{ur}, and it is an algebraic closure of k.

We have a canonical isomorphism $\widehat{\mathbb{Z}} \xrightarrow{\sim} Gal(\overline{k}/k)$ given by $1 \mapsto$ Geometric Frobenius $:= (x \mapsto x^{\#k})^{-1} \in Gal(\overline{k}/k)$.

Definition A Galois representation σ is **unramified** if and only if $I(\overline{F}/F) \subset ker(\sigma)$, i.e., the inertia group acts trivially. This is also sometimes called spherical.

In this case, σ will factor as $Gal(\overline{F}/F) \to Gal(\overline{k}/k) \xrightarrow{\overline{\sigma}} GL(V/\overline{\mathbb{Q}_l})$, and we get a representation $(V, \overline{\sigma}) \in Rep(Gal(\overline{k}/k))$.

Definition If $(V, \sigma) \in Rep(Gal(\overline{F}/F))$ is unramified, then its **Frobenius eigenvalues** are the eigenvalues of $\overline{\sigma}(Frob_k \in Gal(\overline{k}/k)) \in GL(V/\overline{\mathbb{Q}_l})$.

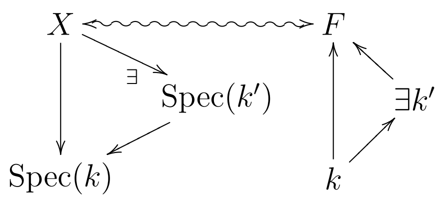
These give us an unordered r-tuple of elements of $\overline{\mathbb{Q}_l}^{\times}$, where $r = rank(V)$. Since we can always factor the Galois representation through some $GL_r(\mathcal{O}_\lambda)$, we have an isomorphism

$$\left\{ \begin{array}{c} \text{isomorphism classes of unramified semisimple} \\ r\text{-dimensional representations of } Gal(\overline{F}/F) \end{array} \right\} \xrightarrow{\sim} S_r \backslash (\overline{\mathbb{Z}}_l^{\times})^r$$

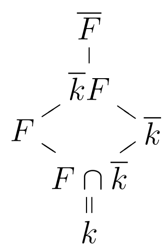
[The Weil group is briefly mentioned here, but it will appear later.]

Function field

Let k be a finite field, and F a function field over k. Assume k is algebraically closed in F (otherwise, we can replace k by its algebraic closure in F



Stein factorization [ref. EGA III 4.3.3] implies the structure map of a smooth curve with function field F factors through Spec of the algebraic closure of k in F). Choose $\overline{F}$, l, and $\overline{\mathbb{Q}_l}$ as above. Let $\overline{k}$ be the algebraic closure of k in $\overline{F}$.



We have an exact sequence of Galois groups:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & Gal(\overline{F}/\overline{k}F) & \longrightarrow & Gal(\overline{F}/F) & \longrightarrow & Gal(\overline{k}F/F) & \longrightarrow & 1 \\
 & & \| & & & & \downarrow \cong & & \\
 & & Gal_{geom}(\overline{F}/F) & & Frob_k & & Gal(\overline{k}/k) & & \\
 & & & & \uparrow & & \uparrow \cong & & \\
 & & & & 1 & & \widehat{\mathbb{Z}} & &
\end{array}$$

We call $Gal_{geom}(\overline{F}/F)$ the **geometric Galois group** of $\overline{F}/F$. The sequence is exact at $Gal(\overline{k}F/k)$ precisely because k is algebraically closed in F.

Let $x \in |F|$ be a place of F. We get F_x the completion of F at x, and κ_x the residue field of F_x. Choose a separable algebraic closure $\overline{F_x}$ of F_x. We get a category $Rep(Gal(\overline{F_x}/F_x))$, F_x^{ur} the maximal unramified extension of F_x in $\overline{F_x}$, and $\overline{\kappa_x}$, the residue field of $\overline{F_x}$. $\overline{\kappa_x}$ is also an algebraic closure of κ_x and of k. We get another exact sequence:

$$1 \to I(\overline{F_x}/F_x) \to Gal(\overline{F_x}/F_x) \to Gal(\overline{\kappa_x}/\kappa_x) \to 1$$

We have a canonical embedding $F \hookrightarrow F_x$. Choose an F-embedding $\overline{F} \overset{j_x}{\hookrightarrow} \overline{F_x}$. This choice induces a well-defined continuous injective homomorphism $Gal(\overline{F_x}/F_x) \overset{(j_x)_*}{\to} Gal(\overline{F}/F)$ given by $g \mapsto (a \mapsto j_x^{-1}(g(j_x(a))))$. The map is injective because of Krasner's lemma ($\overline{F_x} = j_x(\overline{F}) \cdot F_x$). If j'_x is another choice of embedding, then $(j_x)_*$ and $(j'_x)_*$ are conjugate in $Gal(\overline{F}/F)$, i.e., isomorphism classes of pulled-back representations do not change. We get a commutative diagram with exact rows:

$$\begin{array}{ccccccccccc}
1 & \longrightarrow & I(\overline{F_x}/F_x) & \longrightarrow & Gal(\overline{F_x}/F_x) & \longrightarrow & (Gal(\overline{\kappa_x}/\kappa_x) & \overset{\cong}{\longleftarrow} & \widehat{\mathbb{Z}}) & \longrightarrow & 1 \\
 & & \downarrow (j_x)_* & & \downarrow (j_x)_* & & \downarrow & & \downarrow deg(x) & & \\
1 & \longrightarrow & Gal_{geom}(\overline{F}/F) & \longrightarrow & Gal(\overline{F}/F) & \longrightarrow & (Gal(\overline{k}/k) & \overset{\cong}{\longleftarrow} & \widehat{\mathbb{Z}}) & \longrightarrow & 1
\end{array}$$

where $deg(x) = [\kappa_x : k] \in \mathbb{Z}_{>0}$.

Let $(V, \sigma) \in Rep(Gal(\overline{F}/F))$, so for all $x \in |F|$ we get $\sigma_x : Gal(\overline{F_x}/F_x) \overset{(j_x)_*}{\hookrightarrow} Gal(\overline{F}/F) \overset{\sigma}{\to} GL(V/\overline{\mathbb{Q}_l})$. This gives us $(V, \sigma_x) \in Rep(Gal(\overline{F_x}/F_x))$, and if it is unramified, then its Frobenius eigenvalues are called the **Frobenius eigenvalues of** σ **at** x.

Recall that

$$\mathcal{G}^r(F, \overline{\mathbb{Q}_l}) := \left\{ \begin{array}{c} \text{isomorphism classes of continuous irreducible} \\ \text{representations of } Gal(\overline{F}/F) \text{ on a } \overline{\mathbb{Q}_l} \text{ vector space} \\ \text{of rank } r \text{ that are almost everywhere unramified,} \\ \text{and whose determinant characters have finite order} \end{array} \right\}.$$

$\pi \in \mathcal{A}^r(F, \mathbb{C})$ and $\sigma \in \mathcal{G}^r(F, \overline{\mathbb{Q}_l})$ are in Langlands correspondence with respect to $\iota : \overline{\mathbb{Q}_l} \xrightarrow{\sim} \mathbb{C}$ if and only if for any $x \in |F|$ such that both π and σ are unramified at x, one has $\{z_1(\pi_x), \ldots, z_r(\pi_x)\} = \iota\{z_1(\sigma_x), \ldots, z_r(\sigma_x)\}$.

September 27, 2002

Local class field theory

Let F be a nonarchimedean local field, $v : F^\times \twoheadrightarrow \mathbb{Z}$ its valuation, $\mathcal{O}$ its ring of integers (the maximal compact subring of F), $\mathcal{O}^\times = ker(v)$ the ring if units, and k the residue field. We have a short exact sequence of topological groups

$$1 \to \mathcal{O}^\times \to F^\times \to \mathbb{Z} \to 1$$

.

Local Weil Groups

Choose a separable algebraic closure $\overline{F}$ of F. We get F^{ur} the maximal unramified extension of F in $\overline{F}$ and $\overline{k}$, the residue field of F^{ur}, which is also an algebraic closure of k. We have a short exact sequence of profinite groups:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & Gal(\overline{F}/F^{ur}) & \longrightarrow & Gal(\overline{F}/F) & \longrightarrow & Gal(F^{ur}/F) & \longrightarrow & 1 \\
 & & \| & & & & \downarrow \cong & & \\
 & & I(\overline{F}/F) & & & & Gal(\overline{k}/k) & \xleftarrow{\cong} & \widehat{\mathbb{Z}} \\
 & & & & & & & & \\
 & & & & & & Frob_k & \longleftarrow\!\shortmid & 1
\end{array}$$

where $I(\overline{F}/F)$ is called the **absolute inertia group**, and $Frob_k$ is the geometric Frobenius.

The **Weil group** of $\overline{k}/k$ is $W(\overline{k}/k) := (Frob_k)^{\mathbb{Z}} \subset Gal(\overline{k}/k)$. It is isomorphic to $\mathbb{Z}$, with the discrete topology. The **degree homomorphism** is defined as

$$\begin{array}{ccc}
Gal(\overline{F}/F) & \xrightarrow{deg} & \widehat{\mathbb{Z}} \\
\cong \downarrow & & \uparrow \cong \\
Gal(F^{ur}/F) & \xrightarrow{\cong} & Gal(\overline{k}/k)
\end{array}$$

Note that $I(\overline{F}/F) = ker(deg)$.

The **Weil group** of $\overline{F}/F$ is the preimage of $W(\overline{k}/k)$ (or $\mathbb{Z}$, depending on your definition) under deg, given the unique topology such that $I(\overline{F}/F) \subset W(\overline{F}/F)$ is a maximal compact open subgroup.

In fact, $W(\overline{F}/F)$ is noncanonically isomorphic to a semidirect product $I(\overline{F}/F) \rtimes \mathbb{Z}$, with the action given by conjugation in $Gal(\overline{F}/F)$.

Let $IW^{ab}(\overline{F}/F)$ denote the image of $I(\overline{F}/F)$ in $W(\overline{F}/F)^{ab}$. Then we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & I(\overline{F}/F) & \longrightarrow & W(\overline{F}/F) & \xrightarrow{deg} & W(\overline{k}/k) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \cong & & \\ 1 & \longrightarrow & IW^{ab}(\overline{F}/F) & \longrightarrow & W(\overline{F}/F)^{ab} & \xrightarrow{deg} & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

Note that the canonical map $I^{ab} \twoheadrightarrow IW^{ab}$ is not necessarily an isomorphism.

Theorem (local class field theory)

1. There exists a unique homomorphism of (abstract) abelian groups

   $$\psi_F : F^\times \to Gal(\overline{F}/F)^{ab}$$

   such that

   (a) the following diagram commutes

   $$\begin{array}{ccc} F^\times & \xrightarrow{v} & \mathbb{Z} \\ {\scriptstyle \psi_F}\downarrow & & \downarrow \\ Gal(\overline{F}/F)^{ab} & \xrightarrow{deg} & \widehat{\mathbb{Z}} \end{array}$$

   (b) for any finite abelian extension E/F in $\overline{F}/F$, the **norm** group $N_{E/F}(E^\times) \subset F^\times$ lies in the kernel of $F^\times \stackrel{\psi_F}{\to} Gal(\overline{F}/F)^{ab} \twoheadrightarrow Gal(E/F)$.

2. The homomorphism ψ_F has the following properties:

   (a) It is continuous.

   (b) For any finite abelian extension E/F in $\overline{F}/F$, let $\psi_{E/F} : F^\times/N_{E/F}(E^\times) \to Gal(E/F)$ be defined by:

   $$\begin{array}{ccc} F^\times/N_{E/F}(E^\times) & \longrightarrow & Gal(E/F) \\ \uparrow & & \uparrow \\ F^\times & \xrightarrow{\psi_F} & Gal(\overline{F}/F)^{ab} \end{array}$$

   Then $\psi_{E/F}$ is an isomorphism.

3. From 1(a) and 2(a), we get a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{O}^\times & \longrightarrow & F^\times & \xrightarrow{v} & \mathbb{Z} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \psi_F} & & \downarrow{\scriptstyle \psi_F} & & \| & & \\
1 & \longrightarrow & IW^{ab}(\overline{F}/F) & \longrightarrow & W(\overline{F}/F)^{ab} & \longrightarrow & \mathbb{Z} & \longrightarrow & 1
\end{array}$$

Assertion: The homomorphisms $\psi_F : \mathcal{O}^\times \to IW^{ab}(\overline{F}/F)$ and $\psi_F : F^\times \to W(\overline{F}/F)^{ab}$ are isomorphisms of topological abelian groups.

Definition The terms **local reciprocity map** and **local Artin map** refer to $\psi_F : F^\times \xrightarrow{\sim} W(\overline{F}/F)^{ab}$ and anything canonically deduced from it.

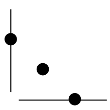
Construction of the homomorphism $\psi_F : F^\times \to Gal(\overline{F}/F)$

Step 1 Theorem The inflation map

$$\begin{array}{ccc}
H^2(F^{ur}/F) & \longrightarrow & H^2(\overline{F}/F) \\
\| & & \| \\
H^2(Gal(F^{ur}/F), (F^{ur})^\times) & & H^2(Gal(\overline{F}/F), \overline{F}^\times) =: Br(F)
\end{array}$$

"Brauer group"

is an **isomorphism**. Note that injectivity follows from Hilbert's Theorem 90. This is proved using a Hochschild-Serre spectral sequence:



Step 2 We have an exact sequence

$$1 \to (\mathcal{O}^{ur})^\times \to (F^{ur})^\times \xrightarrow{v} \mathbb{Z} \to 1$$

Proposition If F_j/F is the unique unramified extension of F of degree j in F^{ur}, then $\widehat{H}^q(Gal(F_j/F), \mathcal{O}_j^\times) = 0$ for all $q \geq 0$.

$\widehat{H}^q$ denotes Tate cohomology, which is defined by:

$$\begin{array}{rcl}
\widehat{H}^q & = & H^q \text{ for } q \geq 1 \\
\widehat{H}^{-q} & = & H_{q-1} \text{ for } q \geq 2 \\
\widehat{H}^0(G,A) & = & A^G/N_G(A) \\
\widehat{H}^{-1}(G,A) & = & \frac{ker(N_G : A \to A)}{I_G(A) := ker(\mathbb{Z}[G] \to \mathbb{Z})}
\end{array}$$

Corollary The map $H^2(F^{ur}/F) \xrightarrow{v_*} H^2(Gal(\overline{k}/k), \mathbb{Z})$ is an isomorphism.

Step 3 Consider the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0.$$

Proposition $\widehat{H}^q(\widehat{\mathbb{Z}}, \mathbb{Q}, \text{trivial action}) = 0$

Corollary The map $H^1(\widehat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}, \text{trivial action}) \xrightarrow{\delta} H^2(\widehat{\mathbb{Z}}, \mathbb{Z}, \text{ trivial action})$ is an isomorphism.

Step 4 The isomorphism inv_F is defined as follows:

$$\begin{array}{ccccc}
Br(F) := H^2(\overline{F}/F) & \xleftarrow[\text{(Step 1)}]{\cong} & H^2(F^{ur}/F) & \xrightarrow[\text{(Step 2)}]{\cong} & H^2(Gal(\overline{k}/k), \mathbb{Z}, \text{trivial action}) \\
{\scriptstyle inv_F}\downarrow\cong & & & & {\scriptstyle \text{(Step 3)}}\uparrow\cong \\
\mathbb{Q}/\mathbb{Z} & \xleftarrow[\text{ev. at } Frob_k]{\cong} & Hom(Gal(\overline{k}/k), \mathbb{Q}/\mathbb{Z}) & = & H^1(Gal(\overline{k}/k), \mathbb{Q}/\mathbb{Z}, \text{trivial action}) \\
 & & \downarrow\cong & & \\
 & & \widehat{\mathbb{Z}} & &
\end{array}$$

Step 5 Theorem For E/F a finite Galois degree n extension in $\overline{F}$, the following diagram commutes:

$$\begin{array}{ccc}
Br(F) & \xrightarrow{res} & Br(E) \\
{\scriptstyle inv_F}\downarrow\cong & & {\scriptstyle inv_E}\downarrow\cong \\
\mathbb{Q}/\mathbb{Z} & \xrightarrow{mult_n} & \mathbb{Q}/\mathbb{Z}
\end{array}$$

Step 6 Theorem The canonical map $H^2(E/F) := H^2(Gal(E/F), E^\times) \to Br(F)$ given E/F as above, induces an isomorphism: $H^2(E/F) \xrightarrow{\sim} ker(Br(F) \xrightarrow{res} Br(E))$

Definition The **fundamental class** $u_{E/F} \in H^2(E/F)$ is characterized by its image: $inv_F(u_{E/F}) = [1/n]$ via the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^2(E/F) & \longrightarrow & Br(F) & \longrightarrow & Br(E) \\
 & & {\scriptstyle inv_F}\downarrow\cong & & {\scriptstyle inv_F}\downarrow\cong & & {\scriptstyle inv_E}\downarrow\cong \\
0 & \longrightarrow & \frac{1}{n}\mathbb{Z}/\mathbb{Z} & \hookrightarrow & \mathbb{Q}/\mathbb{Z} & \xrightarrow{mult_n} & \mathbb{Q}/\mathbb{Z}
\end{array}$$

Step 7 Theorem (Tate) For E/F a finite Galois extension of degree n in $\overline{F}/F$, we have an isomorphism:

$$\widehat{H}^{-2}(Gal(E/F), \mathbb{Z}, \text{trivial action}) \xrightarrow{-\cup u_{E/F}} \widehat{H}^0(Gal(E/F), E^\times)$$

In fact, the map "cup product with $u_{E/F}$" is an isomorphism if we replace 0 by any integer q.

Definition The **local reciprocity map** $\psi_{E/F} : F^\times/N_{E/F}(E^\times) \to Gal(E/F)^{ab}$ for E/F as above is defined by the composition:

$$\begin{array}{ccccc} F^\times/N_{E/F}(E^\times) & \xrightarrow{\psi_{E/F}} & Gal(E/F)^{ab} & \xleftarrow{\cong} & H_1(Gal(E/F), \mathbb{Z}, \text{trivial action}) \\ \| & & & & \| \\ \widehat{H}^0(Gal(E/F), E^\times) & & \xleftarrow[\cong]{\text{(Tate)}} & & \widehat{H}^{-2}(Gal(E/F), \mathbb{Z}, \text{trivial action}) \end{array}$$

Step 8 For $E'/F \supset E/F$ finite Galois extensions in $\overline{F}/F$, the diagram commutes:

$$\begin{array}{ccc} F^\times & & \\ \downarrow & & \\ F^\times/N_{E'/F}(E/^\times) & \xrightarrow[\cong]{\psi_{E'/F}} & Gal(E'/F)^{ab} \\ \downarrow & & \downarrow \\ F\times/N_{E/F}(E^\times) & \xrightarrow[\cong]{\psi_{E/F}} & Gal(E/F)^{ab} \end{array}$$

If we pass to the limit, we have a unique map $\psi_F : F^\times \to Gal(\overline{F}/F)^{ab}$, but it is no longer an isomorphism.

Global class field theory for function fields

Let k be a finite field of characteristic p, let F be a function field over k, let $|F|$ be the set of places of F, and let $\mathbb{A}$ be the ring of adèles. We define the degree homomorphism $\mathbb{A}^\times \to \mathbb{Z}$ by $(a_x)_{x\in|F|} \mapsto \sum_{x\in|F|} deg(\kappa(x)/k)v_x(a_x)$, where $v_x : F_x^\times \twoheadrightarrow \mathbb{Z}$ is the valuation map. Let $(A^\times)^0 := ker(deg)$.

Lemma Let k' be the algebraic closure of k in F. Then the image of $deg : \mathbb{A} \to \mathbb{Z}$ has index $[k' : k]$.

This is not true for number fields. ??

From now on, we assume k is algebraically closed in F. This is saying that the curve $X \to \text{Spec}(k)$ with function field F is geometrically connected. [Something about Stein factorization again here: $X \to \text{Spec}(k') \to \text{Spec}(k)$] We get an exact sequence of locally compact Hausdorff topological groups:

$$1 \to (\mathbb{A}^\times)^0 \to \mathbb{A}^\times \stackrel{deg}{\to} \mathbb{Z} \to 0$$

The topology on $\mathbb{A}^\times$is induced from the topoloical ring structure on $\mathbb{A}$ via $A^\times = \mathbb{G}_m(\mathbb{A})$.

Lemma (Product formula) The diagonal embedding $F^\times \hookrightarrow \mathbb{A}^\times$ factors as $F^\times \hookrightarrow (\mathbb{A}^\times)^0 \hookrightarrow \mathbb{A}^\times$.

Proposition

1. The embedding $F^\times \hookrightarrow (\mathbb{A}^\times)^0$ identifies $F^\times$ as a **discrete** subgroup of $(\mathbb{A}^\times)^0$.
2. The quotient $F^\times\backslash(\mathbb{A}^\times)^0$ with the quotient topology is compact.

Passing to quotients, we get the following exact sequence of topological groups:

$$1 \to F^\times\backslash(\mathbb{A}^\times)^0 \to F^\times\backslash\mathbb{A}^\times \to \mathbb{Z} \to 0$$

where $F^\times\backslash(\mathbb{A}^\times)^0$ is profinite (in particular, compact), $F^\times\backslash\mathbb{A}^\times$ is locally compact and Hausdorff, and $\mathbb{Z}$ is discrete.

Global Weil group

Choose a separable algebraic closure $\overline{F}$ of F, and let $\overline{k}$ be the algebraic closure of k in $\overline{F}$. We have the exact sequence:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Gal(\overline{F}/\overline{k}F) & \longrightarrow & Gal(\overline{F}/F) & \to & Gal(\overline{k}F/F) & \to & 1 \\ & & \| & & & & \downarrow\cong & & \\ & & Gal_{geom}(\overline{F}/F) & & & & Gal(\overline{k}/k) & \overset{\cong}{\longleftarrow} & \widehat{Z} \\ & & & & & & & & \\ & & & & & & Frob_k & \longleftarrow & 1 \end{array}$$

The **degree homomorphism** $deg : Gal(\overline{F}/F) \to \widehat{\mathbb{Z}}$ is defined as the composition $Gal(\overline{F}/F) \twoheadrightarrow Gal(\overline{k}F/F) \cong Gal(\overline{k}/k) \cong \widehat{\mathbb{Z}}$ where the last isomorphism takes $Frob_k$ to 1.

Definition The **Weil Group** of $\overline{F}/F$ is $W(\overline{F}/F) := deg^{-1}(Frob_k^{\mathbb{Z}} \subset Gal(\overline{k}/k)) \subset Gal(\overline{F}/F)$, endowed with the unique topology such that $Gal_{geom}(\overline{F}/F) \subset W(\overline{F}/F)$ is a maximal compact (profinite) open subgroup.

We get an exact sequence:

$$1 \to Gal_{geom}(\overline{F}/F) \to W(\overline{F}/F) \overset{deg}{\to} \mathbb{Z} \to 0.$$

To pass to the abelianization, we let $Gal_{geom}W^{ab}(\overline{F}/F) :=$ image of $Gal_{geom}(\overline{F}/F)$ in $W(\overline{F}/F)^{ab}$. Then we have an exact sequence of abelian topological groups:

$$1 \to Gal_{geom}W^{ab}(\overline{F}/F) \to W(\overline{F}/F)^{ab} \overset{deg}{\to} \mathbb{Z} \to 0$$

Lemma-Definition

- For each $x \in |F|$, we get the completion F_x, and an inclusion $F_x^\times \hookrightarrow \mathbb{A}^\times$.
- Choose a separable closure $\overline{F_x}$ of F_x. Then we get a group $Gal(\overline{F_x}/F_x)$ and a local reciprocity map $\psi_x : F_x^\times \to Gal(\overline{F_x}/F_x)^{ab}$.
- Choose an F-embedding $\overline{F} \overset{j_x}{\hookrightarrow} \overline{F_x}$. This induces homomophisms

$$\begin{array}{rcl} (j_x)_* : Gal(\overline{F_x}/F_x) & \hookrightarrow & Gal(\overline{F}/F) \\ (j_x)_* : Gal(\overline{F_x}/F_x)^{ab} & \to & Gal(\overline{F}/F)^{ab}. \end{array}$$

Then:

1. For each $(a_x)_{x\in|F|} \in \mathbb{A}^\times$, the product $\prod_{x\in|F|}(j_x)_*\psi_x(a_x \in F_x^\times)$ is well-defined, and independent of our choices above.
2. The map $\psi_F : \mathbb{A}^\times \to Gal(\overline{F}/F)^{ab}$ defined by $(a_x) \mapsto \prod_{x\in|F|}(j_x)_*\psi_x(a_x)$ is a homomorphism of (abstract) abelian groups.
3. The homomorphism $\psi_F : \mathbb{A}^\times \to Gal(\overline{F}/F)^{ab}$ is the unique homomorphism of (abstract) abelian groups such that the diagram

$$\begin{array}{ccc} F_x^\times & \xrightarrow{\psi_x} & Gal(\overline{F_x}/F_x)^{ab} \\ \big\downarrow & & \big\downarrow (j_x)_* \\ \mathbb{A}^\times & \xrightarrow{\psi_F} & Gal(\overline{F}/F)^{ab} \end{array}$$

   commutes for all $x \in |F|$ (this is independent of the choice of j_x).

Theorem (Global class field theory)

1. The homomorphism $\psi_F : \mathbb{A}^\times \to Gal(\overline{F}/F)^{ab}$ of (abstract) abelian groups induces a homomorphsim $\psi_F : F^\times\backslash\mathbb{A}^\times \to W(\overline{F}/F)^{ab}$ of abstract abelian groups such that the diagram

$$\begin{array}{ccc} \mathbb{A}^\times & \xrightarrow{\psi_F} & Gal(\overline{F}/F)^{ab} \\ \big\downarrow & & \big\uparrow \text{dense} \\ F^\times\backslash\mathbb{A}^\times & \xrightarrow{\psi_F} & W(\overline{F}/F)^{ab} \end{array}$$

   commutes.
2. $\psi_F : \mathbb{A}^\times \to Gal(\overline{F}/F)$ and $\psi_F : F^\times\backslash\mathbb{A}^\times$ have the following properties:
   (a) They are continuous.

(b) Given any **finite abelian** field extension E/F in $\overline{F}/F$, the norm groups $N_{E/F}(\mathbb{A}_E^\times) \subset \mathbb{A}_F^\times$ and $N_{E/F}(E^\times\backslash\mathbb{A}_E^\times) \subset F^\times\backslash\mathbb{A}_F^\times$ lie in the kernels of the compositions $ker(\mathbb{A}_F^\times \xrightarrow{\psi_F} Gal(\overline{F}/F)^{ab} \to Gal(E/F))$ and $ker(F^\times\backslash\mathbb{A}_F^\times \xrightarrow{\psi_F} W(\overline{F}/F)^{ab} \to Gal(E/F))$, respectively.

(c) The induced map $\psi_{E/F} : F^\times\backslash\mathbb{A}_F^\times/N_{E/F}(\mathbb{A}_E^\times) \to Gal(E/F)$ makes the diagram

$$\begin{array}{ccc} F^\times\backslash\mathbb{A}_F^\times/N_{E/F}(\mathbb{A}_E^\times) & \xrightarrow{\psi_{E/F}} & Gal(E/F) \\ \uparrow & & \uparrow \\ F^\times\backslash\mathbb{A}_F^\times & \xrightarrow{\psi_F} & W(\overline{F}/F)^{ab} \end{array}$$

commute, and $\psi_{E/F}$ is an isomorphism of abelian groups.

3. One obtains from 1 and 2b the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F^\times\backslash\ (\mathbb{A}_F^\times)^0 & \longrightarrow & F^\times\backslash\mathbb{A}_F^\times & \xrightarrow{deg} & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\psi_F} & & \downarrow{\scriptstyle\psi_F} & & \| & & \\ 1 & \longrightarrow & Gal_{geom}W^{ab}(\overline{F}/F) & \longrightarrow & W(\overline{F}/F)^{ab} & \xrightarrow{deg} & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

commutes. The continuous homomorphisms ψ_F are isomorphisms of topological abelian groups.

Definition The terms **global reciprocity map** and **global Artin map** refer to $\psi_F : F^\times\backslash\mathbb{A}^\times \xrightarrow{\sim} W(\overline{F}/F)^{ab}$ and anything canonically deduced from it.

October 4, 2002

$r = 1$ **case of Langlands correspondence** (Existence claims only - uniqueness requires L-function formalism)

Let k be a finite field of characteristic $p > 0$, let F be a function field over k, and let $\mathbb{A}$ be the ring of adèles of F. Choose an algebraic closure $\mathbb{C}$ of $\mathbb{R}$, and get a set $\mathcal{A}^r(F,\mathbb{C})$. Choose $l \neq p$, $\overline{\mathbb{Q}_l}$, and $\overline{F}$ a separable closure of F, and get $\mathcal{G}^r(F,\overline{\mathbb{Q}_l})$.

Theorem

1. For any isomorphism of fields $\iota : \overline{\mathbb{Q}_l} \to \mathbb{C}$, global class field theory in F induces a bijection $\mathcal{A}^1(F,\mathbb{C}) \xleftrightarrow{\sim} \mathcal{G}^1(F,\overline{\mathbb{Q}_l})$. Call this the class field theory correspondence.

2. Let $\pi \in \mathcal{A}^1(F,\mathbb{C})$ and $\sigma \in \mathcal{G}^1(F,\overline{\mathbb{Q}_l})$ be in class field theory correspondence. Then

   (a) $S_\pi = S_\sigma \subset |F|$.

(b) π and σ are in Langlands correspondence with respect to ι.

Proof We may assume k is algebraically closed in F. By global class field theory, we have a map given by the composition:

$$
\begin{array}{ccccc}
GL_1(\mathbb{A}) & & Gal(\overline{F}/F) & & \\
\downarrow & \overset{\psi_F}{\searrow} & \downarrow & & \\
GL_1(F)\backslash GL_1(\mathbb{A}) & \xrightarrow[\text{global CFT iso.}]{\psi_F} & W(\overline{F}/F)^{ab} & \xhookrightarrow[\text{dense}]{} & Gal(\overline{F}/F)^{ab}
\end{array}
$$

Then we have the following bijections:

$$
\mathcal{G}^1(F,\overline{\mathbb{Q}_l}) = \left\{ \begin{array}{c} \text{isomorphism classes of irreducible} \\ \text{continuous representations of} \\ Gal(\overline{F}/F) \text{ on a } \overline{\mathbb{Q}_l} \text{ vector space of} \\ \text{rank 1, almost everywhere} \\ \text{unramified and of finite order} \end{array} \right\}
\qquad
\begin{pmatrix} Gal(\overline{F}/F) \\ \downarrow \\ Gal(\overline{F}/F)^{ab} \\ \downarrow \chi_\sigma \\ \overline{\mathbb{Q}_l}^\times \\ \downarrow \cong \\ V_\sigma \end{pmatrix}
$$

$$
\updownarrow \cong \qquad\qquad \uparrow
$$

$$
\left\{ \begin{array}{c} \text{quasi-characters} \\ Gal(\overline{F}/F)^{ab} \to \overline{\mathbb{Q}_l}^\times \\ \text{of finite order} \end{array} \right\}
\qquad \chi_\sigma
$$

$$
\downarrow \cong \qquad\qquad \downarrow
$$

$$
\chi_\sigma \qquad
\left\{ \begin{array}{c} \text{quasi-characters} \\ W(\overline{F}/F)^{ab} \to \overline{\mathbb{Q}_l}^\times \\ \text{of finite order} \end{array} \right\}
\qquad
\begin{pmatrix} W(\overline{F}/F)^{ab} \\ \downarrow \text{dense} \\ Gal(\overline{F}/F)^{ab} \\ \downarrow \chi_\sigma \\ \overline{\mathbb{Q}_l}^\times \end{pmatrix}
$$

$$
\downarrow \qquad\qquad \text{CFT} \downarrow \cong
$$

$$
\begin{pmatrix} GL_1(F)\backslash GL_1(\mathbb{A}) \\ \cong \downarrow \psi_F \\ W(\overline{F}/F)^{ab} \\ \downarrow \chi_\sigma \\ \overline{\mathbb{Q}_l}^\times \\ \cong \downarrow \iota \\ \mathbb{C}^\times \end{pmatrix}
\qquad
\left\{ \begin{array}{c} \text{quasi-characters} \\ GL_1(F)\backslash GL_1(\mathbb{A}) \to \mathbb{C}^\times \\ \text{of finite order} \end{array} \right\}
\qquad \chi_\pi
$$

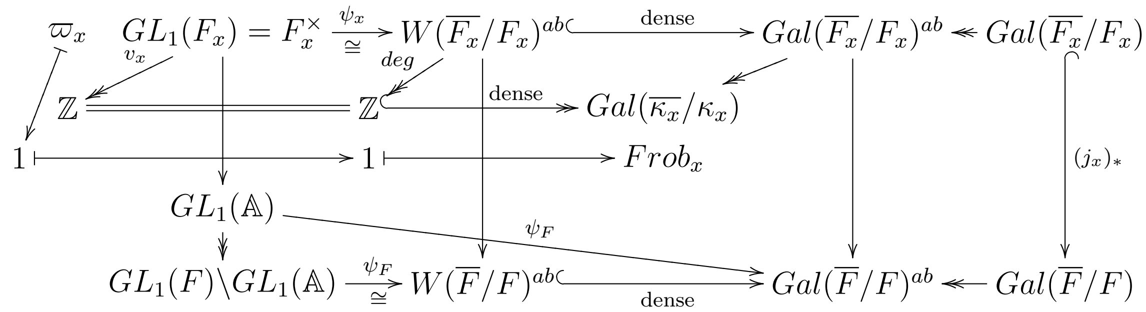
$$
\updownarrow \cong \qquad\qquad \downarrow
$$

$$
\mathcal{A}^1(F,\mathbb{C}) = \left\{ \begin{array}{c} \text{isomorphism classes of (cuspidal)} \\ \text{automorphic irreducible complex} \\ \text{representations of } GL_1(\mathbb{A}) \text{ with a} \\ \text{finite order central character} \end{array} \right\}
\qquad
\begin{pmatrix} \mathbb{C}\cdot(\chi_\pi \text{ as} \\ \text{a function} \\ \text{on } GL_r(\mathbb{A})) \\ \cap \\ \mathcal{A}_{cusp}(GL_r(\mathbb{A}),\mathbb{C}) \end{pmatrix}
$$

Note that the cuspidality condition collapses for $r = 1$. The last correspondence can also be expressed by

$$(V_\pi, \pi) \leftrightarrows \quad \begin{array}{ccc} GL_1(\mathbb{A}) & \xrightarrow{\pi} & GL(V_\pi) \\ \downarrow & & \updownarrow \cong \\ GL_1(F)\backslash GL_1(\mathbb{A}) & \xrightarrow{\chi_\pi} & \mathbb{C}^\times \end{array}$$

For any place $x \in |F|$, we get a completion F_x and a residue field κ_x. Choose a separable algebraic closure $\overline{F_x}$ of F_x. We get $Gal(\overline{F_x}, F_x)$. Choose an F-embedding $j_x : \overline{F} \hookrightarrow \overline{F_x}$. We get $(j_x)_* : Gal(\overline{F_x}/F_x) \hookrightarrow Gal(\overline{F}/F)$. We have the following picture of local-global correspondence (upward arrows are surjections):



We need to check commutativity: Suppose $\pi \in \mathcal{A}^1(F, \mathbb{C})$ and $\sigma \in \mathcal{G}^1(F, \overline{\mathbb{Q}_l})$ are in class field theory correspondence with respect to ι. Then for $x \in |F|$, one has:

$$\begin{aligned} x \notin S_\sigma \quad &\Leftrightarrow \quad \left(\begin{array}{c} \sigma_x : Gal(\overline{F_x}/F_x) \overset{(j_x)_*}{\hookrightarrow} Gal(\overline{F}/F) \to GL(V_\sigma) \text{ factors} \\ \text{through } Gal(\overline{F_x}/F_x) \twoheadrightarrow Gal(\overline{F_x}/F_x)^{ab} \twoheadrightarrow Gal(\overline{\kappa_x}/\kappa_x) \end{array} \right) \\ &\Leftrightarrow \quad \left(\begin{array}{c} GL_1(F_x) = F_x^\times \hookrightarrow GL_1(\mathbb{A}) \twoheadrightarrow GL_1(F)\backslash GL_1(\mathbb{A}) \overset{\pi_x}{\to} GL(\chi_\pi) \\ \text{factors through } GL_1(F_x) = F_x^\times \overset{v_x}{\twoheadrightarrow} \mathbb{Z} \end{array} \right) \\ &\Leftrightarrow \quad x \notin S_\pi \end{aligned}$$

Hence, $S_\sigma = S_\pi \subset |F|$. Choose a uniformizer $\varpi_x \in GL_1(F_x)$. Its image under v_x is 1, and it is sent to $Frob_x \in Gal(\overline{\kappa_x}/\kappa_x)$, the geometric Frobenius.

For each $x \in |F| \setminus (S_\pi = S_\sigma)$, we have

$$\begin{aligned} z_1(\sigma_x) &= \text{eigenvalue of } Frob_x \in Gal(\overline{\kappa_x}/\kappa_x) \text{ under } \sigma_x \\ &= \text{image under } \sigma_x \text{ of any lift of } Frob_x \text{ to } Gal(\overline{F_x}/F_x) \end{aligned}$$

Now, apply $\iota : \overline{\mathbb{Q}_l} \xrightarrow{\sim} \mathbb{C}$.

$$\begin{aligned} \iota(z_1(\sigma_x)) &= \text{ image under } \pi_x \text{ of any uniformizer } \varpi_x \text{ of } F_x \\ &= z_1(\pi_x) \in \mathbb{C}^\times \end{aligned}$$

So much for $r = 1$.

Future Program

1. Rankin-Selberg convolutions, L and ϵ factors
2. Converse theorem of Piatetski-Shapiro
3. Grothendieck's l-adic cohomology
4. Grothendieck-Lefschetz-Verdier trace formula
5. Weil II
6. Product formula of Laumon and Grothendieck, L and ϵ functions
7. Arthur-Selberg trace formula (probably next term)

We have two different theories:

- Automorphic
  - Tate's Thesis - Generalized by Godement-Jacquet (groups of units in simple algebras), gives principal L, ϵ factors for GL_r.
  - Rankin-Selberg convolutions (Jacquet, Piatetski-Shapiro, Shalika) for?? (poles?) of automorphic representations (twistee and twistor), for converse theorem ($GL_r \times GL_{r'}$). ?
  - Langlands-Shahidi method - popular, very general, not necessary for us.
- Galois/cohomological(/motivic)
  - Artin L-functions - Grothendieck (L-factors), Laumon (ϵ-factors)

Tate's thesis (This can be found in the appendix to Cassels & Frohlich. It has been souped up by Godement and Jacquet)

Let F be a nonarchimedean local field, and let $v : F^\times \twoheadrightarrow \mathbb{Z}$ be a normalized valuation. Let $C_c^\infty(F)$ be the complex vector space of functions that are **smooth** (= locally constant) and have **compact support**. Choose a Haar measure $d^\times x$ on $F^\times$. Let $\chi : F^\times \to \mathbb{C}^\times$ be a smooth quasi-character of $F^\times$.

Definition The **zeta integral** of χ, $Z(\chi, -, T) : C_c^\infty(F) \to \mathbb{C}((T))$ is the $\mathbb{C}((T))$-valued distribution on F given by

$$\begin{aligned}\phi \mapsto Z(\chi, \phi, T) &:= \int_{F^\times} \phi(x)\chi(x)T^{v(x)} d^\times x \\ &= \sum_{m\in\mathbb{Z}} T^m \left(\int_{x\in F^\times, v(x)=m} \phi(x)\chi(x) d^\times x \right)\end{aligned}$$

Key proposition Let $P_\chi \subset \mathbb{C}[T] \subset \mathbb{C}((T))$ be the $\mathbb{C}[T]$-ideal

$$P_\chi := \{P(T) \in \mathbb{C}[T] : \forall \phi \in C_c^\infty(F) P(T)Z(\chi, \phi, T) \in \mathbb{C}[T^{\pm 1}] \subset \mathbb{C}((T))\}$$

Assertion: This is a non-zero ideal of $\mathbb{C}[T]$.

Proof Computational. χ is either unramified (look at $1 - \chi(x)T$) or ramified (ideal contains 1).

Note If $Q(T) \in \mathbb{C}[T]$ is such that $TQ(T) \in P_\chi$, then $Q(T) \in P_\chi$, so P_χ is generated by some polynomial with constant term 1.

Definition The **L-factor associated to** χ is the unique element $L(\chi; T) \in \mathbb{C}(T)^\times \cap (1 + T\mathbb{C}[[T]]) \subset \mathbb{C}((T))$ such that $L(\chi; T)^{-1}$ is the generator of P_χ with constant term 1.

Let $\psi : F \to \mathbb{C}^\times$ be a nontrivial additive (unitary) character. This necessarily falls on pth roots of unity.

Definition The **conductor of** ψ is $c(\psi) := \max \{c \in \mathbb{Z} : \psi|_{\mathfrak{m}^{-c}} \text{ is trivial }\}$, where $\mathfrak{m}$ is the maximal ideal in the ring of integers $\mathcal{O} \subset F$.

The conductor is large if and only if the character is more trivial. Note that ψ induces an isomorphism of topological abelian groups $F \xrightarrow{\sim} \widehat{F} := Hom_{top-ab}(F, \mathbb{R}/\mathbb{Z} \subset \mathbb{C}^\times)$ given by $a \mapsto (x \mapsto \psi(ax))$. Let dx be the unique Haar measure self-dual with respect to ψ (given by the normalization $\int_{\mathcal{O}} dx = q^{c(\psi)/2}$, with q the order of the residue field). $U(1)$?

Definition The **Fourier transform on $C_c^\infty(F)$ with respect to** ψ is the map $C_c^\infty(F) \to C_c^\infty(F)$ defined by $\phi \mapsto \widehat{\phi} := (a \mapsto \int_F \phi(x)\psi(ax)dx)$.

The measure dx is self dual with respect to ψ if and only if the Fourier inversion formula takes the form $\widehat{\widehat{\phi}}(x) = \phi(-x)$. If $\chi : F^\times \to \mathbb{C}^\times$ is a smooth quasi-character, then we define the **dual** $\chi^\vee$ of χ by $a \mapsto \chi(a^{-1}) = \chi(a)^{-1}$.

Theorem (Local functional equation) There exists a **unique** function $\epsilon(\chi, \psi; T) \in \mathbb{C}(T) \subset \mathbb{C}((T))$ such that for any $\phi \in C_c^\infty(F)$, one has

$$\frac{Z(\chi^\vee, \widehat{\phi}; \frac{1}{qT})}{L(\chi^\vee; \frac{1}{qT})} = \epsilon(\chi, \psi; T)\frac{Z(\chi, \phi; T)}{L(\chi; T)}$$

Moreover, $\epsilon(\chi, \psi; T) \in \mathbb{C}[T^{\pm 1}]$, i.e., it is a monomial.

Definition The monomial defined above is called the **ϵ-factor associated to χ and ψ**. Choose a $q^{1/2} \in \mathbb{C}^\times$, and write $\epsilon(\chi, \psi; T) = q^{-c(\psi)/2} b(\chi, \psi) T^{a(\chi, \psi)}$. This uniquely defines $b(\chi, \psi) \in \mathbb{C}^\times$, the **local constant of** χ, ψ, and $a(\chi, \psi) \in \mathbb{Z}$, the **local conductor of** χ, ψ.

Note None of $L(\chi;T)$, $\epsilon(\chi,\psi;T)$, $a(\chi,\psi)$, or $b(\chi,\psi)$ depend on our choice of $d^\times x$. Only $b(\chi,\psi)$ depends on our choice of $q^{1/2}$.

Proposition There exists a **unique** nonnegative integer $a(\chi) \in \mathbb{Z}_{\geq 0}$ such that for any choice of nontrivial additive character $\psi : F \to \mathbb{C}$, $a(\chi,\psi) = a(\chi) + c(\psi)$. Call $a(\chi)$ the **local conductor of** χ.

Proposition Let $\chi : F^\times \to \mathbb{C}^\times$ be a smooth quasi-character, and let $\psi : F \to \mathbb{C}^\times$ be a nontrivial continuous additive character. Then:

$$L(\chi;T) = \begin{cases} \frac{1}{1-\chi(\varpi)T} & \text{if } \chi \text{ is unramified} \\ 1 & \text{if } \chi \text{ is ramified} \end{cases}$$

$$a(\chi) = \begin{cases} 0 & \text{if } \chi \text{ is unramified} \\ \min\{a \in \mathbb{Z}_{\geq 1} : \chi|_{1+\mathfrak{m}^a \subset \mathcal{O}^\times \subset F^\times} \text{ is trivial}\} & \text{if } \chi \text{ if ramified} \end{cases}$$

$$b(\chi,\psi) = \begin{cases} \chi(\varpi^{c(\psi)})q^{c(\psi)} & \text{if } \chi \text{ is unramified} \\ \int_{z \in F, v(z) = -a(\chi)-c(\psi)} \chi^{-1}(z)\psi(z)dz & \text{if } \chi \text{ is ramified} \end{cases}$$

Here, we choose dz so $\int_{\mathcal{O}} dz = 1$.

Global story

Let k be a finite field of order q (not the same q as above), let F be a function field over k, and let $\mathbb{A}$ be its ring of adèles. Assume k is algebraically closed in F. Fix a unitary character $\chi : GL_1(F)\backslash GL_1(\mathbb{A}) \to \mathbb{C}^\times$, and let $\psi : F\backslash\mathbb{A} \to \mathbb{C}^\times$ be a nontrivial additive character.

For each $x \in |F|$ we get $\chi_x : F_x^\times \to GL_1(\mathbb{A}) \twoheadrightarrow GL_1(F)\backslash GL_1(\mathbb{A}) \xrightarrow{\chi} \mathbb{C}^\times$, and $\psi_x : F_x \to \mathbb{A} \twoheadrightarrow F\backslash\mathbb{A} \xrightarrow{\psi} \mathbb{C}^\times$ (nontrivial by strong approximation). Also we get local factors: $L(\chi_x;T) \in \mathbb{C}(T)^\times \cap (1 + T\mathbb{C}[[T]]) \subset \mathbb{C}((T))$ and $\epsilon(\chi_x,\psi_x;T) \in \mathbb{C}[T^{\pm 1}]^\times$.

Definition The **global L-function associated to** χ is

$$L(\chi;T) := \prod_{x \in |F|} L(\chi_x; T^{deg(x)}) \in 1 + T\mathbb{C}[[T]].$$

The **global ϵ-function associated to** χ,ψ is

$$\epsilon(\chi,\psi;T) := \prod_{x \in |F|} \epsilon(\chi_x,\psi_x; T^{deg(x)}) \in \mathbb{C}[T^{\pm 1}]^\times.$$

The **global conductor of** χ,ψ is

$$a(\chi,\psi) := \sum_{x \in |F|} deg(x) a(\chi_x,\psi_x).$$

The **global conductor of** χ is

$$a(\chi) := \sum_{x \in |F|} deg(x) a(\chi_x) \in \mathbb{Z}_{\geq 0}.$$

The **global constant** of χ, ψ is

$$b(\chi, \psi) := \sum_{x \in |F|} b(\chi_x, \psi_x) \in \mathbb{C}^\times.$$

These products and sums make sense, because χ is unramified almost everywhere.

Theorem (Global functional equation) Let $\chi : GL_1(F) \backslash GL_1(\mathbb{A}) \to \mathbb{C}^\times$ be a smooth unitary character, and let $\psi : F \backslash \mathbb{A} \to \mathbb{C}^\times$ be a nontrivial character. Then:

1. $L(\chi; T) \in \mathbb{C}(T) \cap (1 + T\mathbb{C}[[T]]) \subset \mathbb{C}((T))$ (rationality, analytic continuation)
2. $L(\chi; T) = \epsilon(\chi, \psi; T) L(\chi^\vee, \frac{1}{qT})$ in $\mathbb{C}(T) \subset \mathbb{C}((T))$ (functional equation)

Note that the functional equation requires k to be algebraically closed in F.

Corollary $\epsilon(\chi, \psi; T) = \epsilon(\chi; T)$ is independent of our choice of additive character ψ.

October 11, 2002

Let k be a finite field of order q, F a function field over k, and let $\mathbb{A}$ be the ring of adèles of F. Assume k is algebraically closed in F. Let $\chi : GL_1(F) \backslash GL_1(\mathbb{A}) \to \mathbb{C}^\times$ be a smooth quasicharacter (called an **idèle class character**), and let $\psi : F \backslash \mathbb{A} \to \mathbb{C}^\times$ be a nontrivial continuous additive character.

For each $x \in |F|$, we get characters $\chi_x : GL_1(F_x) = F_x^\times \to \mathbb{C}^\times$ and $\psi_x : F_x \to \mathbb{C}^\times$ (nontrivial by strong approximation), L-factors $L(\chi_x; T) \in \mathbb{C}(T) \cap (1 + T\mathbb{C}[[T]]) \subset \mathbb{C}((T))$, and ϵ-factors $\epsilon(\chi_x, \psi_x; T) \in \mathbb{C}[T^{\pm 1}]^\times$.

Theorem Let χ be unitary.

1. (rationality, analytic continuation) The global L-function

$$L(\chi; T) := \prod_{x \in |F|} L(\chi_x; T^{deg(x)}) \in 1 + T\mathbb{C}[[T]] \subset \mathbb{C}((T))$$

   actually lies in $\mathbb{C}(T)^\times$ (This doesn't require χ to be unitary).

2. (functional equation) $L(\chi; T) = \epsilon(\chi, \psi; T) L(\chi^\vee, \frac{1}{qT})$, where we define the global ϵ-function to be

$$\epsilon(\chi, \psi; T) := \prod_{x \in |F|} \epsilon(\chi_x, \psi_x; T^{deg(x)}).$$

Corollary The ϵ-function $\epsilon(\chi, \psi; T) = \epsilon(\chi; T)$ is independent of ψ, so $b(\chi, \psi) = b(\chi) = \prod_{x \in |F|} b(\chi_x, \psi_x)$ is independent of ψ.

The sum $\sum_{x \in |F|} deg(x) c(\psi_x)$ is independent of ψ, and equals $2g(F) - 2 \in \mathbb{Z}$. $g(F)$ is called the **genus** of F.

Equivalence of curves and function fields

Let k be any field.

Definition A (proper/smooth) curve over k is a k-scheme $a_X : X \to \mathrm{Spec}(k)$ such that

1. X is integral (the generic point is denoted η_X).
2. The structure morphism a_X is separated and of finite type (proper/smooth if in statement).
3. For any closed point $x \in |X|$, the local ring $\mathcal{O}_{X,x}$ has Krull dimension 1.

The last condition is equivalent to the statement: The residue field at the generic point, $\kappa(X) = \kappa(\eta_X)$ has transcendence degree 1 over k.

Lemma Let X/k be a curve. The following are equivalent:

1. X is normal.
2. X is regular.
3. (when k is perfect) $a_X : X \to \mathrm{Spec}(k)$ is smooth.

Definition A **function field over** k is a k-algebra F that is a field, finitely generated as a k-extension, and of transcendence degree 1 over k.

Theorem Let k be a (perfect) field. The contravariant functor

$$\left\{ \begin{array}{c} \text{Proper (smooth) normal} \\ \text{curves over } k, \\ \text{dominant morphisms over } k \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{Function fields over } k, \\ k\text{-homomorphisms} \end{array} \right\}$$

$$\begin{array}{ccc} X & \mapsto & \kappa(\eta_X) = \kappa(X) \\ \downarrow & & \uparrow \\ Y & \mapsto & \kappa(\eta_Y) \end{array}$$

is an anti-equivalence of categories.

This is well-defined, because any dominant morphism takes the generic point of X to the generic point of Y. It is faithful because X is integral and Y is separated over

k. It is full because Y is locally notherian (see EGA). It is essentially surjective for the following reason: If F is an object on the right hand side, it is a finitely generated extension of k. Let $A = k[x_1, \ldots, x_n]/P$ be a finitely generated domain over k, such that $F \cong \text{Frac}(A)$. Let $U = \text{Spec}(A)$, and $\overline{U}$ = the Zariski closure of U in $\mathbf{P}^n$. If we let Y be the normalization of $\overline{U}$, F is isomorphic to the function field of Y.

Let X/k be a proper normal curve over k, and let $F = \kappa(X)$. Let $x \in X$ be a closed point, so we get a discrete valuation ring $\mathcal{O}_{X,x} \subset F$. We get a valuation map $x() : F^\times \twoheadrightarrow F^\times/\mathcal{O}_{X,x}^\times \xrightarrow{\sim} \mathbb{Z}$.

Proposition The map $|X| \to |F/k|$ given by $x \mapsto x()$ is a bijection.

The map is injective because of the chinese remainder theorem. It is surjective by the valuative criterion of properness and the maximal property of valuation rings with respect to the domination relation in F.

Residues on curves

Let k be a perfect field, X/k a proper smooth curve, and $F := \kappa(X)$, the field of functions.

Theorem (Existence of residues) For any $x \in |X|$, let $\mathcal{O}_x$ be the completion of $\mathcal{O}_{X,x}$ with respect to the x-adic topology, $F_x = \text{Frac}(\mathcal{O}_x)$ the completion of F with respect to x, and $\Omega^1_{F_x/\kappa(x)}$ the F_x-vector space of continuous differentials of F_x over $\kappa(x)$. Then there exists a unique $\kappa(x)$-linear homomorphism of $\kappa(x)$-vector spaces

$$res_x : \Omega^1_{F_x/\kappa(x)} \to \kappa(x)$$

such that for any choice of isomorphism of fields $\kappa(x)((T)) \xrightarrow{\sim} F_x$, and any $f, g \in F_x$ (with images $f(T) := \sum_{i>>-\infty} a_i T^i, g(T) := \sum_{i>>-\infty} b_i T^i \in F_x$, $a_i, B_i \in \kappa(x)$), one has

$$\begin{aligned} res_x(f \cdot dg \in \Omega^1_{F_x/\kappa(x)}) &= \text{ coefficient of } T^{-1} \text{ in } f(T)g'(T) \\ &= \sum_{i+j=0} j a_i b_j \in \kappa(x) \end{aligned}$$

Proposition

1. For any $\omega \in \Omega^1_{\mathcal{O}_x/\kappa(x)} \subset \Omega^1_{F_x/\kappa(x)}$, $res_x(\omega) = 0$.
2. For any $f \in F_x^\times$ and any $n \in \mathbb{Z}$,
$$res_x(f^n df) = \begin{cases} 0 \text{ if } n \neq -1 \\ \text{image of } v_x(f) \text{ in } \kappa(x) \text{ if } n = -1 \end{cases}$$
3. For any uniformizer $t_x \in \mathcal{O}_x \subset F_x$, we get $dt_x \in \Omega^1_{F_x/\kappa(x)}$, which generates the one-dimensional F_x vector space $\Omega^1_{F_x/\kappa(x)}$. For any $\omega \in \Omega^1_{F_x/\kappa(x)}$, and any $f = \sum_{i>>-\infty} a_i t_x^i$, $res_x(f \cdot dt_x) = a_{-1}$.

Theorem (Residue theorem) For any global differential $\omega \in \Omega^1_{F/k}$, we have

$$\sum_{x \in |X|} Tr_{\kappa(x)/k}(res_x(\omega_x)) = 0 \in k,$$

where for each $x \in |X|$, $\omega_x \in \Omega^1_{F_x/\kappa(x)}$ is induced from ω by the commutative diagram

$$\begin{array}{ccc} F & \hookrightarrow & F_x \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d} \\ \Omega^1_{F/k} & \longrightarrow & \Omega^1_{F_x/\kappa(x)} \end{array}$$

Fix a nontrivial additive character $\psi_k : k \to \mathbb{R}/\mathbb{Z}$ (e.g., choose a nontrivial $\psi_{\mathbb{F}_p} : \mathbb{F}_p \to \mathbb{R}/\mathbb{Z}$ and set $\psi_k(-) := \psi_{\mathbb{F}_p}(Tr_{k/\mathbb{F}_p}(-)))$. We avoid $U(1)$ so we don't have to choose an algebraic closure $\mathbb{C}$ of $\mathbb{R}$.

Proposition Fix $x \in |X|$.

1. The homomorphism

   $$\Omega^1_{F_x/\kappa(x)} \to Hom_{\text{top. ab. gp.}}(F_x, \mathbb{R}/\mathbb{Z})$$

   given by

   $$\omega \mapsto (a \mapsto \psi_k(Tr_{\kappa(x)/k}(res_x(a\omega))))$$

   is an isomorphism of topological abelian groups ($\Omega^1_{F_x/\kappa(x)}$ is noncanonically isomorphic to F_x).

2. If $\psi \in Hom_{\text{top. ab. gp.}}(F_x, \mathbb{R}/\mathbb{Z})$ corresponds to $\omega \in \Omega^1_{F_x/\kappa(x)}$, the conductor $c(\psi) := \max \{c \in \mathbb{Z} : \psi|_{\mathfrak{m}^{-c}} \text{ is trivial } \}$ is equal to $v_x(\omega)$, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_x$ in F_x, $v_x(f dt) := v_x(f)$ if $f, t \in F_x$ and t is a uniformizer.

Proposition

1. The homomorphism

   $$\Omega^1_{F/k} \to Hom_{\text{top. ab. gp.}}(F\backslash\mathbb{A}, \mathbb{R}/\mathbb{Z})$$

   given by

   $$\omega \mapsto (a \mapsto \psi_k(\sum_{x \in |X|} Tr_{\kappa(x)/k}(res_x(a_x\omega_x))))$$

   is an isomorphism of **discrete** topological abelian groups (note that the sum is well-defined by the residue theorem, and the sequence $0 \to F \to \mathbb{A} \to F\backslash\mathbb{A} \to 0$ is Pontryagin self-dual).

2. For any $x \in |X|$, the diagram

$$\begin{array}{ccc} Hom_{\text{top. ab. gp.}}(F\backslash\mathbb{A}, \mathbb{R}/\mathbb{Z}) & \longrightarrow & Hom_{\text{top. ab. gp.}}(F_x, \mathbb{R}/\mathbb{Z}) \\ \cong \uparrow & & \uparrow \cong \\ \Omega^1_{F/k} & \longrightarrow & \Omega^1_{F_x/\kappa(x)} \end{array}$$

commutes, with the top arrow taking ψ to ψ_x.

3. If $\psi \in Hom_{\text{top. ab. gp.}}(F\backslash\mathbb{A}, \mathbb{R}/\mathbb{Z})$ corresponds to a nonzero $\omega \in \Omega^1_{F/k}$, then $\sum_{x\in|F|} deg(x)c(\psi) = \sum_{x\in|X|} deg(x)v_x(\omega_x)$ is independent of ψ, and equal to $C\cdot(2g-2) = k$-degree of the canonical divisor K of X, where if k' is $H^0(X, \mathcal{O}_X)$, the algebraic closure of k in F, then $C = [k' : k] = dim_k H^0(X, \mathcal{O}_X)$, $g = dim_{k'} H^1(X, \mathcal{O}_X)$, and $C \cdot g = dim_k H^1(X, \mathcal{O}_X)$.

Switch to $GL_r, r \geq 2$

The notable fact about the $r \geq 2$ case is that the groups are non-abelian. For $r = 2$, the unipotent radical has the form $\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \cong \mathbb{G}_a$ which is abelian, so Fourier analysis is straightforward. This is not so for $r \geq 3$, and we use **Whittaker models** instead.

Let $r \geq 1$ be fixed. We get algebraic groups $GL_r \supset B_r \supset N_r =$ upper triangulars with ones on the diagonal. These are functors from commutative rings to groups. Let $\begin{cases} F \text{ be a nonarchimedean local field.} \\ \mathbb{A} \text{ be the adèles of a function field } F/k, k \text{ finite.} \end{cases}$ Choose an algebraic closure $\mathbb{C}$ of $\mathbb{R}$, so we get an isomorphism $\mathbb{R}/\mathbb{Z} \xrightarrow{\sim} U(1) \subset \mathbb{C}^\times$. Let $\begin{cases} \psi : F \to \mathbb{C}^\times \\ \psi : F\backslash\mathbb{A} \to \mathbb{C}^\times \end{cases}$ be a nontrivial additive character. Extend ψ to $\begin{cases} N_r(F) \to \mathbb{C}^\times \\ N_r(\mathbb{A}) \to \mathbb{C}^\times \end{cases}$ by

$$\begin{pmatrix} 1 & x_{1,2} & & * \\ & \ddots & \ddots & \\ & & \ddots & x_{r-1,r} \\ 0 & & & 1 \end{pmatrix} \mapsto \psi(x_{1,2} + \cdots + x_{r-1,r}).$$

Definition A (smooth) **Whittaker function on** $\begin{cases} GL_r(F) \\ GL_r(\mathbb{A}) \end{cases}$ **with respect to** ψ is a function $\begin{cases} w : GL_r(F) \to \mathbb{C} \\ w : GL_r(\mathbb{A}) \to \mathbb{C} \end{cases}$ that is uniformly locally constant (i.e., the smoothness is invariant under right translation by some open subgroup) such that for all

$$\begin{cases} n \in N_r(F), x \in GL_r(F), \\ n \in N_r(\mathbb{A}), x \in GL_r(\mathbb{A}), \end{cases} \quad w(nx) = \psi(n)w(x) \in \mathbb{C}.$$

From now until the next theorem, we will just use local notation for convenience. For the global statements, just replace F by $\mathbb{A}$.

Let $\mathcal{W}(GL_r(F), \psi) := \{$all Whittaker functions on $GL_r(F)$ with respect to $\psi\}$. This is a complex vector space, and $GL_r(F)$ acts on it by right translation:

$$\begin{array}{rcl} R_\psi : GL_r(F) & \to & GL(\mathcal{W}(GL_r(F), \psi)) \\ g & \mapsto & (w \mapsto (x \mapsto w(xg))) \end{array}$$

We get $(\mathcal{W}(GL_r(F), \psi), R_\psi) = Ind_{N_r(F)}^{GL_r(F)}(\mathbb{C}_\psi, \psi) \in Rep_{smooth}(GL_r(F))$.

Definition Let $(V_\pi, \pi) \in Rep_{smooth}(GL_r(F))$ be an irreducible representation. If there exists a nontrivial homomorphism in Rep:

$$(V_\pi, \pi) \to (\mathcal{W}(GL_r(F), \psi), R_\psi)$$

(necessarily injective) then the image of (V_π, π) is a **Whittaker model of** (V_π, π), denoted $\mathcal{W}(\pi, \psi) \subset \mathcal{W}(GL_r(F), \psi)$, and (V_π, π) is called **generic**. There is a uniqueness statement coming later.

Definition Let $(V_\pi, \pi) \in Rep_{smooth}(GL_r(F))$ be irreducible. A **Whittaker functional on** (V_π, π) **with respect to** ψ is a $\mathbb{C}$-linear homomorphism of complex vector spaces $\Lambda : V_\pi \to \mathbb{C}$ such that for all $n \in N_r(F), \xi \in V_\pi$, one has $\Lambda(\pi(n) \cdot \xi) = \psi(n) \cdot \Lambda(\xi)$, i.e., $\Lambda \in Hom_{Rep_{smooth}(N_r(F))}(Res_{N_r(F)}^{Gl_r(F)}(V_\pi, \pi), (\mathbb{C}_\psi, \psi))$.

Proposition (Frobenius Reciprocity) The complex linear maps

$$Hom_{Rep(N_r(F))}(Res_{N_r(F)}^{GL_r(F)}(V_\pi, \pi), (\mathbb{C}_\psi, \psi)) \rightleftarrows$$
$$\rightleftarrows Hom_{Rep(GL_r(F))}((V_\pi, \pi), Ind_{N_r(F)}^{GL_r(F)}(\mathbb{C}_\psi, \psi))$$

are inverse isomorphisms via

$$\begin{array}{ccc} \Lambda & \mapsto & (\xi \mapsto w_\xi := (x \mapsto \Lambda(\pi(x) \cdot \xi))) \\ (\xi \mapsto w_\xi(1 \in GL_r(F))) & \leftarrow & \begin{pmatrix} w_- : (V_\pi, \pi) \to (\mathcal{W}(GL_r(F), \psi), R_\psi) \\ \xi \mapsto w_\xi \end{pmatrix} \end{array}$$

Theorem (Uniqueness of local Whittaker models - Gelfand-Kazhdan for nonarchimedean local fields, Shalika for archimedean case) Let $(V_\pi, \pi) \in Rep_{smooth}(GL_r(F))$ be irreducible, and let $\psi : F \to \mathbb{C}^\times$ be a nontrivial character. Then

$$dim_{\mathbb{C}} Hom_{Rep(N_r(F))}(Res_{N_r(F)}^{GL_r(F)}(V_\pi, \pi), (\mathbb{C}_\psi, \psi)) \leq 1.$$

The proof uses a detailed analysis of the Bruhat decomposition of GL_r.

Corollary (Uniqueness of global Whittaker models) Let $\psi : F\backslash\mathbb{A} \to \mathbb{C}^\times$ be a nontrivial character. Then

$$dim_{\mathbb{C}} Hom_{Rep(N_r(\mathbb{A}))}(Res^{GL_r(\mathbb{A})}_{N_r(\mathbb{A})}(V_\pi, \pi), (\mathbb{C}_\psi, \psi)) \leq 1.$$

Proof (main steps) Let $\Lambda : V_\pi \to \mathbb{C}_\psi$ be a nontrivial global Whittaker functional. Put it in the fridge for a while (we'll use it later). By Flath,

$$(V_\pi, \pi) \subset \bigotimes_{x\in|X|} ((V_{\pi_x}, \pi_x), \overset{0}{\xi}_x \in V_{\pi_x}),$$

where $(V_{\pi_x}, \pi_x) \in Rep_{adm}(GL_r(F_x))$ is irreducible (recall that Jacquet showed that any smooth representation of GL_r is admissible), and we may take $\overset{0}{\xi}_x \in V_{\pi_x}$ to be fixed by $GL_r(\mathcal{O}_x)$ whenever π_x is unramified, i.e., whenever $dim_{\mathbb{C}} V_{\pi_x}^{GL_r(\mathcal{O}_x)} = 1$ (note that π_x is unramified for almost all x).

Adjust the distinguished vectors $\overset{0}{\xi}_x \in V_{\pi_x}$ at finitely many $x \in |F|$. Then we may assume $\overset{0}{\xi} = \bigotimes_{x\in|F|} \overset{0}{\xi}_x \in V_\pi$ is such that $\Lambda(\overset{0}{\xi}) = 1 \in \mathbb{C}$.

For each $x \in |F|$, we get an embedding $V_{\pi_x} \hookrightarrow V_\pi$ given by

$$\xi_x \mapsto \xi_x \otimes (\bigotimes_{x'\in|F|, x'\neq x} \overset{0}{\xi}_{x'}).$$

Precompose to get the local Whittaker functional: $\Lambda_x : V_{\pi_x} \hookrightarrow V_\pi \xrightarrow{\Lambda} \mathbb{C}$ of V_{π_x} with respect to ψ_x satisfying $\Lambda_x(\overset{0}{\xi}_x) = 1$. By local uniqueness, Λ is equal to $\bigotimes_{x\in|F|} \Lambda_x :$ $V_\pi \to \mathbb{C}$ given by $\bigotimes_{x\in|F|} \xi_x \mapsto \prod_{x\in|F|} \Lambda_x(\xi_x)$, where $\xi_x = \overset{0}{\xi}_x$ for almost all x. This gives us global uniqueness.

October 18, 2002

Remark Let $(V_\pi, \pi) \in Rep_{adm}(GL_r(\mathbb{A}))$ be irreducible and generic, and let $\Lambda : V_\pi \to \mathbb{C}$ be a nontrivial Whittaker functional with respect to ψ. Choose a decomposition $(V_\pi, \pi) \cong \bigotimes_{x\in|F|}(V_{\pi_x}, \pi_x; \overset{0}{\xi}_x \in V_{\pi_x})$ such that $\Lambda(\bigotimes_{x\in|F|} \overset{0}{\xi}_x) = 1 \in \mathbb{C}_\psi$. For each $x \in |F|$, we have $\Lambda_x : V_{\pi_x} \hookrightarrow V_\pi \xrightarrow{\Lambda} \mathbb{C}$ a nontrivial local Whittaker functional with respect to ψ_x, that sends $\overset{0}{\xi}_x \in V_{\pi_x}$ to $1 \in \mathbb{C}_{\psi_x}$. We get a global Whittaker model $\mathcal{W}_\Lambda(\pi, \psi) := \{w_\xi := (g \mapsto \Lambda(\pi(g) \cdot \xi))\}_{\xi \in V_\pi}$, and local Whittaker models $\mathcal{W}_{\Lambda_x}(\pi_x, \psi_x) := \{w_{\xi_x} := (g_x \mapsto \Lambda_x(\pi_x(g_x) \cdot \xi_x))\}_{\xi_x \in V_{\pi_x}}$.

For a pure tensor $\xi = \bigotimes_{x\in|F|} \zeta_x \in V_\pi$ with $\xi_x = \zeta_x$ for almost all $x \in |F|$, using global uniqueness, we get $w_\xi = \prod_{x\in|F|} w_{\xi_x}$. This is an equality of functions on $GL_r(\mathbb{A})$ fix via $w_\xi(g) = \prod w_{\xi_x}(g)$.

Definition Let $\psi : F\backslash\mathbb{A} \to \mathbb{C}^\times$ be a nontrivial additive character. For a cusp form $\psi \in \mathcal{A}_{cusp}(GL_r(A), \mathbb{C})$ (in particular, an actual automorphic uniformly locally constant admissible cuspidal function on $GL_r(\mathbb{A})$), its **associated Whittaker function** is $W_\phi := W_{\phi,\psi} : GL_r(\mathbb{A}) \to \mathbb{C}$ defined by

$$g \mapsto W_{\phi,\psi}(g) := \int_{N_r(F)\backslash N_r(\mathbb{A})} \phi(ng)\psi^{-1}(n)\frac{dn}{d\mu},$$

where $\frac{dn}{d\mu}$ is the **right** $N_r(\mathbb{A})$-invariant measure on the compact space $N_r(F)\backslash N_r(\mathbb{A})$ giving it total volume 1.

One has $W_{\phi,\psi} \in \mathcal{W}(GL_r(\mathbb{A}), \psi)$, i.e., this **is** an actual Whittaker function, but it **may** be 0. The $\mathbb{C}$-linear homomorphism $\mathcal{A}_{cusp}(GL_r(\mathbb{A})) \to \mathcal{W}(GL_r(\mathbb{A}), \psi)$ given by $\phi \mapsto W_\phi$ is GL_r-invariant.

Theorem (Fourier-Whittaker expansion of cusp forms - Piatetski-Shapiro, Shalika) Let $\psi : F\backslash\mathbb{A} \to \mathbb{C}$ be any nontrivial additive character, $r \geq 2$. Then for any $\phi \in \mathcal{A}_{cusp}(GL_r(\mathbb{A}))$ with its associated ψ-Whittaker function $W_{\phi,\psi} \in \mathcal{W}(GL_r(\mathbb{A}), \psi)$, we have

$$\phi(g) = \sum_{\gamma\in N_{r-1}(F)\backslash GL_{r-1}(F)} W_{\phi,\psi}(\left(\begin{array}{c|c} \gamma & \\ \hline & 1 \end{array}\right) = g) \in \mathbb{C}$$

This is locally uniformly absolutely convergent in $\mathbb{C}$. (Note that the γ in the matrix is $r-1$ by $r-1$.)

Corollary (Existence of global Whittaker models) Let $(V_\pi, \pi) \in Rep_{adm}(GL_r(\mathbb{A}))$ be a **cuspidal** automorphic irreducible representation. Then

$$dim_{\mathbb{C}} Hom_{Rep_{smooth}(N_r(\mathbb{A}))}(Res^{GL_r(\mathbb{A})}_{N_r(\mathbb{A})}(V_\pi, \pi), (\mathbb{C}_\psi, \psi)) = 1,$$

i.e., the space of ψ-Whittaker functionals on V_π has dimension one.

Proof The function $\Lambda : V_\pi \to \mathbb{C}$ defined by

$$\phi \mapsto W_{\phi,\psi}(1 \in GL_r(\mathbb{A})) = \int_{N_r(F)\backslash N_r(\mathbb{A})} \phi(n)\psi^{-1}(n)\frac{dn}{d\mu}$$

is a nonzero element of the space of ψ-Whittaker functionals on V_π.

Theorem (Weak multiplicity one) Let $(V_\pi, \pi) \in Rep_{adm}(GL_r(\mathbb{A}))$ be a cuspidal automorphic irreducible representation. Then

$$dim_{\mathbb{C}} Hom_{Rep_{smooth}(GL_r(\mathbb{A}))}((V_\pi, \pi), (\mathcal{A}_{cusp}(GL_r(\mathbb{A})), R_{cusp})) = 1.$$

Proof Suppose (V_π, π) has two embeddings (V_{π_1}, π_1) and (V_{π_2}, π_2) in $(\mathcal{A}_{cusp}, R_{cusp})$. For $\phi \in V_\pi$, let $\phi_i \in V_{\pi_i}$ be the image (an actual cusp form) under each embedding. Choose any nontrivial $\psi : F\backslash\mathbb{A} \to \mathbb{C}^\times$, and get Whittaker functionals on V_{π_i} with respect to ψ given by $\Lambda_i : V_{\pi_i} \to \mathbb{C}$, $\phi \mapsto W_{\phi,\psi,i}$, which are nonzero by existence of global Whittaker models. By uniqueness of global Whittaker models, $\Lambda_1 = c\Lambda_2$ for some $c \in \mathbb{C}^\times$, so for any $g \in GL_r(\mathbb{A})$, and any $\phi \in V_\pi$, $W_{\phi,\psi,1}(g) = \Lambda_1(\pi(g)\cdot\phi) = c\Lambda_2(\pi(g)\cdot\phi) = cW_{\phi,\psi,2}(g)$. By the Fourier-Whittaker expansion formula, we have for any $g \in GL_r(\mathbb{A})$ and any $\phi \in V_\pi$,

$$\phi_1(g) = \sum_\gamma W_{\phi,\psi,1}(\left(\begin{array}{c|c} \gamma & \\ \hline & 1 \end{array}\right) g) = c\sum_\gamma W_{\phi,\psi,2}(\left(\begin{array}{c|c} \gamma & \\ \hline & 1 \end{array}\right) g) = c\phi_2(g)$$

Hence, $V_{\pi_1} = V_{\pi_2}$ as subspaces of $\mathcal{A}_{cusp}$.

Not-so-crucial part Mirabolic subgroups (miracle-parabolic)

Let $\psi : F\backslash\mathbb{A} \to \mathbb{C}^\times$ be a nontrivial character, and extend it to $\psi : N_r(\mathbb{A}) \to \mathbb{C}^\times$. Let $P_r \subset GL_r$ be the mirabolic subgroup.

Definition A **mirabolic subgroup** of GL_r is the stabilizer group of **any** nonzero element of the standard or dual representation of GL_r

$$P_n = \{\left(\begin{array}{c|c} h & y \\ \hline 0 & 1 \end{array}\right) : h \in GL_{r-1}, y \in \mathbb{A}^{r-1}_{col}\} \cong Y_r.GL_{r-1}$$

where $Y_r = \{\left(\begin{array}{c|c} I_{r-1} & y \\ \hline 0 & 1 \end{array}\right) : y \in \mathbb{A}^{r-1}\} \cong \mathbb{G}_a^{r-1}$. We have a unipotent radical $N_r \subset P_r$, but P_r is **not** actually parabolic, since it does not contain a Borel (bottom right entry can't vary).

Definition A **cuspidal function** on $P_r(\mathbb{A})$ is a function $\phi : P_r(\mathbb{A}) \to \mathbb{C}$ that is:

1. (automorphic) For any $\gamma \in P_r(F)$ and any $x \in P_r(\mathbb{A})$, $\phi(\gamma x) = \phi(x)$
2. (uniformly locally constant) There exists an open subgroup $K \subset P_r(\mathbb{A})$ such that for any $k \in K$ and any $x \in P_r(\mathbb{A})$, $\phi(xk) = \phi(x)$.
3. (cuspidal) For any $x \in P_r(\mathbb{A})$ and any $U \subset P_r$ standard unipotent subgroup,

$$\int_{U(F)\backslash U(\mathbb{A})} \phi(ux)\frac{du}{d\mu} = 0.$$

Facts If $\phi \in \mathcal{A}_{cusp}(GL_r(\mathbb{A}))$, we can restrict ϕ to $\underline{\phi}$ a cuspidal function of $P_r(\mathbb{A})$. Since $N_r \subset P_r$, we can define Whittaker functions associated to any cuspidal function of $P_r(\mathbb{A})$.

Main theorem For any $r \geq 1$, any cuspidal function ϕ on $P_r(\mathbb{A})$, and any $x \in P_r(\mathbb{A})$,

$$\phi(x) = \sum_{\gamma \in N_{r-1}(F)\backslash GL_{r-1}(F)} W_{\phi,\psi}(\left(\begin{array}{c|c} \gamma & 0 \\ \hline 0 & 1 \end{array}\right) \cdot x) \in \mathbb{C}$$

Rankin-Selberg theory (local theory) ref: Jacquet, Piatetskii-Shapiro, Shalika. "Rankin-Selberg convolutions" Amer. J. Math 105 #2 (1983)

Let F be a nonarchimedean local field. We get $1 \to \mathcal{O}^\times \to F^\times \xrightarrow{v_F} \mathbb{Z} \to 0$. Let k be the residue field, of order q and characteristic p. Let $\psi : F \to \mathbb{C}^\times$ be a nontrivial additive unitary character, and extend it to $N_r(F)$ for all $r \geq 1$. At some point in the future, we will choose $p^{1/2} \in \mathbb{C}^\times$, and get $q^{1/2} \in \mathbb{C}^\times$.

Definition Let $r \geq r' \geq 1$ be integers. We get groups GL_r and GL'_r. Let $(V_\pi, \pi) \in Rep_{smooth}(GL_r(F))$ and $(V_{\pi'}, \pi') \in Rep_{smooth}(GL_{r'}(F))$ be irreducible and generic. We get Whittaker models $\mathcal{W}(\pi, \psi)$ and $\mathcal{W}(\pi', \psi^{-1})$. These are spaces of Whittaker functions. We also have the Schwartz-Bruhat space $C_c^\infty(F^r_{rows})$. Choose left Haar measures dg' on $GL_{r'}(F)$ and dn on $N_r(F)$. If $r > r' \geq 1$, define a $\mathbb{C}$-bilinear map:

$$\begin{array}{ccccc} \mathcal{W}(\pi,\psi) & \times & \mathcal{W}(\pi',\psi^{-1}) & \to & \mathbb{C}((T)) \\ w & & w' & \mapsto & \Psi(w,w';T) \end{array}$$

where

$$\begin{aligned} \Psi(w,w';T) &:= \int_{N_{r'}(F)\backslash GL_{r'}(F)} w(\left(\begin{array}{c|c} g' & \\ \hline & 1_{r-r'} \end{array}\right)) w'(g')(q^{\frac{r-r'}{2}} \cdot T)^{v_F(det(g'))} \frac{dg'}{dn} \\ &= \sum_{v \in \mathbb{Z}} T^v \int_{g \in N_{r'}(F)\backslash GL_{r'}(F), v_F(det(g'))=v} w(\left(\begin{array}{c|c} g' & \\ \hline & 1_{r-r'} \end{array}\right)) w'(g') q^{\frac{r-r'}{2}\cdot v} \frac{dg'}{dn} \end{aligned}$$

Note that ψ^{-1} is to cancel the contribution of $N_{r'}(F)$. If $r = r' \geq 1$, define a $\mathbb{C}$-trilinear map:

$$\begin{array}{ccccccc} \mathcal{W}(\pi,\psi) & \times & \mathcal{W}(\pi',\psi^{-1}) & \times & C_c^\infty(F_{rows}) & \to & \mathbb{C}((T)) \\ w & & w' & & \Phi & \mapsto & \Psi(w,w',\Phi;T) \end{array}$$

where

$$\Psi(w,w',\Phi;T) := \int_{N_r(F)\backslash GL_r(F)} w(g)w'(g)\Phi(e_r g) T^{v_F(det(g))} \frac{dg}{dn}$$

Here, $e_r = (0, \ldots, 0, 1)$ is the last standard basis vector.

Lemma These are well-defined. (some kind of convergence argument)

Theorem (Rationality of Rankin-Selberg integrals)

1. Let $w \in \mathcal{W}(\pi, \psi)$, and $w' \in \mathcal{W}(\pi', \psi^{-1})$. If $r = r'$, let $\Phi \in C_c^\infty(F^r_{rows})$ be given. There exists $\epsilon > 0$ and nontrivial functions $Q(w, w'; T) \in \mathbb{C}(T)$, and (if $r = r'$) $Q(w, w', \Phi; T) \in \mathbb{C}(T)$ such that on the open punctured disc $\mathbb{D} := \{t \in \mathbb{C} : 0 < |t| < \epsilon\}$,
   (a) The integrals $\Psi(w, w'; t)$ and $\Psi(w, w', \Phi; t)$ converge locally uniformly and absolutely on $\mathbb{D}$
   (b) On $\mathbb{D}$, one has an equality of analytic functions:
   $$\begin{aligned} \Psi(w, w'; t) &= Q(w, w'; t) \\ \Psi(w, w', \Phi; t) &= Q(w, w', \Phi; t) \end{aligned}$$
2. (same as Tate's thesis) Let $\mathcal{I}(\pi, \pi') \subset \mathbb{C}(T)$ be the complex vector space, defined by
   $$(r > r') \quad \mathcal{I}(\pi, \pi') := \mathbb{C}\{Q(w, w'; T) \in \mathbb{C}(T) : w \in \mathcal{W}(\pi, \psi), w' \in \mathcal{W}(\pi', \psi^{-1})\}$$
   $$(r = r') \quad \mathcal{I}(\pi, \pi') := \mathbb{C}\left\{ Q(w, w', \Phi; T) \in \mathbb{C}(T) : \begin{array}{c} w \in \mathcal{W}(\pi, \psi) \\ w' \in \mathcal{W}(\pi', \psi^{-1}) \\ \Phi \in C_c^\infty(F^r_{rows}) \end{array} \right\}$$
   (a) Then, $\mathcal{I}(\pi, \pi') \subset \mathbb{C}(T)$ is a $\mathbb{C}[T^{\pm 1}]$-fractional ideal, independent of any choice of dg', dn, and ψ, i.e., the Qs have bounded denominators, except for powers of T.
   (b) There exist choices $w \in \mathcal{W}(\pi, \psi)$, $w' \in \mathcal{W}(\pi', \psi^{-1})$, $\Phi \in C_c^\infty(F^r_{rows})$ such that $Q(w, w'; T) = 1$ and $Q(w, w', \Phi; T) = 1$, so the ideals $\mathcal{I}(\pi, \pi')$ are nonzero.

Definition The **L-factor associated to the pair** (π, π') is the **unique** element $L(\pi \times \pi'; T) \in \mathbb{C}(T)^\times \cap (1 + T\mathbb{C}[[T]]) \subset \mathbb{C}((T))^\times$ such that we have an equality of free ideals $\mathbb{C}[T, T^{-1}]L(\pi \times \pi'; T) = \mathcal{I}(\pi, \pi') \subset \mathbb{C}(T)$.

Note $L(\pi \times \pi'; T)^{-1} \in \mathbb{C}[T]$ is a polynomial.

Remark The L-factor depends on $p^{1/2}$, but not on dg', dn, or ψ.

Remark One can verify that if $r = r'$, then $L(\pi \times \pi'; T) = L(\pi' \times \pi; T)$.

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Let F be a nonarchimedean local field, and let k be its residue field of characteristic p and order q. Choose $\mathbb{C}$ and $p^{1/2} \in \mathbb{C}^\times$. We get $q^{1/2} \in \mathbb{C}$. Let $\psi : F \to \mathbb{C}^\times$ be a nontrivial additive character. Fix $r \geq r' \geq 1$.

Let $(V_\pi, \pi) \in Rep_{smooth}(GL_r(F))$ and $(V_{\pi'}, \pi') \in Rep_{smooth}(GL_{r'}(F))$ be irreducible and generic. Extend ψ to $N_r(F)$ and $N_{r'}(F)$ as before. We get Whittaker models $\mathcal{W}(\pi,\psi)$ and $\mathcal{W}(\pi',\psi^{-1})$, which are spaces of functions. If $r = r'$, we also get the Schwartz-Bruhat space $C_c^\infty(F^r_{rows})$.

Choose Haar measures dg' on $GL_{r'}(F)$ and dn on $N_r(F)$. Let dx be the Haar measure on $M_{(r-r'-1)\times r}(F)$ (for $r > r'$) that is self-dual with respect to the pairing $M \times M \to F$ given by $x \cdot y \mapsto Tr(x^t y)$.

Definition(Rankin-Selberg integrals) For $r > r'$, we get two homomorphisms:

$$\begin{array}{ccccc} \mathcal{W}(\pi,\psi) & \times & \mathcal{W}(\pi',\psi^{-1}) & \to & \mathbb{C}((T)) \\ w & & w' & \mapsto & \Psi(w,w';T) \\ w & & w' & \mapsto & \widetilde{\Psi}(w,w';T) \end{array}$$

given by

$$\Psi(w,w';T) := \int_{N_{r'}(F)\backslash GL_{r'}(F)} w\left(\begin{array}{c|c} g' & \\ \hline & 1_{r-r'} \end{array}\right) w'(g')(q^{\frac{r-r'}{2}}T)^{v_F(det(g'))}\frac{dg'}{dn}$$

$$\widetilde{\Psi}(w,w';T) := \int_{N_{r'}(F)\backslash GL_{r'}(F)} \left[\int_{M_{r-r'-1\times r'}(F)} w\left(\begin{array}{c|c|c} g' & & \\ \hline x & 1 & \\ \hline & & 1 \end{array}\right) dx\right] \times$$
$$\times\, w'(g')(q^{\frac{r-r'}{2}}T)^{v_F(det(g'))}\frac{dg'}{dn}$$

For $r = r'$, we have a trilinear map:

$$\begin{array}{ccccccc} \mathcal{W}(\pi,\psi) & \times & \mathcal{W}(\pi',\psi^{-1}) & \times & C_c^\infty(F^r_{rows}) & \to & \mathbb{C}((T)) \\ w & & w' & & \Phi & \mapsto & \Psi(w,w',\Phi;T) \end{array}$$

defined by

$$\int_{N_r(F)\backslash GL_r(F)} w(g)w'(g)\Phi(e_r g)T^{v_F(det(g))}\frac{dg}{dn}$$

Recall the rationality theorem:

1. For any $w \in \mathcal{W}(\pi,\psi)$, $w' \in \mathcal{W}(\pi',\psi^{-1})$, and $\Phi \in C_c^\infty(F^r_{rows})$, the functions $\Psi(w,w';T)$, $\widetilde{\Psi}(w,w';T)$, and $\Psi(w,w',\Phi;T)$ are elements of $\mathbb{C}(T)$.
2. Let $\mathcal{I}(\pi,\pi')$ be the complex vector space spanned by all of these Ψs. Then $\mathcal{I}(\pi,\pi')$ is a $\mathbb{C}[T,T^{-1}]$-fractional ideal, and contains $1 \in \mathcal{I} \subset \mathbb{C}(T)$.

Definition An L-factor of the pair (π,π') is the unique element $L(\pi\times\pi';T) \in \mathbb{C}(T)^\times \cap (1 + T\mathbb{C}[[T]]) \subset \mathbb{C}((T))^\times$ such that $\mathbb{C}[T,T^{-1}]\cdot L(\pi\times\pi';T) = \mathcal{I}(\pi,\pi')$.

Remember that $L(\pi\times\pi';T)$ has the form 1/(polynomial with constant term 1).

Definition For $w \in \mathcal{W}(\pi, \psi)$, set $\widetilde{w} : GL_r(F) \to \mathbb{C}$ by $g \mapsto w(w_r \cdot {}^t g^{-1})$, where $w_r := \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}$ is the longest element of the Weyl group. Define $\widetilde{w'} : GL_{r'}(F) \to \mathbb{C}$ for each $w \in \mathcal{W}(\pi', \psi^{-1})$ the same way.

Lemma $\widetilde{w} \in \mathcal{W}(\pi^\vee, \psi^{-1})$, and $\widetilde{w'} \in \mathcal{W}(\pi'^\vee, \psi)$.

Remark $GL_r(F)$ acts on $\mathcal{W}(\pi, \psi)$ and on $\mathcal{W}(\pi^\vee, \psi^{-1})$ by right translation ρ. Set

$$w_{r,r'} := \left(\begin{array}{c|c} 1_{r'} & \\ \hline & w_{r-r'} \end{array}\right) \in GL_r(F),$$

where $w_{r-r'}$ is the square matrix of size $r - r'$ with ones on the antidiagonal and zeroes elsewhere. Then $\rho(w_{r,r'}) \cdot \widetilde{w}$ is the function $GL_r(F) \to \mathbb{C}$ given by

$$g \mapsto \widetilde{w}(g \cdot w_{r,r'}) := w(w_r \cdot {}^t g^{-1} w_{r,r'}).$$

Definition The **Fourier transform on $C_c^\infty(F^r_{rows})$ with respect to ψ** is the map

$$\begin{array}{ccc} C_c^\infty(F^r_{rows}) & \to & C_c^\infty(F^r_{rows}) \\ \Phi & \mapsto & \widehat{\Phi} := (y \mapsto \int_{F^r} \Phi(x)\psi(Tr(x \cdot {}^t y))dx) \end{array}$$

Here, dx is the (unique) self-dual Haar measure on F^r with respect to ψ, i.e., the Fourier inversion formula $\widehat{\widehat{\Phi}}(x) = \Phi(-x)$.

Theorem (Functional equation) There exists a **unique** rational function $\gamma(\pi \times \pi'; \psi; T) \in \mathbb{C}(T)$ such that for any $w \in \mathcal{W}(\pi, \psi)$, $w' \in \mathcal{W}(\pi', \psi^{-1})$, and $\Phi \in C_c^\infty(F^r_{rows})$ (giving functions $\widetilde{w} \in \mathcal{W}(\pi^\vee, \psi^{-1})$, $\widetilde{w'} \in \mathcal{W}(\pi'^\vee, \psi)$, and $\widehat{\Phi} \in C_c^\infty(F^r_{rows})$), one has the following equalities in $\mathbb{C}(T)$:

$$\widetilde{\Psi}(\rho(w_{r,r'})\widetilde{w}, \widetilde{w'}; \frac{1}{qT}) = \chi_{\pi'}(-1)^{r-1}\gamma(\pi \times \pi', \psi; T)\Psi(w, w'; T) \qquad r > r'$$

$$\Psi(\widetilde{w}, \widetilde{w'}, \widehat{\Phi}; \frac{1}{qT}) = \chi_{\pi'}(-1)^{r-1}\gamma(\pi \times \pi', \psi; T)\Psi(w, w', \Phi; T) \qquad r = r'$$

Definition The **ϵ-factor associated to (π, π') and ψ** is defined by:

$$\epsilon(\pi \times \pi', \psi; T) := \gamma(\pi \times \pi', \psi; T)\frac{L(\pi \times \pi'; T)}{L(\pi^\vee \times \pi'^\vee; \frac{1}{qT})} \in \mathbb{C}(T)^\times$$

Corollary (Local Functional Equation) For any $w \in \mathcal{W}(\pi, \psi)$, $w' \in \mathcal{W}(\pi', \psi^{-1})$, and $\Phi \in C_c^\infty(F^r_{rows})$, get the corresponding $\widetilde{w}$, $\widetilde{w'}$, and $\widehat{\Phi}$. Then one has

$$\frac{\widetilde{\Psi}(\rho(w_{r,r'})\widetilde{w}, \widetilde{w'}; \frac{1}{qT})}{L(\pi^\vee \times \pi'^\vee; \frac{1}{qT})} = \epsilon(\pi \times \pi', \psi; T)\frac{\Psi(w, w'; T)}{L(\pi \times \pi'; T)} \in \mathbb{C}(T)$$

If we throw some Φs in the numerators and denominators, we get a similar result for $r = r'$.

Corollary $\epsilon(\pi \times \pi', \psi; T) \cdot \epsilon(\pi^\vee \times \pi'^\vee, \psi^{-1}; \frac{1}{qT}) = 1 \in \mathbb{C}(T)^\times$, and $\epsilon(\pi \times \pi', \psi; T) \in \mathbb{C}[T^{\pm 1}]^\times \subset \mathbb{C}(T)^\times$

Proof The first equation is clear. For the second statement, note that $L(\pi \times \pi'; T)$ generates $\mathcal{I}(\pi, \pi')$, so $L(\pi \times \pi'; T) = \sum_i \Psi(w_i, w_i'; T)$ for some w_i, w_i'. Hence, $1 = \sum_i \frac{\Psi()}{L()}$. We get $\epsilon(\pi \times \pi', \psi; T) = \chi_{\pi'}(-1)^{r-1} \sum_i \frac{\Psi(\widetilde{w_i}, \widetilde{w_i'}; \frac{1}{qT})}{L(\pi^\vee \times \pi'^\vee; \frac{1}{qT})} \in \mathbb{C}[T^{\pm 1}]^\times$.

We have a pairing:

$$\begin{array}{ccccc} \mathcal{W}(\pi, \psi) & \times & \mathcal{W}(\pi', \psi^{-1}) & \to & \mathbb{C}(T) \\ w & & w' & \mapsto & \Psi(w, w'; T) \end{array}$$

View this as a well-defined $GL_r \times GL_{r'}$-quasi-invariant.

Now, before we can calculate the global functional equation, we need to compute the L and ϵ factors. In particular, we need to show that the ϵ-factors are almost all 1. We can exclude places where ψ is ramified and blah blah about π ?

Definition Let $(V_\pi, \pi) \in \begin{cases} Rep_{smooth}(GL_r(F)) \\ Rep_{smooth}(GL_r(\mathbb{A})) \end{cases}$ be a representation (where F is a local nonarchimedean field, and $\mathbb{A}$ is the ring of adèles of a global function field), and let $\chi : \begin{cases} \mathbb{G}_m(F) \\ \mathbb{G}_m(\mathbb{A}) \end{cases} \to \mathbb{C}^\times$ be a quasi-character. Then $(V_\pi, \chi\pi) \in \begin{cases} Rep_{smooth}(GL_r(F)) \\ Rep_{smooth}(GL_r(\mathbb{A})) \end{cases}$ is the representation on the same space V_π, with $\chi\pi$ defined by $g \mapsto \chi(det(g)) \cdot \pi(g)$. $(V_\pi, \chi\pi)$ is called the **χ-twist of** π.

Theorem Let $r \geq r' \geq 1$, and let $(V_\pi, \pi) \in Rep_{smooth}(GL_r(F))$ and $(V_{\pi'}, \pi') \in Rep_{smooth}(GL_{r'}(F))$ be irreducible and generic. Assume that π and π' are both unramified, and let $\chi, \chi' : \mathbb{G}_m(F) \to \mathbb{C}^\times$ be quasi-characters. Then

1. (a)

$$L(\chi\pi \times \chi'\pi'; T) = \begin{cases} 1 \text{ if } \chi\chi' \text{ is ramified} \\ \prod_{1 \leq i \leq r, 1 \leq j \leq r'} (1 - z(\chi\chi') z_i(\pi) z_j(\pi') T)^{-1} \text{ otherwise} \end{cases}$$

where $z_1(\pi), \ldots, z_r(\pi), z_1(\pi'), \ldots z_{r'}(\pi')$ are the Satake parameters of π and π', and $z(\chi\chi') = \chi\chi'(\varpi)$ is the Satake parameter of $\chi\chi'$.

(b) If $\psi : F \to \mathbb{C}^\times$ is an unramified (i.e., trivial on $\mathcal{O}$) nontrivial character (so we get $\pi \cong \mathcal{W}(\pi, \psi)$, $\pi' \cong \mathcal{W}(\pi', \psi^{-1})$, and $dim_{\mathbb{C}} \mathcal{W}(\pi, \psi)^{K = GL_r(\mathcal{O})} =$

$dim_{\mathbb{C}}\mathcal{W}(\pi', \psi^{-1})^{K'=GL_{r'}(\mathcal{O})} = 1)$. Then there exist nonzero $w_0 \in \mathcal{W}(\pi, \psi)^K$, $w_0' \in \mathcal{W}(\pi', \psi^{-1})^{K'}$ such that $w_0(1) = w_0'(1) = 1$, and

$$L(\pi \times \pi'; T) = \begin{cases} \Psi(w_0, w_0'; T) & r > r' \\ \Psi(w_0, w_0', \Phi_0; T) & r = r' \end{cases}$$

where Φ_0 is the characteristic function of $\mathcal{O}^r_{rows} \subset F^r_{rows}$.

2. If χ, χ', ψ are all unramified, (also π, π'), then $\epsilon(\chi\pi \times \chi'\pi', \psi; T) = 1$(This is also true for extremely ramified χ, π. Twisting with these can be very informative.)

Proof (sketch for 1(a).) We assume $r = r'$ and $\chi = \chi' = 1$ for simplicity. Pick $\chi : F \to \mathbb{C}^\times$ nontrivial unramified (i.e., $\psi|_{\mathcal{O}} = 1$). Let $w_0 \in \mathcal{W}(\pi, \psi)^K$, $w_0' \in \mathcal{W}(\pi', \psi^{-1})^{K'}$ as before, and $\Phi_0 \in C_c^\infty(F^r_{rows})$. w_0 and w_0' are invariant under right translation by $K = K' = GL_r(\mathcal{O}) = GL_{r'}(\mathcal{O})$, and they transform by ψ under left translation by $N_r(F) = N_{r'}(F)$. By the Iwasawa decomposition: $GL_r(F) = N_r(F) \cdot A_r(F) \cdot K$, w_0 and w_0' are determined by their values on $A_r(F)$, the group of diagonal matrices.

We want a formula for $w_0(\varpi^J := \begin{pmatrix} \varpi^{j_1} & & \\ & \ddots & \\ & & \varpi^{j_r} \end{pmatrix})$ where $J = (j_1, \dots, j_r) \in \mathbb{Z}^r$.

Theorem(Shintani's formula - Casselman, Shalika) Let $T(r) := \{J = (j_1, \dots, j_r) \in \mathbb{Z}^r : j_1 \geq \dots \geq j_r\}$. For $J \in T(r)$, let P_J be the set of irreducible rational representations of GL_r with highest weight J (i.e., $\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_r \end{pmatrix} \mapsto t_1^{j_1} \dots t_r^{j_r}$). Let $A\pi := \begin{pmatrix} z_1(\pi) & & \\ & \ddots & \\ & & z_r(\pi) \end{pmatrix} \in GL_r(\mathbb{C})$, and let $\delta_{B_r(F)} : B_r(F) \to \mathbb{R}_{>0}$ be a modular character of $B_r(F)$. Then for any $J \in \mathbb{Z}^r$,

$$w_0(\varpi^J) = \begin{cases} \delta^{1/2}_{B_r(F)}(\varpi^J) \cdot Tr(P_J(A\pi)) \text{ if } J \in T(r) \\ 0 \text{ otherwise} \end{cases}$$

We compute:

$$
\begin{aligned}
\Psi(w_0, w_0', \Phi; T) &= \int_{N_r(F)\backslash GL_r(F)} w_0(g)w_0'(g)\Phi(e_r \cdot g)T^{v_F(det(g))}\frac{dg}{dn} \\
&\stackrel{(1)}{=} \sum_{J\in T(r)} w_0(\varpi^J)w_0'(\varpi^J)\Phi(0,\ldots,0,\varpi^{j_r})T^{|J|}\delta_{B_r(F)}^{-1}(\varpi^J) \\
&= \sum_{J\in T(r), j_r\geq 0} w_0(\varpi^J)w_0'(\varpi^J)T^{|J|}\delta_{B_r(F)}^{-1}(\varpi^J) \\
&\stackrel{(2)}{=} \sum_{J\in T(r), j_r\geq 0} Tr(\rho_J(A_\pi))Tr(\rho_J(A_{\pi'}))T^{|J|} \\
&= \sum_{J\in T(r), j_r\geq 0} Tr(\rho_J(A_\pi)\otimes\rho_J(A_{\pi'}))T^{|J|} \\
&\stackrel{(3)}{=} \sum_{d\geq 0} Tr(Sym^d(A_\pi\otimes A_{\pi'}))T^d \\
&= det(1 - A_\pi\otimes A_{\pi'}\cdot T)^{-1} \\
&= \prod_{1\leq i\leq r, 1\leq j\leq r'} (1 - z_i(\pi)z_j(\pi')T)^{-1}
\end{aligned}
$$

(1) is from the Iwasawa decomposition, (2) is from Shintani's formula, and (3) comes from Weyl's formula:

$$\sum_{J\in T(r), j_r\geq 0, |J|=d} Tr(\rho_J(A_\pi)\otimes\rho_J(A_{\pi'})) = Tr(Sym^d(A_\pi\otimes A_{\pi'}))$$

Next time: global situation.

November 1, 2002

Let F be a nonarchimedean local field, and $\psi : F \to \mathbb{C}^\times$ a nontrivial unitary character. For $r \geq r' \geq 1$, we get linear algebraic groups $GL_r \supset B_r \supset N_r$. Extend ψ to $N_r(F) \to \mathbb{C}^\times$. Let $(V_\pi, \pi) \in Rep_{smooth}(GL_r(F))$ and $(V_{\pi'}, \pi') \in Rep_{smooth}(GL_{r'}(F))$ be irreducible and generic. We get Whittaker models $\mathcal{W}(\pi, \psi)$ and $\mathcal{W}(\pi', \psi^{-1})$.

For $r > r'$, let $Y_{r,r'}$ be the unipotent radical of the strict parabolic subgroup of GL_r associated to the $(r'+1, 1, \ldots, 1)$ partition of r, i.e.,

$$Y_{r,r;} = \left\{ \left(\begin{array}{c|ccc} 1_{r'+1} & & * & \\ \hline & 1 & & * \\ 0 & & \ddots & \\ & 0 & & 1 \end{array} \right) \right\} \subset GL_r.$$

Theorem (Bernstein, Zelevinsky) Given π, π' as above, fix $t \in \mathbb{C}^\times$, and consider the space of $\mathbb{C}$-bilinear maps $B_t : \mathcal{W}(\pi, \psi) \times \mathcal{W}(\pi', \psi^{-1}) \to \mathbb{C}$ satisfying the following properties:

1. For any $w \in \mathcal{W}(\pi, \psi)$, any $w' \in \mathcal{W}(\pi', \psi^{-1})$, and any $g' \in GL_{r'}(F)$,
$$B_t(\pi \begin{pmatrix} g' & \\ & 1_{r-r'} \end{pmatrix} w, \pi'(g')w') = q^{-\frac{r-r'}{2}} t)^{v_F(det(g'))} B(w, w') \in \mathbb{C}.$$

2. For any $w \in \mathcal{W}(\pi, \psi)$, any $w' \in \mathcal{W}(\pi', \psi^{-1})$, and any $y \in Y_{r,r'}(F)$,
$$B_t(\pi(y)w, w') = \psi(y)B_t(w, w') \in \mathbb{C}.$$

Then there exists a finite subset $Exc \subset \mathbb{C}$ such that for all $t \in \mathbb{C} \setminus Exc$, the complex vector space $\{B_t \text{ satisfying 1 and 2 }\}$ has complex dimension at most 1.

The point of this is that the maps $(w, w') \mapsto \Psi(w, w'; t \in \mathbb{C}) \in \mathbb{C}$ satisfy conditions 1 and 2 for those $t \in \mathbb{C}$ that are not poles of Ψ. Similarly, the maps $(w, w') \mapsto \Psi(\rho(w_{r,r'})\widetilde{w}, \widetilde{w'}; \frac{1}{qt}) \in \mathbb{C}$ satisfy 1 and 2 for ...

Rankin-Selberg convolutions: global theory

Let k be a finite field of order q and characteristic p. Let F be a function field over k, and let A be the ring of adèles of F. Assume k is algebraically closed in F. We get an exact sequence:

$$1 \to \mathbb{G}_m(F)\backslash\mathbb{G}_m(\mathbb{A})^{deg0} \to \mathbb{G}_m(F)\backslash\mathbb{G}_m(\mathbb{A}) \overset{deg}{\to} \mathbb{Z} \to 0$$

where the last part is surjective because k is algebraically closed in F. Choose $\mathbb{C}$, $p^{1/2} \in \mathbb{C}$, and get $q^{1/2} \in \mathbb{C}$. Let $\psi : \mathbb{A} \twoheadrightarrow F\backslash\mathbb{A} \to \mathbb{C}^\times$ be a nontrivial unitary character. Let $r \geq r' \geq 1$, and let $(V_\pi, \pi) \in Rep_{adm}(GL_r(\mathbb{A}))$ and $(V_{\pi'}, \pi') \in Rep_{adm}(GL_{r'}(\mathbb{A}))$ be cuspidal automorphic irreducible representations. By Flath, we get factorizations:

$$\begin{aligned} (V_\pi, \pi) &= \bigotimes_{x\in|F|}(V_{\pi_x}, \pi_x, \overset{0}{\xi}_x \in V_{\pi_x}) \quad \text{where} \quad (V_{\pi_x}, \pi_x) \in Rep_{adm}(GL_r(F_x)) \\ (V_{\pi'}, \pi') &= \bigotimes_{x\in|F|}(V_{\pi'_x}, \pi'_x, \overset{0'}{\xi}_x \in V_{\pi'_x}) \quad \text{where} \quad (V_{\pi'_x}, \pi'_x) \in Rep_{adm}(GL_{r'}(F_x)) \\ \psi &\cong \bigotimes_{x\in|F|} \psi_x \quad \text{where} \quad \psi_x : F_x \to \mathbb{C}^\times \text{ nontrivial, unitary} \end{aligned}$$

By existence of global Whittaker models (Fourier-Whittaker expansion formula), (V_π, π) and $(V_{\pi'}, \pi')$ are generic, so for all $x \in |F|$, (V_{π_x}, π_x) and $(V_{\pi'_x}, \pi'_x)$ are generic. Thus, we get $L(\pi_x \times \pi'_x; T) \in \mathbb{C}(T)^\times \cap (1+T\mathbb{C}[[T]]) \subset \mathbb{C}((T))^\times$ and $\epsilon(\pi_x \times \pi'_x, \psi_x; T) \in \mathbb{C}[T^{\pm 1}]^\times \subset \mathbb{C}((T))^\times$.

Definition The **global L-function associated to** (π, π') is

$$L(\pi \times \pi'; T) := \prod_{x\in|F|} L(\pi_x \times \pi'_x; T^{deg(x)}) \in 1 + T\mathbb{C}[[T]]$$

The **global ϵ-function associated to** π, π', and ψ is

$$\epsilon(\pi \times \pi', \psi; T) := \prod_{x \in |F|} \epsilon(\pi_x \times \pi'_x, \psi_x; T^{deg(x)}) \in \mathbb{C}[T^{\pm 1}]^\times$$

These are well-defined.

Key theorem (Rationality and functional equation)

1. The global L-function $L(\pi \times \pi', T)$ is a **rational function**, i.e., $L(\pi \times \pi'; T) \in \mathbb{C}(T)^\times \cap (1 + T\mathbb{C}[[T]]) \subset \mathbb{C}((T))^\times$.
2. Assume π and π' are unitarizable (this is a normalizing condition). Then $L(\pi \times \pi'; T) = \epsilon(\pi \times \pi', \psi; T) L(\pi^\vee \times \pi'^\vee; \frac{1}{qT}) \in \mathbb{C}(T)^\times$.

Key steps of proof (for $r > r'$)

Step 1 Projection operators of cuspidal functions. Let $Y_{r,r'} \subset GL_r$ be the unipotent radical of the parabolic associated to the partition $(r'+1, 1, \ldots, 1)$ of r, i.e.,

$$Y_{r,r'} = \left\{ \left(\begin{array}{c|c} 1_{r'+1} & * \\ \hline 0 & N_{r-r'-1} \end{array} \right) \right\}$$

Extend $\psi : F\backslash\mathbb{A} \to \mathbb{C}^\times$ to $N_r(\mathbb{A}) \to \mathbb{C}^\times$ by

$$\begin{pmatrix} 1 & x_{1,2} & & \\ & \ddots & \ddots & \\ & & \ddots & x_{r-1,r} \\ & & & 1 \end{pmatrix} \mapsto \psi(x_{1,2} + \cdots + x_{r-1,r}),$$

and restrict to $\psi : Y_{r,r'} \to \mathbb{C}^\times$.

Note

1. $Y_{r',r}$ is normalized by $GL_{r'+1} \subset GL_r$, i.e., $\left(\begin{array}{c|c} GL_{r'+1} & * \\ \hline 0 & 1 \end{array} \right)$ lies in the normalizer.
2. $GL_{r'+1}$ acts on $Y_{r,r'}$ by conjugation inside GL_r.
3. The stabilizer in $GL_{r'+1}(\mathbb{A})$ of the character ψ of $Y_{r,r'}(\mathbb{A})$ is the **mirabolic** subgroup $P_{r'+1} = \left\{ \left(\begin{array}{c|c} GL_{r'} & * \\ \hline 0 & 1 \end{array} \right) \right\} \subset GL_{r'+1}$

Definition The **projection operator** $\mathbb{P}^r_{r'}$ is the complex linear map

$$\begin{array}{rcl} \mathbb{P}^r_{r'} : \mathcal{A}_{cusp}(GL_r(\mathbb{A})) & \to & Cts(P_{r'+1}(\mathbb{A})) \\ \phi & \mapsto & \mathbb{P}^r_{r'}\phi \end{array}$$

where $\mathbb{P}^r_{r'}\phi : P_{r'+1}(\mathbb{A}) \to \mathbb{C}$ is defined by:

$$p \mapsto q^{\frac{r-r'-1}{2} deg(det(p))} \int_{Y_{r,r'}(F)\backslash Y_{r,r'}(\mathbb{A})} \phi(y \left(\begin{array}{c|c} p & 0 \\ \hline 0 & 1_{r-r'-1} \end{array}\right))\psi^{-1}(y)\frac{dy}{d\mu}$$

and $\frac{dy}{d\mu}$ is the right $Y_{r,r'}$-invariant measure on $Y_{r,r'}(F)\backslash Y_{r,r'}(\mathbb{A})$ with volume 1.

Lemma For any $\phi \in \mathcal{A}_{cusp}(GL_r(\mathbb{A}))$, $\mathbb{P}^r_{r'}\phi \in Cts(P_{r'+1}(\mathbb{A}))$ is a **cuspidal factor** of $P_{r'+1}(\mathbb{A})$, i.e., it is:

1. invariant under left translation by $P_{r'+1}(F)$
2. uniformly locally constant
3. cuspidal for $P_{r'+1}$: for any $p \in P_{r'+1}(\mathbb{A})$ and any standard unipotent subgroup $U \subset P_{r'+1}$, $\int_{U(F)\backslash U(\mathbb{A})}(\mathbb{P}^r_{r'}\phi)(up)\frac{du}{d\mu} = 0$.

Step 2 Fourier-Whittaker expansion formula for $P_{r'+1}$. For $\phi \in \mathcal{A}_{cusp}(GL_r(\mathbb{A}))$, we get $W_{\phi,\psi} \in \mathcal{W}(GL_r(\mathbb{A}), \psi)$, which is a map $GL_r(\mathbb{A}) \to \mathbb{C}$ given by $g \mapsto W_{\phi,\psi}(g) := \int_{N_r(F)\backslash N_r(\mathbb{A})} \phi(ng)\psi^{-1}(n)\frac{dn}{d\mu}$ where $\frac{dn}{d\mu}$ is the right-$N_r(\mathbb{A})$-invariant measure with total volume 1.

Lemma For any $\phi \in \mathcal{A}_{cusp}(GL_r(\mathbb{A}))$ and any $g' \in GL_{r'}(\mathbb{A})$,

$$(\mathbb{P}^r_{r'}\phi)\left(\begin{array}{c|c} g' & 0 \\ \hline 0 & 1 \end{array}\right) = q^{\frac{r-r'-1}{2}deg(det(g'))} \sum_{\gamma \in N_{r'}(F)\backslash GL_{r'}(F)} W_{\phi,\psi}(\left(\begin{array}{c|c} \gamma & 0 \\ \hline 0 & 1_{r-r'} \end{array}\right)\left(\begin{array}{c|c} g' & 0 \\ \hline 0 & 1_{r-r'} \end{array}\right))$$

Step 3 Global cusp form integral. Choose a Haar measure dg' on $GL_{r'}(\mathbb{A})$. Let $d\mu$ be the counting measure on $GL_{r'}(F)$.

Definition We define a complex-linear map

$$\begin{array}{ccccc} \mathcal{A}_{cusp}(GL_r(\mathbb{A})) & \times & \mathcal{A}_{cusp}(GL_{r'}(\mathbb{A})) & \to & (Anal(\mathbb{C}^\times), \text{ pole at } 0) \\ \phi & & \phi' & \mapsto & \mathcal{I}(\phi, \phi'; t) \end{array}$$

where

$$\mathcal{I}(\phi, \phi'; t) := \int_{GL_{r'}(F)\backslash GL_{r'}(\mathbb{A})} (\mathbb{P}^r_{r'}\phi)\left(\begin{array}{c|c} g' & 0 \\ \hline 0 & 1 \end{array}\right)\phi'(g')q^{deg(det(g'))/2}t^{deg(det(g'))}\frac{dg'}{d\mu}$$

One should verify that these integrals converge on $\mathbb{C}^\times$, with only a pole at 0 (not an essential singularity).

Theorem (Functional equation for $\mathcal{I}$s) One can similarly define $\widetilde{\mathcal{I}}(-,-;t)$ (e.g., by defining $\widetilde{\ }$-projection operators) such that for any $\phi \in \mathcal{A}_{cusp}(GL_r(\mathbb{A}))$ and $\phi' \in \mathcal{A}_{cusp}(GL_{r'}(\mathbb{A}))$, $\mathcal{I}(\phi,\phi';t) = \widetilde{\mathcal{I}}(\widetilde{\phi},\widetilde{\phi'};t)$, where $\widetilde{\phi} := (g \mapsto \phi({}^t g^{-1}))$. A key point here is the existence of the outer automorphism of GL_r given by $g \mapsto {}^t g^{-1}$. This makes it difficult to do this construction for other groups.

Put this in the fridge.

$$Y_{r,r'} = \left(\begin{array}{c|c} 1_{r'+1} & * \\ \hline 0 & N_{r-r'-1} \end{array}\right) \quad \widetilde{Y_{r,r'}} = \left(\begin{array}{c|c} 1_{r'+1} & 0 \\ \hline * & N_{r-r'-1} \end{array}\right)$$

$$P_{r'+1} = \left(\begin{array}{c|c} GL_{r'} & * \\ \hline 0 & 1 \end{array}\right) \quad \widetilde{P_{r'+1}} = \left(\begin{array}{c|c} GL_{r'} & 0 \\ \hline * & 1 \end{array}\right)$$

$\mathcal{I}$ and $\widetilde{\mathcal{I}}$ have possible poles at 0 and ∞, and are analytic elsewhere. $\lim_{t\to\infty} \mathcal{I}(\phi,\phi';t)$ exists in $\overline{\mathbb{C}}$.

Step 4 Global Rankin-Selberg integrals. Let dg' be the same Haar measure on $GL_{r'}(\mathbb{A})$ as above. Define a complex linear map

$$\begin{array}{ccccc} \mathcal{W}(GL_r(\mathbb{A}),\psi) & \times & \mathcal{W}(GL_{r'}(\mathbb{A}),\psi^{-1}) & \to & Germ_{mero}(\mathbb{C})_0 := \left\{\begin{array}{c} \text{Germs of} \\ \text{meromorphic} \\ \text{functions} \\ \text{at } 0 \in \mathbb{C} \end{array}\right\} \\ w & & w' & \mapsto & \Psi(w,w';t) \end{array}$$

given by:

$$\Psi(w,w';t) := \int_{N_{r'}(\mathbb{A})\backslash GL_{r'}(\mathbb{A})} w\left(\begin{array}{c|c} g' & 0 \\ \hline 0 & 1_{r-r'} \end{array}\right) w'(g') q^{\frac{r-r'}{2} deg(det(g'))} t^{deg(det(g'))} \frac{dg'}{dn'}$$

where dn' is the left Haar measure on $N_{r'}(\mathbb{A})$ such that $\frac{dn'}{d\mu}$ on $N_{r'}(F)\backslash N_{r'}(\mathbb{A})$ has volume 1. One has to verify that the integral converges in a punctured neighborhood of 0, and has a pole at 0.

Proposition Let $w \in \mathcal{W}(GL_r(\mathbb{A}),\psi)$, and $w' \in \mathcal{W}(GL_{r'}(\mathbb{A}),\psi^{-1})$. Suppose

$$w = \prod_{x\in|F|} w_x \qquad \text{and} \qquad w' = \prod_{x\in|F|} w'_x,$$

where $w_x \in \mathcal{W}(GL_r(F_x),\psi_x)$ and $w'_x \in \mathcal{W}(GL_{r'}(F_x),\psi_x^{-1})$. Then

$$\Psi(w,w';t) = \prod_{x\in|F|} \Psi(w_x,w'_x;t),$$

where the global integral is with respect to the Haar measure dg', the local integrals are with respect to Haar measures dg'_x, and $dg' = \prod_{x\in|F|} dg'_x$.

Step 5 Equate $\mathcal{I}$ with Ψ.

Theorem (Euler factorization) For $\phi \in \mathcal{A}_{cusp}(GL_r(\mathbb{A}))$, $\phi' \in \mathcal{A}_{cusp}(GL_{r'}(\mathbb{A}))$ (giving us the associated Whittaker functions $W_{\phi,\psi} \in \mathcal{W}(GL_r(\mathbb{A}),\psi)$ and $W_{\phi',\psi^{-1}} \in \mathcal{W}(GL_{r'}(\mathbb{A}),\psi^{-1})$), one has $\mathcal{I}(\phi,\phi';t) = \Psi(W_{\phi,\psi}, W_{\phi',\psi^{-1}};t)$ in $Germ_{mero}(\mathbb{C})_0$. If ϕ and ϕ' are chosen so that $W_{\phi,\psi} = \prod_{x\in|F|} w_x$ and $W_{\phi',\psi^{-1}} = \prod_{x\in|F|} w'_x$, then $\mathcal{I}(\phi,\phi';t) = \prod_{x\in|F|} \Psi(w_x, w'_x; t^{deg(x)})$ in $Germ_{mero}(\mathbb{C})_0$.

Step 6 Local to Global considerations. Let $(V_\pi,\pi) \in Rep_{adm}(GL_r(\mathbb{A}))$ and $(V_{\pi'},\pi') \in Rep_{adm}(GL_{r'}(\mathbb{A}))$ be cuspidal automorphic irreducible representations, and let $S = \{x \in |F| : \pi_x, \pi'_x, \text{ or } \psi_x \text{ is ramified}\}$, a finite subset of $|F|$.

We know that for any $x \in |F| \setminus S$, we can choose $w_x \in \mathcal{W}(\pi_x,\psi_x)$ and $w'_x \in \mathcal{W}(\pi'_x,\psi_x^{-1})$ such that $L(\pi_x,\pi'_x;t) = \Psi(w_x,w'_x;t)$ as germs of meromorphic functions on $\mathbb{C}$ at 0, and that for any $x \in S$, we can choose $w_x \in \mathcal{W}(\pi_x,\psi_x)$ and $w'_x \in \mathcal{W}(\pi'_x,\psi_x^{-1})$ such that $\Psi(w_x,w'_x;t) \not\equiv 0$ as germs of meromorphic functions on $\mathbb{C}$ at 0 (i.e., don't do anything stupid at the ramified places).

We get $w := \prod_{x\in|F|} w_x \in \mathcal{W}(\pi,\psi)$ and $w' := \prod_{x\in|F|} w'_x \in \mathcal{W}(\pi',\psi^{-1})$. This is special to Whittaker functions and cannot be done with cusp forms in general.

By the Fourier expansion formula, we get $\phi \in V_\pi \subset \mathcal{A}_{cusp}(GL_r(\mathbb{A}))$ and $\phi' \in V_{\pi'} \subset \mathcal{A}_{cusp}(GL_{r'}(\mathbb{A}))$ such that $W_{\phi,\psi} = w$, and $W_{\phi',\psi^{-1}} = w'$.

By step 5 (Euler factorization),

$$\begin{aligned}\mathcal{I}(\phi,\phi';t) &= \prod_{x\in|F|} \Psi(w_x,w'_x;t^{deg(x)}) \\ &= L(\pi\times\pi';t)\prod_{x\in S}\frac{\Psi(w_x,w'_x;t^{deg(x)})}{L(\pi\times\pi';t^{deg(x)})}\end{aligned}$$

The product on the end is a correction factor for the ramified places. Call the quotients in the product $e(w_x,w'_x;t^{deg(x)})$.

Local Rankin-Selberg theory implies $e(w_x,w'_x;T) \in \mathbb{C}[T^{\pm 1}] \setminus \{0\}$. Hence, $L(\pi \times \pi';t) = \frac{\mathcal{I}(\phi,\phi';t)}{\prod_{x\in S} e(w_x,w'_x;t^{deg(x)})}$. The numerator is analytic in $\mathbb{C}^\times$ with a possible pole at 0, while the denominator is a finite product of Laurent polynomials, so $L(\pi\times\pi';t)$ is meromorphic on $\mathbb{C}$.

From the functional equation of $\mathcal{I}$ and local functional equations of the Ψs, we get $L(\pi\times\pi';t) = \epsilon(\pi\times\pi',\psi;t)L(\pi^\vee\times\pi'^\vee;\frac{1}{qt})$ as meromorphic functions on $\mathbb{C}^\times$. Since $\lim_{t\to(0 \text{ or } \infty)} L(\pi\times\pi';t)$ exists in $\overline{\mathbb{C}}$, $L(\pi\times\pi') \in Mero(\overline{\mathbb{C}}) \cong \mathbb{C}(t)$.

November 8, 2002

No notes (no class?)

November 15, 2002

Wrap up Rankin-Selberg integration, then move on to converse theorem.

Let k be a finite field of order q and characteristic p, let F be a function field over k, and let $\mathbb{A}$ be its ring of adèles. Assume k is algebraically closed in F, so we get an exact sequence:

$$1 \to \mathbb{G}_m(F)\backslash\mathbb{G}_m(\mathbb{A})^0 \to \mathbb{G}_m(F)\backslash\mathbb{G}_m(\mathbb{A}) \stackrel{deg}{\to} \mathbb{Z} \to 0.$$

Choose an algebraic closure $\mathbb{C} \cong \mathbb{R}[i]$, and choose a square root $p^{1/2} \in \mathbb{C}^\times$, which gives us $(\#\kappa(x))^{1/2} \in \mathbb{C}^\times$ for all $x \in |F|$. Let $\psi : F\backslash\mathbb{A} \to \mathbb{C}^\times$ be a nontrivial unitary character, and let $r \geq r' \geq 1$ be integers. Let $(V_\pi, \pi) \in Rep_{adm}(GL_r(\mathbb{A}), \psi)$ and $(V_{\pi'}, \pi') \in Rep_{adm}(GL_{r'}(\mathbb{A}), \psi^{-1})$ be cuspidal automorphic irreducible representations. We get $L(\pi \times \pi'; T) \in 1 + T\mathbb{C}[[T]] \subset \mathbb{C}((T))^\times$ and $\epsilon(\pi \times \pi', \psi; T) \in \mathbb{C}[T^{\pm 1}]^\times \subset \mathbb{C}((T))^\times$.

Theorem

1. $L(\pi \times \pi'; T)$ is rational, i.e., it lies in $\mathbb{C}(T) \cap (1 + T\mathbb{C}[[T]]) \subset \mathbb{C}((T))^\times$.

2. Assume π and π' are unitarizable. Then

$$L(\pi \times \pi'; T) = \epsilon(\pi \times \pi', \psi; T)L(\pi^\vee, \pi'^\vee; \frac{1}{qT}).$$

Theorem (Location of poles of global L-functions - a bit like the Riemann Hypothesis) Assume π and π' are unitarizable.

1. If $r > r'$, then $L(\pi \times \pi'; T)$ has no pole in $\mathbb{C}$, i.e., $L \in 1 + T\mathbb{C}[T]$.

2. Suppose $r = r'$. Let $T(\pi, \pi') := \{\alpha \in U(1) \subset \mathbb{C}^\times : \pi \cong \pi'^\vee \otimes \alpha^{deg(det(-))}\}$ (so if $\alpha \in \mathbb{C}^\times$, then $GL_r(\mathbb{A}) \to \mathbb{C}^\times$ given by $g \mapsto \alpha^{deg(det(g))}$ is a quasi-character of $GL_r(\mathbb{A})$). Then the poles of $L(\pi \times \pi'; T)$ in $\mathbb{C}$ are $\{\frac{1}{\alpha} : \alpha \in T(\pi, \pi')\} \cup \{\frac{1}{q\alpha} : \alpha \in T(\pi, \pi')\}$, and each is a **simple pole**.

Corollary Assume π and π' are unitarizable. Then $T = 1$ and $T = \frac{1}{q}$ are poles of $L(\pi \times \pi'^\vee; T)$ if and only if $\pi \cong \pi'$.

Theorem (Strong multiplicity one) Let $r \geq 1$ be an integer, and let $(V_\pi, \pi), (V_{\pi'}, \pi') \in Rep_{adm}(GL_r(\mathbb{A}))$ be unitarizable cuspidal automorphic irreducible representations. Suppose there exists a finite set of places $S \subset |F|$, such that for any $x \in |F| \setminus S$, $\pi_x \cong \pi'_x$ in $Rep_{adm}(GL_r(F_x))$. Then $\pi \cong \pi'$ in $Rep_{adm}(GL_r(\mathbb{A}))$.

Proof (Looks like Čebotarev) Define $L^S(-;T) := \prod_{x \in |F| \setminus S} L(-_x, T^{deg(x)}) \in 1 + T\mathbb{C}[[T]]$. Then by assumption, $L^S(\pi \times \pi'^\vee; T) := \prod_{x \in |F| \setminus S} L(\pi_x \times \pi_x'^\vee; T^{deg(x)})$ is equal to $L^S(\pi \times \pi^\vee; T) := \prod_{x \in |F| \setminus S} L(\pi_x \times \pi_x^\vee; T^{deg(x)})$. For $x \in S$, consider $L(\pi_x \times \pi_x'^\vee; T^{deg(x)})$ as $\frac{1}{\text{element of } 1+T\mathbb{C}[T]}$, so it has no zeroes in $\mathbb{C}$. Furthermore, this has no poles in $\{t \in \mathbb{C} : |t| \leq \frac{1}{q}\}$ (this is nontrivial - it uses norm estimates of local Rankin-Selberg integrals). Hence, $L^S(\pi \times \pi'^\vee; T)$ has a pole at $T = \frac{1}{q}$ if and only if $L(\pi \times \pi'^\vee; T)$ does, and $L^S(\pi \times \pi^\vee; T)$ has a pole at $T = \frac{1}{q}$ if and only if $L(\pi \times \pi^\vee; T)$ does. As we noted above, $L^S(\pi \times \pi'^\vee; T) = L^S(\pi \times \pi^\vee; T)$, and by the corollary, $L(\pi \times \pi^\vee; T)$ has a pole at $T = \frac{1}{q}$, so $L(\pi \times \pi'^\vee; T)$ also has a pole there. By the corollary again, this implies $\pi \cong \pi'$ in $Rep_{adm}(GL_r(\mathbb{A}))$.

Main converse theorem references: Cogdell, Piatetski-Shapiro. Publ. Math. IHES 79 (1994), and J. Reine Angew. Math. 517 (1999).

We have the same situation as before, but no representations. Let $r \geq r' \geq 1$ be integers, so we get groups $GL_r \supset B_r \supset N_r$ with maximal torus (and Levi quotient) $A_r \subset B_r$. Extend the additive character ψ to $N_r(\mathbb{A}) \to \mathbb{C}^\times$ via

$$\begin{pmatrix} 1 & x_{1,2} & & \\ & \ddots & \ddots & \\ & & \ddots & x_{r-1,r} \\ & & & 1 \end{pmatrix} \mapsto \psi(x_{1,2} + \cdots + x_{r-1,r}),$$

Recall that if $(V_\pi, \pi) \in Rep_{adm}(GL_r(\mathbb{A}))$ and $(V_{\pi'}, \pi') \in Rep_{adm}(GL_r(\mathbb{A}))$ are irreducible **generic** representations, then we can define $L(\pi \times \pi'; T) \in 1 + T\mathbb{C}[[T]] \subset \mathbb{C}((T))^\times$ and $\epsilon(\pi \times \pi', \psi; T) \in \mathbb{C}[T^{\pm 1}]^\times \subset \mathbb{C}((T))^\times$ in the same way as when π and π' are cuspidal automorphic.

Theorem (Converse theorem) Let $r \geq 2$, and let $(V_\Pi, \Pi) \in Rep_{adm}(GL_r(\mathbb{A}))$ be irreducible and generic with respect to ψ. Assume:

1. The **central quasi-character** of Π is automorphic, i.e., $\chi_\Pi : Z(GL_r(\mathbb{A})) \cong \mathbb{G}_m(\mathbb{A}) \to \mathbb{C}^\times$ factors through $Z(GL_r(F)) \backslash Z(GL_r(\mathbb{A}))$.

2. Suppose there exists a finite set $S \subset |F|$ of places, such that for any $r' < r$ (strictly) and any cuspidal automorphic irreducible $(V_{\pi'}, \pi') \in Rep_{adm}(GL_{r'}(\mathbb{A}))$ that is unramified **at** S, the L-function $L(\Pi \times \pi'; T)$ is **nice**, i.e., $L(\Pi \times \pi'; T)$ and $L(\Pi^\vee \times \pi'^\vee; T)$ lie in $1 + T\mathbb{C}[T]$ (instead of just $1 + T\mathbb{C}[[T]]$) and satisfy the functional equation $L(\Pi \times \pi'; T) = \epsilon(\Pi \times \pi', \psi; T) L(\Pi^\vee \times \pi'^\vee; \frac{1}{qT})$.

Then there exists an automorphic irreducible $(V_\pi, \pi) \in Rep_{adm}(GL_r(\mathbb{A}))$ such that for any $x \in |F| \setminus S$, one has $(V_\Pi, \Pi) \cong (V_\pi, \pi)$ in $Rep_{adm}(GL_r(F_x))$. **Furthermore**, if the finite set $S \subset |F|$ is actually empty, i.e., if for any $r' < r$ and any cuspidal automorphic irreducible $(V_{\pi'}, \pi') \in Rep_{adm}(GL_{r'}(\mathbb{A}))$, $L(\Pi \times \pi'; T)$ is nice, then $(V_\Pi, \Pi) \cong (V_\pi, \pi)$ and is also cuspidal.

Note that on the Galois side we have $L(\Sigma \otimes \sigma'; T) = \frac{1 - TFrobH^1_c(x \otimes T, \Sigma \otimes \sigma)}{H^0_c(\dots)H^2_c(\dots)}$ with $H^2_c(\Sigma \otimes \sigma)^{blah} \cong Hom(\Sigma \otimes \sigma'^{\vee})^{(-1)}$ or something like that. ?

Suppose $(V_\Pi, \Pi) \in Rep_{adm}(GL_r(\mathbb{A}))$ were actually cuspidal automorphic. Embed $V_\Pi \hookrightarrow \mathcal{A}_{cusp}(GL_r(\mathbb{A}))$, and get a Fourier-Whittaker expansion: For any $\phi \in \mathcal{A}_{cusp}(GL_r(\mathbb{A}))$ and any $g \in GL_r(\mathbb{A}))$,

$$\begin{aligned}\phi(g) &= \sum_{\gamma \in N_{r-1}(F)\backslash GL_{r-1}(F)} W_{\phi,\psi}(\left(\begin{array}{c|c} \gamma & 0 \\ \hline 0 & 1 \end{array}\right) g) \\ &= \sum_{\delta \in N_r(F)\backslash P_r(F)} W_{\phi,\psi}(\delta \cdot g),\end{aligned}$$

where $W_{\phi,\psi} : GL_r(\mathbb{A}) \to \mathbb{C}$ is given by $g \mapsto \int_{N_r(F)\backslash N_r(\mathbb{A})} \phi(ng)\psi^{-1}(n)\frac{dn}{d\mu}$, and

$$P_r = \left\{ \left(\begin{array}{c|c} * & * \\ \hline 0 & 1 \end{array}\right) \right\} \subset GL_r$$

is the mirabolic subgroup. We have maps:

$$\begin{array}{ccccc} \mathcal{A}_{cusp}(GL_r(\mathbb{A})) & \overset{W_{-,\psi}}{\to} & \mathcal{W}(GL_r(\mathbb{A}), \psi) & \overset{FW-exp.}{\to} & \mathcal{A}_{cusp}(GL_r(\mathbb{A})) \\ \phi & \mapsto & W_{\phi,\psi} & \mapsto & \phi \end{array}$$

Now, **stop** assuming (V_Π, Π) is cuspidal.

Start with (V_Π, Π) a generic irreducible representation of $GL_r(\mathbb{A})$. We get a global Whittaker model (embedding) $(V_\Pi, \Pi) \overset{\sim}{\hookrightarrow} \mathcal{W}(\Pi, \psi) \subset \mathcal{W}(Gl_r(\mathbb{A}), \psi)$ given by $\xi \mapsto w_\xi$. the basic idea is to use the Fourier-Whittaker expansion formula to define the embedding of V_Π into $\mathcal{A}_{cusp}$.

The mirabolic embeds in the parabolic subgroup of GL_r associated to the partition $r = (r-1) + 1$:

$$P_r := \left\{ \left(\begin{array}{c|c} * & * \\ \hline 0 & 1 \end{array}\right) \right\} \subset P'_r := \left\{ \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right) \right\}$$

Define for each $\xi \in V_\Pi$, a function $u_\xi : GL_r(\mathbb{A}) \to \mathbb{C}$ by

$$\begin{aligned}u_\xi(g) &= \sum_{\delta \in N_r(F)\backslash P_r(F)} w_\xi(\delta \cdot g) \\ &= \sum_{\gamma \in N_{r-1}(F)\backslash GL_{r-1}(F)} w_\xi(\left(\begin{array}{c|c} \gamma & 0 \\ \hline 0 & 1 \end{array}\right) g)\end{aligned}$$

Lemma The above sums are **finite sums** for any given $g \in GL_r(\mathbb{A})$, so u_ξ is well-defined.

Proposition u_ξ is invariant under left translation by $P_r(F)$ (this is clear), and by $Z(GL_r(F))$ (because w_ξ is, by assumption 1), so u_ξ is invariant under left translation by $P_r'(F)$. u_ξ is invariant under right translation by some open subgroup of $GL_r(\mathbb{A})$ (because the defined sum is finite, and the w_ξ are smooth).

Lemma For any $g \in GL_r(\mathbb{A})$, such that $w_\xi(g) \neq 0$, there exists $n \in N_r(\mathbb{A})$ such that $u_\xi(ng) \neq 0$. In particular, if $\xi \neq 0$ in V_Π, then u_ξ is not the zero function.

Idea (opposite construction to bring out $\Pi^\vee$) Let $w_r := \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix} \in GL_r(F)$ be the longest element of the Weyl group, and let

$$\alpha_r := w_r \left(\begin{array}{c|c} w_{r-1} & 0 \\ \hline 0 & 1 \end{array}\right) = \left(\begin{array}{ccc|c} & & & 1 \\ \hline 1 & & & \\ & \ddots & & \\ & & 1 & \end{array}\right) \in GL_r(F).$$

We have:

$$\begin{array}{lllll} \overline{P_r} & := & {}^t(P_r)^{-1} & = & \left\{\left(\begin{array}{c|c} * & 0 \\ \hline * & 1 \end{array}\right)\right\} \quad \text{opposite mirabolic} \\ \overline{P_r'} & := & {}^t(P_r')^{-1} & = & \left\{\left(\begin{array}{c|c} * & 0 \\ \hline * & * \end{array}\right)\right\} \quad \text{opposite parabolic} \\ \overline{N_r} & := & \alpha_r^{-1} N_r \alpha_r & = & \left\{\left(\begin{array}{ccc|c} 1 & & * & \\ & \ddots & & 0 \\ 0 & & 1 & \\ \hline & * & & 1 \end{array}\right)\right\} \subset \overline{P_r} \end{array}$$

Definition For any $\xi \in V_\Pi$, consider the map $\widetilde{w_\xi} : GL_r(\mathbb{A}) \to \mathbb{C}$ defined by $g \mapsto w_\xi(w_r {}^t g^{-1})$. One checks that $\widetilde{w_\xi} \in \mathcal{W}(GL_r(\mathbb{A}), \psi^{-1})$. Define for any $\xi \in V_\Pi$ the funtion $v_\xi : GL_r(\mathbb{A}) \to \mathbb{C}$ given by: g'?

$$\begin{aligned} v_\xi(g) &:= \sum_{\delta \in \overline{N_r}(F) \backslash \overline{P_r}(F)} w_\xi(\alpha_r \cdot \delta g) \\ &= \sum_{\gamma \in N_{r-1}(F) \backslash GL_{r-1}(F)} w_\xi(\alpha_r \left(\begin{array}{c|c} \gamma & 0 \\ \hline 0 & 1 \end{array}\right) g) \\ &= \sum_{\gamma' \in N_{r-1}(F) \backslash GL_{r-1}(F)} \widetilde{w_{\xi'}}(\left(\begin{array}{c|c} \gamma' & 0 \\ \hline 0 & 1 \end{array}\right) g') \end{aligned}$$

Now, everything works as before by formal comparison. Previous lemmata hold with u replaced by v, and bars placed over P_r, P_r', and N_r.

Key technical theorem Under the hypotheses of the converse theorem, assumption 2 (niceness), we have: If $\xi = \otimes_{x \in |F|} \xi_x \in V_\Pi$ is a decomposable factor, and if for any $x \in S \subset |F|$, $\xi_x \in V_{\Pi_x}$ is fixed by $GL_{r-1}(\mathcal{O}_x) \subset GL_r(\mathcal{O}_x)$, then $u_\xi(1 \in GL_r(F)) = v_\xi(1 \in GL_r(F))$ in $\mathbb{C}$.

We assume the key theorem above, and finish the proof in the case $S = \emptyset$. It is clear that $P'_r(F)$ and $\overline{P'_r}(F)$ generate $GL_r(F)$. By the key theorem above, since $S = \emptyset$, for any decomposable vector $\xi = \otimes \xi_x \in V_\Pi$, $u_\xi(1) = v_\xi(1) \in \mathbb{C}$. Hence, for any $g \in GL_r(\mathbb{A})$, $u_\xi(g) = u_{\Pi(g)\xi}(1) = v_{\Pi(g)\xi}(1) = v_\xi(g)$, so $u_\xi = v_\xi$ as functions on $GL_r(\mathbb{A})$, so they are invariant under left translation by $GL_r(\mathbb{A})$. Also, they are invariant under right translation by some open subgroup of $GL_r(\mathbb{A})$. From the definition of u_ξ as a Fourier-Whittaker expansion, one can show that u_ξ is a cuspidal function, and u_ξ generates an admissible subrepresentation of $C(GL_r(\mathbb{A}), \mathbb{C})$. Hence, u_ξ **is** a cusp form on $GL_r(\mathbb{A})$ and lies in $\mathcal{A}_{cusp}(GL_r(\mathbb{A}), \mathbb{C})$, so the map $\xi \mapsto u_\xi$ defined on decomposable ξ extends to a $GL_r(\mathbb{A})$-equivariant embedding $V_\Pi \hookrightarrow \mathcal{A}_{cusp}(GL_r(\mathbb{A}))$. Thus, (V_Π, Π) is cuspidal automorphic.

Next time: Grothendieck's six operations.

November 22, 2002

Definition An **additive category** is a category $\mathcal{C}$ equipped with abelian group structures on all hom-sets, such that:

1. there exists a zero object 0, i.e., for any object $X \in \mathcal{C}$ there exist unique morphisms $X \to 0$ and $0 \to X$.
2. all binary (hence finite) products and coproducts exist.
3. composition is a bilinear map on hom-sets.

[The abelian group structure on hom-sets is not actually an additional piece of data. If we decree the obvious map $X \oplus X \to X \times X$ defined by the diagonal matrix of identity maps to be an isomorphism for all X, then addition of maps $f, g : X \to Y$ arises from $X \xrightarrow{(id,id)} X \times X \xrightarrow{\sim} X \oplus X \xrightarrow{diag(f,g)} Y \times Y \xrightarrow{\sim} Y \oplus Y \to Y$. Then we need only ask for existence of additive inverses.] A functor between additive categories is **additive** if it maps zero objects to zero objects, and takes hom-sets to hom-sets via homomorphisms of abelian groups.

Definition A **triangulated category** $D = (D, \{[n]\}_{n \in \mathbb{Z}}, \Delta)$ is a triple, where:

1. D is an additive category (objects are called "complexes" but shouldn't be mistaken for them).
2. For all $n \in \mathbb{Z}$, $[n] : D \to D$ is a covariant additive functor [most people ask for this to be an equivalence, but Cheewhye seemed to think it detrimental], such that for all $m, n \in \mathbb{Z}$, we have an **equality** of functors $[n] \circ [m] = [n + m]$.

3. Δ is a collection of diagrams in D of the form $K \xrightarrow{u} L \xrightarrow{v} M \xrightarrow{w} K[1]$, called **distinguished triangles**.

satisfying the following axioms:

1. For any $K \in D$, $K \xrightarrow{id_k} K \xrightarrow{0} 0 \xrightarrow{0} K[1]$ is a distinguished triangle.

2. If an arbitrary triangle $K \to L \to M \to K[1]$ is isomorphic to a distinguished triangle, then it is also distinguished. Note that morphisms between triangles are given by the obvious commutative diagram. One could study categories of triangles etc., but it's not important here.

3. Any morphism $K \xrightarrow{u} L$ in D can be completed to a distinguished triangle $K \xrightarrow{u} L \to M \to K[1]$. The object M is often written $cone(u)$, and axiom 5 implies the completion is unique up to non-unique isomorphism.

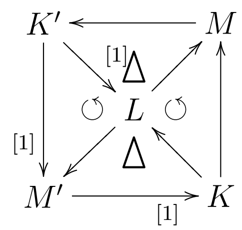
4. If $K \xrightarrow{u} L \xrightarrow{v} M \xrightarrow{w} K[1]$ is distinguished, then $L \xrightarrow{v} M \xrightarrow{w} K[1] \xrightarrow{-u[1]} L[1]$ is distinguished. The minus sign is very important.

5. If

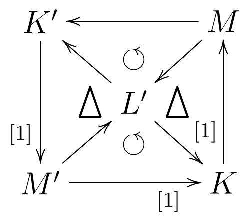
$$\begin{array}{ccccccc} K & \longrightarrow & L & \longrightarrow & M & \longrightarrow & K[1] \\ \downarrow f & & \downarrow g & & & & \downarrow f[1] \\ K' & \longrightarrow & L' & \longrightarrow & M' & \longrightarrow & K'[1] \end{array}$$

   is a commutative diagram, whose rows are distinguished triangles, then there exists a morphism $h : M \to M'$ making the diagram commute. Note that h is **not necessarily unique**.

6. (Octahedron axiom) Given an upper cap:

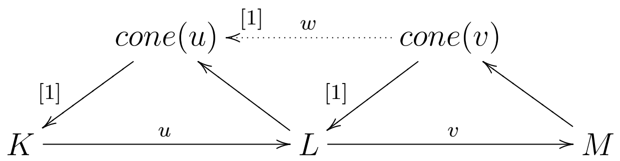


   there exists a lower cap:

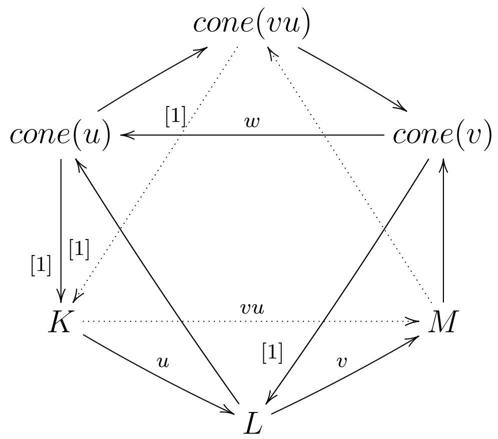


where ↻ denotes a commuting triangle, △ denotes a distinguished triangle, and [1] indicates a degree shift.

[The last axiom gave me a headache. Here's an alternative formulation: For any pair of morphisms $K \xrightarrow{u} L \xrightarrow{v} M$, if we take cones:



then the morphism $w : cone(v) \to cone(u)[1]$ defined by composing the maps in the middle satisfies $cone(w) \cong cone(vu)[1]$. Here, a [1] over an arrow indicates a degree shift. One can make an octahedral diagram by wrapping the above diagram about a vertical axis and adding a few arrows:



so that $K \to L \to M$ forms a commutative triangle on the bottom, and $cone(u) \to cone(vu) \to cone(v) \xrightarrow{w} cone(u)[1]$ forms a distinguished triangle on the top. Unfortunately, most of the problems people have understanding this axiom seem to come from thinking in terms of octahedra. It may be better to think of the octahedron axiom as some vague composition law on triangles, or as the statement that $M/L \cong (M/K)/(L/K)$, which is true if u and v are monomorphisms and quotients exist - often the category is nice enough that one can apply some kind of cofibrant replacement to L and M, e.g., by mapping cylinders, to make this hold in general (see also [BBD] and Verdier's article in SGA 4.5 for other presentations). The octahedron axiom is apparently not strong enough for some purposes, and this seems to come from cones being insufficiently well-defined. Several attempts have been made to fix this, but I won't mention them here.]

Definition If D and D' are triangulated categories, a covariant functor $F : D \to D'$ is **triangulated** if

1. F is additive.

2. F commutes with $[n]$ for all $n \in \mathbb{Z}$, i.e., $F \circ [n] = [n]' \circ F$.

3. F takes distinguished triangles to distinguished triangles.

Definition Let D be triangulated, and A abelian. A **cohomological functor** $H : D \to A$ is a covariant additive functor such that for any distinguished triangle $K \xrightarrow{u} L \xrightarrow{v} M \xrightarrow{w} K[1]$, the sequence $H(K) \xrightarrow{H(u)} H(L) \xrightarrow{H(v)} H(M)$ is exact in A.

Definition Given a homological functor H, and an integer n, we write $H^n(-) := H(-[n])$

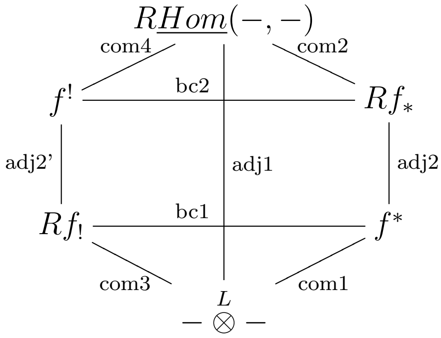
This gives rise to long exact sequences, spectral sequences, and spectral objects (introduced by Verdier, but not widely used).

Grothendieck's formalism of six operations This isn't really written down anywhere, so use at your own risk. I heard it from Deligne, who heard it from Grothendieck.

Let $\mathcal{S}$ be a category with fiber products (e.g., $\mathcal{S} = (Sch/S)$ or some suitable subcategory). Consider the following data:

- For each object $X \in \mathcal{S}$, give a triangulated category $D(X)$ (e.g., $D(X)$ will eventually be $D^b_c(X, \overline{\mathbb{Q}_l})$, the bounded derived category of constructible $\overline{\mathbb{Q}_l}$-sheaves).
- Give 6 families of functors:
  - For any object $X \in \mathcal{S}$, $- \overset{L}{\otimes}_X - : D(X) \times D(X) \to D(X)$, "tensor product"
  - For any object $X \in \mathcal{S}$, $R\underline{Hom}_X(-,-) : D(X)^{op} \times D(X) \to D(X)$, "internal hom"
  - For any morphism $f : X \to Y$ in $\mathcal{S}$, $f^* : D(Y) \to D(X)$, "inverse image"
  - For any morphism $f : X \to Y$ in $\mathcal{S}$, $Rf_* : D(X) \to D(Y)$, "direct image"
  - For any morphism $f : X \to Y$ in $\mathcal{S}$, $Rf_! : D(X) \to D(Y)$, "direct image with proper support"
  - For any morphism $f : X \to Y$ in $\mathcal{S}$, $Rf^! : D(Y) \to D(X)$, "extraordinary inverse image"
- For each object $X \in \mathcal{S}$, define canonical objects:
  - $R_X \in D(X)$, "constant sheaf"
  - $D_X \in D(X)$, "dualizing sheaf"

These data must satisfy the following axioms A, B, C, D, E, F: fix



- Adjunction. $f : X \to Y \in S,\ K, K_1, K_2 \in D(X),\ L, L_1, L_2 \in D(Y))$
  - global:
    - adj1: $Hom_X(K_1 \overset{L}{\otimes}_X -, K_2) \cong Hom_X(K_1, Hom_X(-, K_2))$.
    - adj2: $Hom_X(f^*L, K) \cong Hom_Y(L, Rf_*K)$.
    - adj2': $Hom_Y(Rf_!K, L) \cong Hom_X(K, f^!L)$.
  - local:
    - adj1: $R\underline{Hom}_X(K_1 \overset{L}{\otimes}_X -, K_2) = R\underline{Hom}_X(K_1, R\underline{Hom}_X(-, K_2))$.
    - adj2: $Rf_*R\underline{Hom}_X(f^*L, K) = R\underline{Hom}_Y(L, Rf_*K)$.
    - adj2': $R\underline{Hom}_Y(Rf_!K, L) = Rf_*R\underline{Hom}_X(K, f^!L)$.
- Base change. If

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

  is cartesian (i.e., a pullback diagram) in $\mathcal{S}$, then
  - bc1: $g^*Rf_! = Rf_!g^*$ as functors $D(X) \to D(Y')$ (proper base change).
  - bc2: $g^!Rf_* = Rf_*g^!$ as functors $D(X) \to D(Y')$ (smooth base change).

  Equality here is actually only up to canonical isomorphism.
- Commutativity.
  - com0: $(K_1 \overset{L}{\otimes}_X -) \overset{L}{\otimes}_X K_2 = K_1 \overset{L}{\otimes}_X (- \overset{L}{\otimes}_X K_2)$.
  - com1: $f^*(L_1 \overset{L}{\otimes}_Y L_2) = f^*L_1 \overset{L}{\otimes}_X f^*L_2$.
  - com2: $Rf_*R\underline{Hom}_X(f^*L, K) = R\underline{Hom}_Y(L, Rf_*K)$. (same as local adj2)

- com3: $Rf_!(f^*L \overset{L}{\otimes}_X K) = L \overset{L}{\otimes}_Y Rf_!K$.
- com3': $Rf_!(K \overset{L}{\otimes}_X f^*L) = Rf_!K \overset{L}{\otimes}_Y L$.
- com4: $f^!R\underline{Hom}_Y(L_1, L_2) = R\underline{Hom}_X(f^*L_1, f^!L_2)$.

- Duality. For any $X \in \mathcal{S}$, define a **dualizing functor** $\mathbb{D} : D(X)^{op} \to D(X)$ by $K \mapsto R\underline{Hom}_X(K, D_X)$. There is an isomorphism of functors $id \overset{\sim}{\leftrightarrow} \mathbb{D} \circ \mathbb{D}$.

- Exchange. $\mathbb{D}$ exchanges $R\underline{Hom}_X(K, -)$ with $K \overset{L}{\otimes}_X -$, Rf_* with $Rf_!$, and f^* with $f^!$, e.g. $\mathbb{D}(Rf_*K) = Rf_!(\mathbb{D}K)$. In particular, $\mathbb{D}$ exchanges axioms: adj1 with com0, adj2 (= com2) with com3, adj2' with com3', com1 with com4, and bc1 with bc2.

- Functoriality. For $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{S}$,

  - $(g \circ f)^* = f^* \circ g^*$, and $id^* = id$
  - $R(g \circ f)_* = Rg_* \circ Rf_*$, and $R(id)_* = id$
  - $R(g \circ g)_! = Rg_! \circ Rf_!$, and $R(id)_! = id$
  - $(g \circ f)^! = f^! \circ g^!$, and $id^! = id$

  The functors:

  - $K \mapsto R_X \overset{L}{\otimes}_X K$
  - $K \mapsto R\underline{Hom}_X(R_X, K)$

  are isomorphic to the identity functor $D(X) \to D(X)$.

A large portion of SGA 4 and 5 is devoted to showing that $D^b_c(X, \overline{\mathbb{Q}_l})$ satisfies the formalism of six operations given above, and Deligne showed that the category $D^b_m(X, \overline{\mathbb{Q}_l})$ of mixed sheaves admits the formalism in section 6 of Weil II.

Now we can put SGA 4 away.

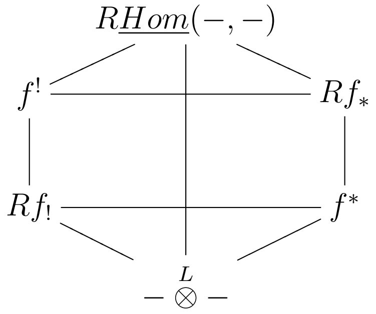
November 29, 2002

Thanksgiving break.

December 6, 2002

Some quick notes about last time:

- Recall the hexagon:



- $\mathbb{D}$ exchanges ! with $*$.
- When you have a bifunctor, $\mathbb{D}$ only cares about the **second** variable.
- If you want to get local monodromy a la Weil I and II, you need more input than just six operations.
- [Ogus asks if $\overset{L}{\otimes}$ satisfies the pentagon axiom instead of com0.] We can pretend the = signs are actually equality, but that would be too strong for the examples in nature. The truth requires isomorphisms and a big mess of compatibility conditions that haven't been written down. This issue is **not** raised in the literature! [Ogus remarks that this sort of negligence has led to a number of proofs of wrong theorems to be published.]

Profinite fundamental groups (SGA 1)

Let X be a connected scheme, and let $\overline{\eta} \to X$ be a geometric point. Let $FinEt_X$ be the category of X-schemes which are finite ètale over X. Then we have a "fiber" functor $fib_{\overline{\eta}} : FinEt_X \to FinSets$ given by $(Y \to X) \mapsto Hom_X(\overline{\eta}, Y) = Y(\overline{\eta}) = fib_{\overline{\eta}}(Y)$.

Definition The profinite fundamental group of $(X, \overline{\eta})$ is

$$\pi_1(X, \overline{\eta}) := Aut_{FinEt_X \to FinSets}(fib_{\overline{\eta}}),$$

the automorphism group of the fiber functor.

This is a profinite group, as it is a subgroup of $\prod_{Y \in FinEt_X} Perm(fib_{\overline{\eta}}(Y))$ defined by closed conditions.

Theorem A group G is compact and totally disconnected if and only if there exists a system $\{G_i\}$ of finite groups such that $G \cong \varprojlim G_i$.

Note that it is **not true** that every finite index subgroup of a profinite group is open, i.e., there may exist discontinuous homomorphisms to finite groups. Possibly the simplest example comes from $Gal(\mathbb{Q}(\{\sqrt{p}\})/\mathbb{Q}) \cong \prod_p \mathbb{Z}/2$, where p runs over all primes. There are uncountably infinitely many index 2 subgroups of this group, since such subgroups are in natural bijection with non-zero elements of the dual space of an infinite dimensional vector space over $\mathbb{F}_2$. By the fundamental theorem of Galois theory, the index 2 closed (hence open) subgroups of the Galois group are in bijection with the degree 2 extensions of $\mathbb{Q}$ lying in $\mathbb{Q}(\{\sqrt{p}\})$, and there are only countably many such extensions.

Let $\pi := \pi_1(X, \overline{\eta})$, and let $\pi - FinSets$ be the category of finite sets with **continuous** π-action. The objects are pairs (E, ρ), where E is a finite set and $\rho : \pi \to Perm(E)$ is a continuous homomorphism. The morphisms are π-equivariant maps ot sets. Then $fib_{\overline{\eta}} : FinEt_X \to FinSets$ factorizes as:

$$\begin{array}{ccccc} FinEt_X & & \xrightarrow{\quad fib_{\overline{\eta}} \quad} & & FinSets \\ & \searrow^{\widetilde{fib_{\overline{\eta}}}} & & \nearrow^{\text{forget}} & \\ Y & & \pi - FinSets & & fib_{\overline{\eta}}(Y) \\ & \searrow & & \nearrow & \\ & & (fib_{\overline{\eta}}(Y), ev_Y) & & \end{array}$$

where $ev_Y : \pi \to Perm(fib_{\overline{\eta}}(Y))$ is the **evaluation homomorphism**. This factorization is an **equality** of functors.

Theorem (Grothendieck) The functor $\widetilde{fib_{\overline{\eta}}} : FinEt_X \to \pi - FinSets$ is an equivalence of categories.

Properties of π_1

1. Functoriality. π_1 is a functor:

$$\left\{ \begin{array}{c} \text{geometrically pointed} \\ \text{connected schemes with} \\ \text{pointed morphisms} \end{array} \right\} \xrightarrow{\pi} \left\{ \begin{array}{c} \text{profinite groups} \\ \text{with continuous} \\ \text{homomorphisms} \end{array} \right\}$$

2. Independence of base point. If $\overline{\eta}' \to X$ is another geometric point, then:
   (a) The functors $fib_{\overline{\eta}}, fib_{\overline{\eta}'} \in (FinEt_X \to FinSets)$ are isomorphic.
   (b) Any choice of isomorphism $\alpha : fib_{\overline{\eta}} \xrightarrow{\sim} fib_{\overline{\eta}'}$ induces an isomorphism $\alpha_* : \pi_1(X, \overline{\eta}) \xrightarrow{\sim} \pi_1(X, \overline{\eta}')$ given by $g \mapsto \alpha \circ g \circ \alpha^{-1}$.
   (c) If α and β are two choices of isomorphisms $fib_{\overline{\eta}} \xrightarrow{\sim} fib_{\overline{\eta}'}$, then there exists a (nonunique) $\gamma \in \pi_1(X, \overline{\eta}')$ such that $\alpha_* = conj_\gamma \circ \beta_*$. In fact, there exists a unique $\gamma \in \pi_1(X, \overline{\eta}') = Aut(fib_{\overline{\eta}'})$ such that $\alpha = \gamma \circ \beta$.

Proposition Let k be a field, $X := \text{Spec}(k)$, and $\overline{\eta} \to X$ a geometric point. Then there is a canonical isomorphism $\pi_1(X, \overline{\eta}) \to Gal(k^{sep}/k)$, where k^{sep} is the separable closure of k in $\Gamma(\overline{\eta})$.

Analogy with GAGA: Consider a scheme $X \to \text{Spec}(\mathbb{C})$, connected and locally of finite type, and let $\pi := \pi_1(X^{an} = X(\mathbb{C}), \overline{\eta})$ be the topological fundamental group. Then $\pi_1(X, \overline{\eta}) = \hat{\pi}$, the profinite completion. [Note that this could be trivial, if π is not residually finite. See Raynaud's comments regarding Higman's group and Toledo's construction at the end of SGA 1, Exp XII (new edition).]

Proposition Let X be a normal irreducible scheme, and $F = \kappa(X)$ its function field. Let $\overline{F}$ be **some** algebraically closed extension of F. This defines a geometric point $\overline{\eta} \to \eta := \text{Spec}(F) \hookrightarrow X$.

1. The homomorphism $Gal(F^{sep}/F) = \pi_1(\eta, \overline{\eta}) \to \pi_1(X, \overline{\eta})$ is surjective (normalization is a functor).

2. Let $F^{ur/X} \subset F^{sep}$ be the maximal unramified extension of F in F^{sep} that is unramified over X (a finite extension F'/F is unramified over X if and only if the normalization:

$$\begin{array}{ccccc} \text{Spec}(F') = \eta' & \hookrightarrow & X' & = & \text{normalization of } X \text{ in } \eta' \\ \downarrow \scriptstyle\text{finite} & & \downarrow \scriptstyle\text{finite} & & \\ \text{Spec}(F) = \eta & \hookrightarrow & X & & \end{array}$$

   of X in F' is ètale over X). Then $ker(Gal(F^{sep}/F) \twoheadrightarrow \pi_1(X, \overline{\eta}))$ is the normal subgroup $Gal(F^{sep}/F^{ur/X})$.

Corollary Let X be a proper smooth connected curve over a finite field. Let $U \underset{open,dense}{\hookrightarrow} X \underset{closed}{\hookleftarrow} S = (X - U)_{red}$. For each $x \in |X|$, choose a separable closure $\overline{F_x}$ of F_x, choose an F-embedding $\alpha_x : \overline{F} \hookrightarrow \overline{F_x}$, and get $(\alpha_x)_* : Gal(\overline{F_x}/F_x) \hookrightarrow Gal(F^{sep}/F)$. Then $ker(Gal(F^{sep}/F) \twoheadrightarrow \pi_1(U, \overline{\eta}))$ is the normal subgroup generated by $(\alpha_x)_*(I(\overline{F_x}/F_x))$ as x runs over $|U| = |X| \setminus S$, where $I(\overline{F_x}/F_x) = Gal(\overline{F_x}/F_x^{ur})$ is the inertia group.

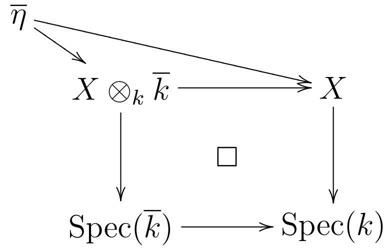
This corollary says that to study Langlands correspondence, we should try studying $\pi_1(U, \overline{\eta})$ for some U. There exist representations which are everywhere ramified, but we ignore those. One reason why we have the condition that S is finite is because then U exists.

Theorem Let k be a field, and X a quasi-compact k-scheme. Let $\overline{\eta} \to X$ be a geometric point, and let $\overline{k}$ be the separable closure of k in $\Gamma(\overline{\eta})$. Assume X is geometrically connected (i.e., $X \otimes_k \overline{k}$ is connected). Then the following sequence is exact:

$$1 \to \pi_1(X \otimes_k \overline{k}, \overline{\eta}) \to \pi_1(X, \overline{\eta}) \to \pi_1(\mathrm{Spec}(k), \overline{\eta}) \to 1$$

The first term is called the geometric fundamental group, the second term is the arithmetic fundamental group, and the last term is just $Gal(\overline{k}/k)$.

Corollary Let F be any field extension of k, and take $X = \mathrm{Spec}(F)$. Let $\overline{F}$ be **any** separably closed extension of F, and take $\overline{\eta} = \mathrm{Spec}(\overline{F})$. Let $\overline{k}$ be the separable closure of k in $\overline{F}$. Assume k is separably closed in F (so that $\Gamma(X \otimes_k \overline{k}) = F \otimes_k \overline{k} = F\overline{k}$ in $\overline{F}$).



Then the above sequence becomes:

$$1 \to Gal(\overline{F}/F\overline{k}) \to Gal(\overline{F}/F) \to Gal(F\overline{k}/F) = Gal(\overline{k}/k) \to 1$$

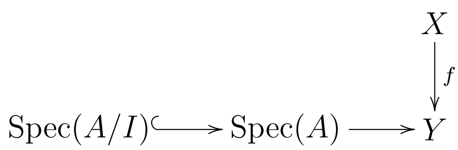
The moral is: π_1 is much better behaved than Gal, even though they are the same thing.

Etale sheaves

Let X be a quasi-compact quasi-separated scheme.

Definition The **(small) étale site** of X is the category $X_{\acute{e}t}$ whose objects are X-schemes that are étale over X and whose morphisms are X-morphisms (naturally étale be a special property of étale maps), together with a notion of **covering**, where a family of étale morphisms $(U_i \xrightarrow{\phi_i} U)_{i \in I}$ is a covering if and only if $U = \bigcup_{i \in I} \phi_i(U_i)$.

What **is** an étale morphism? $\begin{array}{c} X \\ \downarrow f \\ Y \end{array}$ is $\begin{cases} \text{smooth} \\ \text{étale} \\ \text{unramified} \end{cases}$ if and only if f is locally of finite presentation, and for any artinian ring A, any ideal $I \subset A$ satisfying $I^2 = 0$, and any map $\mathrm{Spec}(A) \to Y$, the canonical map $X(A) \to X(A/I)$ is a $\begin{cases} \text{surjection} \\ \text{bijection} \\ \text{injection} \end{cases}$.



Incidentally, this can be rephrased in terms of complete dvrs.

étale	means	deformations lift uniquely
smooth	means	deformations have lifts
unramified	means	if a deformation lifts, the lift is unique

An map is étale if and only if the target can be covered by affines, such that the restricted maps have invertible derivatives. More precisely,

$$\begin{array}{ccccc} U = \mathrm{Spec}(A) & \hookrightarrow & X \times_Y V & \longrightarrow & X \\ & \searrow & \downarrow & \square & \downarrow \\ & & V = \mathrm{Spec}(B) & \hookrightarrow & Y \end{array}$$

A/B is an extended étale algebra if and only if $A = B[T_1, \ldots, T_n]/(f_1, \ldots f_n)$ and the matrix $\left(\frac{\partial f_i}{\partial T_j}\right)_{1 \le i,j \le n}$ is invertible.

For convenience, we're going to change the definition of **proper** to require finite presentation, instead of just finite type.

Definition A **presheaf** on $X_{\acute{e}t}$ (also called an **étale presheaf of sets**) is a contravariant functor $X_{\acute{e}t} \to Sets$. A **morphism** of presheaves is a natural transformation. We get a category $Preshv(X_{\acute{e}t}, Sets)$.

Definition A **Sheaf** on $X_{\acute{e}t}$ is a presheaf $\mathcal{F}$ on $X_{\acute{e}t}$ such that for any covering $(U_i \xrightarrow{\phi_i} U)_{i \in I}$ in $X_{\acute{e}t}$, the following sequence is exact:

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \underset{[j]}{\overset{[i]}{\rightrightarrows}} \prod_{i,j \in I} \mathcal{F}(U_i \times_U U_j)$$

This means the image of the first map is $\{x : [i]x = [j]x\}$.

We get a full subcategory $Shv(X_{\acute{e}t}, Sets)$ of $Preshv(X_{\acute{e}t}, Sets)$, called the **étale topos of** X. The "s" in "topos" is silent. There exists a left adjoint of the inclusion functor, called the "associated sheaf" functor. It is exact, but inclusion is only left exact.

Definition Assume X is locally noetherian, and let $\mathcal{F} \in Shv(X_{\acute{e}t}, Sets)$.

1. $\mathcal{F}$ is **constant** if and only if $\mathcal{F}$ is the sheaf associated to the presheaf given by $Y \mapsto E$ for some fixed set E. This is the sheaf $X_{\acute{e}t} \to Sets$ given by $Y \mapsto E^{\pi_0(Y)}$.

2. $\mathcal{F}$ is **locally constant** if and only if there exists a covering $(U_i \to X)_{i \in I}$ in $X_{\acute{e}t}$ such that $\mathcal{F}|_{U_i}$ is constant.

3. $\mathcal{F}$ is **finite** if for any $U \in X_{\acute{e}t}$, $\mathcal{F}(U)$ is a finite set. [This may not be quite correct.]

4. If $\overline{\eta} \to X$ is a geometric point, then the **stalk** (or **fiber**) of $\mathcal{F}$ at $\overline{\eta}$ is $\mathcal{F}_{\overline{\eta}} := \varinjlim \mathcal{F}(U)$, where the limit is taken over all diagrams:

$$\begin{array}{ccc} & & U \\ & \nearrow & \downarrow \text{étale} \\ \overline{\eta} & \longrightarrow & X \end{array}$$

   $\mathcal{F}$ has **finite fibers** if for any geometric point $\overline{\eta} \to X$, $\mathcal{F}_{\overline{\eta}}$ is a finite set. [This also may not be quite correct.]

5. $\mathcal{F}$ is **lisse** if and only if $\mathcal{F}$ is locally constant and $\begin{cases} \text{finite} \\ \text{finite fibers} \end{cases}$. Either one works. [The correct definition: there exists a covering $(U_i \to X)_{i \in I}$ in $X_{ét}$ such that $\mathcal{F}|_{U_i}$ is constant, and each $\mathcal{F}(U_i)$ is a finite set.]

6. $\mathcal{F}$ is **constructible** if and only if X can be partitioned into a disjoint union $\coprod_{i \in I} X_i$ of locally closed subschemes $X_i \subset X$, such that each $\mathcal{F}|_{X_i}$ is lisse in $Shv((X_i)_{ét}, Sets)$. Such a partition is called a (quasi-)stratification.

What is $\mathcal{F}|_{X_i}$ for nonopen X_i? Consider $X_i \overset{i}{\hookrightarrow} X$ locally closed, and let $f : U \to X_i$ be étale. Commutative diagrams of the form

$$\begin{array}{ccc} U & \xrightarrow{j} & V_j \\ f \downarrow ét & & g \downarrow ét \\ X_i & \overset{i}{\hookrightarrow} & X \end{array}$$

form the objects of a filtered subcategory of $X_{ét}$, so we can define a presheaf by ${}^{pre}(\mathcal{F}|_{X_i})(U) := \varinjlim \mathcal{F}(V_j)$, and take the associated sheaf. check

Proposition Let X be a connected normal scheme (locally noetherian), and $\overline{\eta} \to X$ a geometric point (giving us $\pi_1(X, \overline{\eta})$). Then the functor

$$\begin{array}{rcl} \{\text{lisse sheaves on } X_{ét}\} & \to & \pi_1(X, \overline{\eta}) - FinSets \\ \mathcal{F} & \mapsto & (\mathcal{F}_{\overline{\eta}}, \text{automorphism}) \end{array}$$

is an equivalence of categories.

The point is that every lisse sheaf is representable.

Proposition The functors

$$\begin{array}{rccc} & \left\{ \begin{array}{c} X\text{-schemes} \\ \text{étale over } X \end{array} \right\} & \to & \left\{ \begin{array}{c} \text{locally constant} \\ \text{sheaves on } X_{ét} \end{array} \right\} \\ & \cup & & \cup \\ FinEt_X = & \left\{ \begin{array}{c} X\text{-schemes finite} \\ \text{étale over } X \end{array} \right\} & \to & \{\text{lisse sheaves on } X_{ét}\} \\ & Y & \mapsto & Hom_X(-, Y) \end{array}$$

are equivalences of categories.

Variant Replace "Sets" everywhere above with "R-modules" for a fixed commutative ring R (keep $\mathbb{Z}_l$ and $\mathcal{O}_\lambda \subset K/\mathbb{Q}_l$ in mind).

- We get R-linear abelian categories $Shv(X_{ét}, R-mod) \subset Preshv(X_{ét}, R-mod)$. Once we have nice abelian categories, we can do cohomology.
- We get a full subcategory $Shv_c(X_{ét}, R-mod) \subset Shv(X_{ét}, R-mod)$ of **constructible** sheaves. This is also R-linear and abelian, but we can't do cohomology, since there aren't enough injectives.
- We get a full subcategory $LisseShv(X_{ét}, R-mod) \subset Shv_c(X_{ét}, R-mod)$ of lisse sheaves. This category is R-linear, abelian, and equivalent to the category of π_1-representations on (set-theoretically) finite R-modules.

December 13, 2002

No notes (finals week?)

January 28, 2003

Recall the program from last term:

- Class field theory (Langlands over GL_1)
- Cuspidal automorphic representations of $GL_r(\mathbb{A})$
- L- and ϵ-factors of automorphic representations
- Converse theorem
- Galois representations
- l-adic cohomology theory
- L- and ϵ-factors of Galois representations
- Twisting results (Deligne, et al.)
- Product formula (Laumon, et al.)
- $\mathcal{G} \to \mathcal{A}$ and $\mathcal{A} \to \mathcal{G}$

Recall the diagram:

Lafforgue, Arthur-Selberg trace formula	(moduli stacks of shtukas)	l-adic cohomology, Grothendieck-Lefshetz-Verdier trace formula
	Geometric	
	$\nearrow$ $\searrow$	
(automorphic representations)	Spectral Algebraic	(Galois representations)
	$\nwarrow$ $\swarrow$	
	Analytic	
converse theorem, Rankin-Selberg convolutions	(L-functions, ϵ-factors)	Deligne's Weil II (RH), product formula

Bottom row: Rankin-Selberg theory lets us construct automorphic representations from L- and ϵ-factors and functions. In order to construct such representations, we need information on L- and ϵ-factors, given by Grothendieck's and Laumon's product formula.

Let X be a connected normal scheme. Arithmetic π_1 is defined by $\pi_1(X, \overline{\eta}) = Aut_{FinEt_X \to FinSets}(fib_{\overline{\eta}})$, where $fib_{\overline{\eta}}$ is defined by:

$$\begin{array}{ccc} FinEt_X & \stackrel{fib_{\overline{\eta}}}{\longrightarrow} & FinSets \\ \begin{pmatrix} Y \\ \downarrow \\ X \end{pmatrix} & \mapsto & Y(\overline{\eta})_X := Hom_X(\overline{\eta}, Y) \end{array}$$

This factors through $\widetilde{fib_{\overline{\eta}}} : FinEt_X \to \pi_1 - FinSets$, whose image is $Y(\overline{\eta})_X$ endowed with an action of π_1.

Let R be a commutative ring. Then we can define R-linear abelian categories: $Shv(X_{\acute{e}t}, R - mod) \underset{\text{full}}{\supset} Shv_c(X_{\acute{e}t}, R - mod) \underset{\text{full}}{\supset} Shv_{lisse}(X_{\acute{e}t}, R - mod)$, with the last equivalent to the category of set theoretically finite R-modules with continuous π_1-action.

Now we fit this formalism into the framework of 6 operations.

Derived category of an abelian category

Let A be an additive category. $Cplx(A)$ is the additive category of complexes over A. We get a canonical functor $A \to Cplx(A)$ taking K to $\cdots \to 0 \to K \to 0 \to \dots$, where K is placed in the degree 0 spot. If A is abelian, we get cohomological functors: $H^i : Cplx(A) \to A$ defined by $K \mapsto H^i(K) := \frac{ker(d^i)}{im(d^{i-1})}$ for all $i \in \mathbb{Z}$.

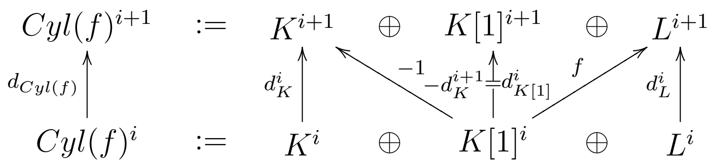
Definition Let A be an abelian category. A morphism $f : K \to L$ in $Cplx(A)$ is a **quasi-isomorphism** if and only if for $i \in \mathbb{Z}$, $H^i(f) : H^i(K) \to H^i(L)$ is an isomorphism in A.

Definition Let A be additive. For $n \in \mathbb{Z}$, the shift functor $[n] : Cplx(A) \to Cplx(A)$ is defined by:

$$\begin{array}{ccc} K = (K^i, d_K^i) & \xrightarrow{\quad f \quad} & L = (L^i, d_L^i) \\ & \Big\downarrow [n] & \\ K[n] & \xrightarrow[\quad f[n] \quad]{} & L[n] \end{array}$$

where $K[n]^i = K^{n+i}$, $d^i_{K[n]} = (-1)^n d_K^{n+i}$, and $f[n]^i = f^{n+i}$.

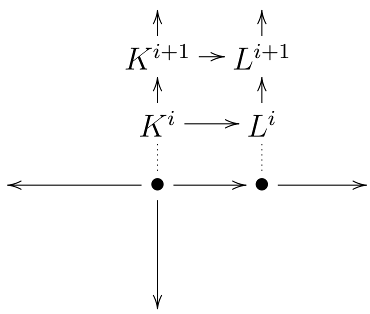
Definition If $f : K \to L$ is a morphism in $Cplx(A)$, then we define the **mapping cylinder** of f, written $Cyl(f) \in Cplx(A)$, by:



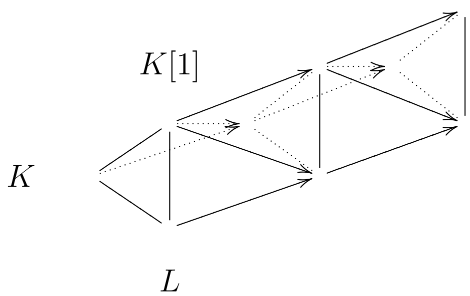
We can write the differential in shorthand as:

$$((k^i, k^{i+1}, l^i) \mapsto (dk^i - k^{i+1}, -dk^{i+1}, f(k^{i+1}) + dl^i)).$$

There is a spectral sequence picture.



One reason to call it a cylinder:



The motivation from topology comes from a "top hat" picture. K is the top. L is the brim.

Definition If $f : K \to L$ is a morphism in $Cplx(A)$, the **mapping cone** of f, $C(f) \in Cplx(A)$ is defined by:

$$\begin{array}{ccccc} C(f)^{i+1} & := & K[1]^{i+1} & \oplus & L^{i+1} \\ \uparrow d_{C(f)} & & \uparrow d^i_{K[1]} = -d^{i+1}_K & \nearrow f & \uparrow d^i_L \\ C(f)^i & := & K[1]^i & \oplus & L^i \end{array}$$

We have a collection of canonical morphisms:

$$\begin{array}{lll} \overline{f} : K \to Cyl(f) & \text{given by} & k \mapsto (k, 0, 0) \\ \alpha : L \to Cyl(f) & & l \mapsto (0, 0, l) \\ \beta : Cyl(f) \to L & & (k^i, k^{i+1}, l^i) \mapsto f(k^i) + l^i \\ \pi : Cyl(f) \to C(f) & & (k, k', l) \mapsto (k', l) \\ i : L \to C(f) & & l \mapsto (0, l) \\ \partial : C(f) \to K[1] & & (k, l) \mapsto k \end{array}$$

so every morphism $f : K \to L$ in $Cplx(A)$ gives rise to a diagram:

$$\begin{array}{ccccccccc} K & \xrightarrow{f} & L & & & & & & \\ \| & & \uparrow \beta & & & & & & \\ K & \xrightarrow{\overline{f}} & Cyl(f) & \xrightarrow{\pi} & C(f) & \longrightarrow & 0 & & \\ & & \uparrow \alpha & & \| & & & & \\ 0 & \longrightarrow & L & \xrightarrow{i} & C(f) & \xrightarrow{\partial} & K[1] & \longrightarrow & 0 \end{array}$$

and if A is abelian, then the rows are exact, and α and β are quasi-isomorphisms.

Definition A **distinguished triangle** in $Cplx(A)$ is a sequence of maps of the form

$$K \xrightarrow{\overline{f}} Cyl(f) \xrightarrow{\pi} C(f) \xrightarrow{\partial} K[1]$$

for some morphism f in $Cplx(A)$. We **do not** allow isomorphic triangles to be distinguished. [what goes wrong?]

We get $\Delta_{Cplx(A)}$, a collection of distinguished triangles in $Cplx(A)$.

Note $(Cplx(A), \{[n]\}_{n \in \mathbb{Z}}, \Delta_{Cplx(A)})$ is not in general a triangulated category.

References: Verdier's thesis, Des categories abeliennes et des categories derivees. Asterisque 2??. BBD, Faisceaux Pervers. Asterisque 100.

Working with derived categories is like quantum mechanics. An object in a derived category is like a wave function, and taking homology is like making an observation - it messes up the wave function. No matter how many observations you make by taking spectral sequences, you can't reconstruct the original object.

February 4, 2003

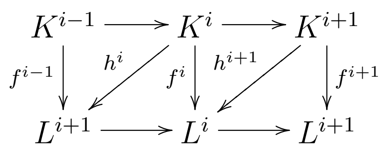
Let A be an additive category. Last time, we defined the additive category $Cplx(A)$ of complexes in A, with a shift functors $[n], n \in \mathbb{Z}$, and a class $\Delta_{Cplx(A)}$ of distinguished triangles.

Definition The **homotopy category** $K(A)$ has the same objects as $Cplx(A)$, and morphisms are defined by

$$Hom_{K(A)}(K, L) := Hom_{Cplx(A)}(K, L)/(\text{null-homotopies}),$$

where $f : K \to L$ is a null-homotopy if and only if there exists a sequence of morphisms $h^i : K^i \to L^{i-1}$ in A for all $i \in \mathbb{Z}$, such that $f^i = h^{i+1} \circ d_K^i + d_L^{i-1} \circ h^i$.

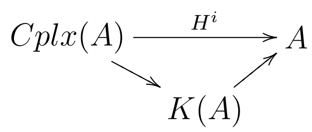
This weakens the notion of isomorphism. Here's a potentially helpful diagram:



We get a canonical functor $Cplx(A) \to K(A)$ that is identity on objects. We can define shift functors $[n] : K(A) \to K(A)$ via pushforward. Let $\Delta_{K(A)}$:= essential image of $\Delta_{Cplx(A)}$. This means we take anything isomorphic to an image object.

Proposition $(K(A), \{[n]\}_{n\in\mathbb{Z}}, \Delta_{K(A)})$ is a triangulated category.

From now on, we assume A is an abelian category. We get factorizations of cohomology functors for all $i \in \mathbb{Z}$:



This is an **equality** of functors. Thus, it makes sense to speak of quasi-isomorphisms in $K(A)$.

Big Theorem-Definition There exists an additive category $D(A)$ with an additive functor $\delta : K(A) \to D(A)$ such that:

1. δ is identity on objects.

2. For any quasi-isomorphism f in $K(A)$, $\delta(f)$ is an isomorphism in $D(A)$.
3. If $F : K(A) \to \mathcal{E}$ is an additive functor to an additive category $\mathcal{E}$, such that for any quasi-isomorphism f in $K(A)$, $F(f)$ is an isomorphism in $\mathcal{E}$, then there exists a unique functor $\overline{F} : D(A) \to \mathcal{E}$ such that $F = \overline{F} \circ \delta$:

$$\begin{array}{ccc} D(A) & \xrightarrow{\overline{F}} & \mathcal{E} \\ & {\scriptstyle\delta}\nwarrow \quad \nearrow{\scriptstyle F} & \\ & K(A) & \end{array}$$

   Note that this is an **equality** of functors, which is necessary for uniqueness to make sense.

$D(A)$ is called the **derived category** of A.

References: Verdier's thesis. Asterisque 2??. Gelfand and Manin wrote two books on homological algebra. The thin one has lots of errors and no proofs. The thick one has an even higher error density than the thin one.

The shift functors $[n]$ in $D(A)$ are defined by the following diagram commuting as an **equality** of functors:

$$\begin{CD} D(A) @>[n]>> D(A) \\ @A{\delta}AA @AA{\delta}A \\ K(A) @>[n]>> K(A) \end{CD}$$

$\Delta_{D(A)}$ is defined to be the essential image of $\Delta_{K(A)}$ under δ.

Proposition $(D(A), \{[n]\}_{n\in\mathbb{Z}}, \Delta_{D(A)})$ is a triangulated category.

Proposition The inclusion functor $A \overset{deg\,0}{\to} Cplx(A) \to K(A) \to D(A)$ is fully faithful, the essential image is $D(A)^{deg\,0} := \{K \in D(A) : H^i(K) = 0 \text{ for all } i \neq 0\}$, and $H^0 : D(A) \to A$ is the quasi-inverse.

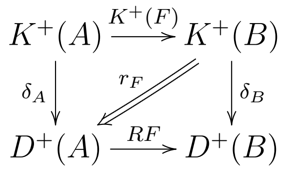
There are some variants to the basic derived category. Let $* \in \{\emptyset, +, -, b\}$. then we can define $Cplx^*(A)$, $K^*(A)$, and $D^*(A)$ by restricting to complexes that are:

$\emptyset$	arbitrary	$(\cdots \to * \to * \to * \to \dots)$
$+$	bounded below	$(\cdots \to 0 \to 0 \to * \to * \to \dots)$
$-$	bounded above	$(\cdots \to * \to * \to 0 \to 0 \to \dots)$
b	bounded	$+ \cap -$

Each of the categories $K^*(A)$ and $D^*(A)$ is triangulated.

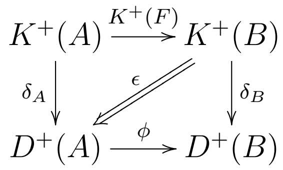
Let A and B be abelian categories, and let $F : A \to B$ be an additive functor. For $* \in \{\emptyset, +, -, b\}$, F induces $Cplx^*(F) : Cplx^*(A) \to Cplx^*(B)$ by mapping componentwise. This respects null-homotopy, so it induces $K^*(F) : K^*(A) \to K^*(B)$.

Definition Suppose F is **left-exact**. Then the **right derived functor** of F (unique if it exists) consists of a triangulated functor $RF : D^+(A) \to D^+(B)$, and a morphism of functors $r_F : \delta_B \circ K^+(F) \Rightarrow RF \circ \delta_A$ (the arrow goes between the functors, not the categories)

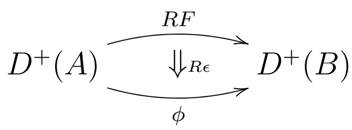


with the following universal property:

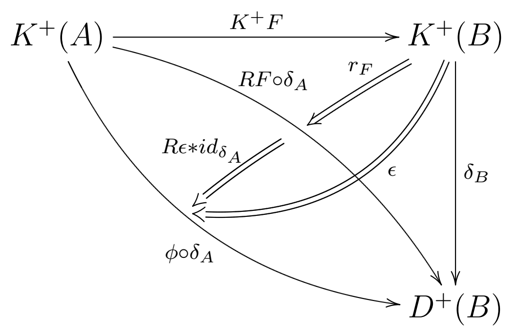
- For any triangulated functor $\phi : D^+(A) \to D^+(B)$, and any morphism of functors $\epsilon : \delta_B \circ K^+(F) \Rightarrow \phi \circ \delta_A$:



  there exists a unique morphism of functors $R\epsilon : RF \Rightarrow \phi$:



  such that $\epsilon = (R\epsilon * id_{\delta_A}) \circ r_F$:



The $*$ in the last equation denotes horizontal composition of natural transformations, which composes maps on functors between different categories.

There is an analogous definition for left derived functors of right exact functors. [In fact, $LF = R(F^{op})^{op}$, where $-^{op}$ denotes the functor naturally induced on opposite categories. Neither definition actually need the exactness properties, but they are necessary in order for the next definition to agree with the usual hyper-derived functors.]

Definition $R^iF(-) := H^i_{D^+(B)} \circ RF(-)$. The image lies in B.

Proposition Suppose A has **enough injectives**, and B is abelian. Then for any left exact functor $F : A \to B$, the right derived functor $RF : D^+(A) \to D^+(B)$ exists. [Dually, the left derived functor of a right exact functor exists, given enough projectives.]

So much for abstract nonsense.

Definition and Theorems for 6 operations in l-adic cohomology

Let S be a notherian, separated, regular scheme of dimension $leq1$. Let (Sch/S) be the category of S-schemes which are separated of finite type/presentation over S, with S-morphisms. Let l be a prime, and assume l is invertible in S. Let $\mathcal{O}_\lambda$ be the ring of integers in some finite extension E_λ of $\mathbb{Q}_l$. Let $R = \mathcal{O}_\lambda/\lambda^n$ for some fixed $n \geq 1$. Then R is a finite order torsion local ring with residue characteristic l.

For any $X \in (Sch/S)$, we get the topos $X_{\acute{e}t}$ and the category $Shv(X_{\acute{e}t}, R)$ which is abelian and has enough injectives and projectives. We get the derived category $D(Shv(X_{\acute{e}t}, R))$, and $D^b_c(X_{\acute{e}t}, R)$, the full subcategory of $D^b(Shv(X_{\acute{e}t}, R))$ whose objects K are such that for all $i \in \mathbb{Z}$, $H^i(K) \in Shv_c(X_{\acute{e}t}, R)$, i.e., complexes of sheaves with constructible cohomology. $D^b_c(X_{\acute{e}t}, R)$ is triangulated.

One $- \overset{L}{\otimes} -$. Start from

$$\begin{array}{ccccc} Shv(X_{\acute{e}t}, R) & \times & Shv(X_{\acute{e}t}, R) & \to & Shv(X_{\acute{e}t}, R) \\ \mathcal{F} & & \mathcal{G} & \mapsto & \mathcal{F} \otimes \mathcal{G} \end{array}$$

where $\mathcal{F} \otimes \mathcal{G}$ is the sheaf associated to the presheaf $X_{\acute{e}t} \to (R - mod)$ given by $U \mapsto \mathcal{F}(U) \otimes \mathcal{G}(U)$.

If we fix one variable, this functor is right exact in the other, so we get the left derived functor $D^-(X_{\acute{e}t}, R) \times D^-(X_{\acute{e}t}, R) \to D^-(X_{\acute{e}t}, R)$ (actually, we would need to verify its niceness as a bifunctor first).

Theorem This induces a functor $D^b_c(X_{\acute{e}t}, R) \times D^b_c(X_{\acute{e}t}, R) \overset{\overset{L}{\otimes}}{\to} D^b_c(X_{\acute{e}t}, R)$.

[Someone asks about the boundedness in the case $X = \mathrm{Spec}(\mathbb{C})$, $R = \mathbb{Z}/l^2$, $\mathbb{Z}/l \otimes_R \mathbb{Z}/l$ in degree 0.]

Two $R\underline{Hom}(-,-)$. Start from

$$\begin{array}{ccccc} Shv(X_{\acute{e}t}, R)^{op} & \times & Shv(X_{\acute{e}t}, R) & \to & Shv(X_{\acute{e}t}, R) \\ \mathcal{F} & & \mathcal{G} & \mapsto & \underline{Hom}(\mathcal{F}, \mathcal{G}) \end{array}$$

where $\underline{Hom}(\mathcal{F}, \mathcal{G})$ is the the sheaf (not just a presheaf - by descent) $X_{ét} \to (R-mod)$ given by $U \mapsto Hom_{Shv(U_{ét},R)}(\mathcal{F}|_U, \mathcal{G}|_U)$.

If we fix one variable, the functor is left exact in the other, so we get the right derived functor $D^-(X_{ét}, R)^{op} \times D^+(X_{ét}, R) \overset{R\underline{Hom}}{\to} D^+(X_{ét}, R)$.

Next week: finiteness statements.

February 11, 2003

No notes (no class?)

February 18, 2003

More abstract nonsense. No l-adic sheaves for a little while.

Proposition-Definition Let $\mathcal{C}$ be any category, and let $S \subset Mor(\mathcal{C})$ be a collection of morphisms. Then there exists a category $\mathcal{C}[S^{-1}]$ and a functor $i : \mathcal{C} \to \mathcal{C}[S^{-1}]$ with the universal property that:

1. i is the identity on objects.
2. For any morphism $f \in S$, $i(f)$ is an isomorphism in $\mathcal{C}[S^{-1}]$.
3. If $F : \mathcal{C} \to \mathcal{E}$ is a functor such that $F(S) \subset Isom(\mathcal{E})$, then there exists a unique functor $\overline{F} : \mathcal{C}[S^{-1}] \to \mathcal{E}$ such that $F = \overline{F} \circ i$, i.e., the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C}[S^{-1}] & \xrightarrow{\overline{F}} & \mathcal{E} \\ {\scriptstyle i}\uparrow & \nearrow_{F} & \\ \mathcal{C} & & \end{array}$$

$\mathcal{C}[S^{-1}]$ is called the **localization of $\mathcal{C}$ with respect to** S.

Example Let A be an abelian category, $\mathcal{C} = Cplx(A)$ or $K(A)$, and $S = \{$quasi-isomorphisms in $\mathcal{C}\}$. Then $\mathcal{C}[S^{-1}]$ is the derived category $D(A)$ of A.

Definition $S \subset Mor(\mathcal{C})$ is a **localizing class** if and only if:

1. $id_X \in S$ for all $X \in Ob(\mathcal{C})$, and S is closed under composition (i.e., S is a subcategory of $\mathcal{C}$, bijective on objects).
2. For any $f \in Mor(\mathcal{C})$ and $s \in S$ such that

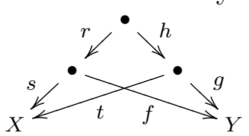
$$\begin{array}{ccc} & & Z \\ & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{f} & Y \end{array}, \text{ resp. } \begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle s}\downarrow & & \\ Z & & \end{array}$$

there exist $g \in Mor(\mathcal{C})$ and $t \in S$ such that

$$\begin{array}{ccc} W & \xrightarrow{g} & Z \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{f} & Y \end{array} \text{, resp. } \begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ Z & \xrightarrow{g} & W \end{array} \text{ commute.}$$

3. For any $f, g : X \to Y$ in $Mor(\mathcal{C})$, there exists $s \in S$ such that $sf = sg$ if and only if there exists $t \in S$ such that $ft = gt$.

Proposition (Verdier) If $S \subset Mor(\mathcal{C})$ is localizing, then $\mathcal{C}[S^{-1}]$ has the following description:

1. $\mathcal{C}[S^{-1}]$ has objects = objects $[\mathcal{C}]$.
2. A morphism $X \to Y$ in $\mathcal{C}[S^{-1}]$ is an equivalence class of "roof diagrams" $X \xleftarrow{s} \bullet \xrightarrow{f} Y$ with $s \in S$, $f \in Mor(\mathcal{C})$, where two roofs $X \xleftarrow{s} \bullet \xrightarrow{f} Y$ and $X \xleftarrow{t} \bullet \xrightarrow{g} Y$ are equivalent if and only if there exists an upper roof giving a commutative diagram  .
3. id_X in $\mathcal{C}[S^{-1}]$ is the equivalence class of the "identity roof" $X \xleftarrow{id_X} X \xrightarrow{id_X} X$
4. Composition of the morphisms $\left(X \xleftarrow{s} \bullet \xrightarrow{f} Y \right)$ followed by $\left(Y \xleftarrow{t} \bullet \xrightarrow{g} Z \right)$ is described by applying axiom 2 to the diagram $\bullet \xrightarrow{f} Y \xleftarrow{t} \bullet$ to get $\begin{array}{ccccc} & & \bullet & & \\ & {\scriptstyle r}\swarrow & & \searrow{\scriptstyle h} & \\ \bullet & & & & \bullet \\ & {\scriptstyle f}\searrow & & \swarrow{\scriptstyle t} & \\ & & Y & & \end{array}$, then composing to get $X \xleftarrow{sr} \bullet \xrightarrow{gh} Z$

From now on, the category A is abelian.

Note that in $Cplx(A)$, $S :=$ {quasi-isomorphisms } is **not** a localizing class.

Proposition In $K(A)$, $S :=$ {quasi-isomorphisms in $K(A)$} **is** a localizing class, so $K(A)[S^{-1}]$ is **isomorphic** to $D(A)$.

Definition Let $F : A \to B$ be a $\begin{cases} \text{left} \\ \text{right} \end{cases}$ -exact functor between abelian categories. A collection R of objects is called **adapted to** F if and only if:

1. R is closed under $\oplus$.

2. F maps exact objects in $Cplx^{\pm}(R)$ to exact objects in $Cplx(B)$.

3. Any object of A is a $\begin{cases} \text{sub-} \\ \text{quotient} \end{cases}$ object of some object of R (i.e., A has enough R).

Example If F is exact, then $Ob(A)$ is adapted to F.

Regard R as a full subcategory of A. By property 1, R is an additive subcategory. We get a category $K^{\pm}(R)$ and an inclusion functor $i : K^{\pm}(R) \to K^{\pm}(A)$. Let $S_R = \{f \in Mor(K^{\pm}(R)) : i(f) \in Mor(K^{\pm}(A))$ is a quasi-isomorphism$\}$

Theorem

1. S_R is a localizing class.

2. The induced functor $K^{\pm}(R)[S_R^{-1}] \to D^{\pm}(A)$ is an equivalence.

3. The derived functor $\begin{cases} RF \\ LF \end{cases} : D^{\pm}(A) \to D^{\pm}(B)$ exists, and is constructed by:

   - Choose an equivalence $D^{\pm}(A) \to K^{\pm}(R)[S_R^{-1}]$.
   - Apply $K(F)$ "term-by-term" to get $K^{\pm}(R)[S_R^{-1}] \to K^{\pm}(B)$.
   - Follow by $K^{\pm}(B) \to D^{\pm}(B)$.

If A has enough $\begin{cases} \text{injectives} \\ \text{projectives} \end{cases}$, then $R = \{\text{all} \begin{cases} \text{injectives} \\ \text{projectives} \end{cases}\}$ is adapted to any additive functor $F : A \to B$.

So much for abstract nonsense.

Definitions and theorems of 6 operations functors and l-adic cohomology

Let S be a noetherian separated regular scheme of dimension at most 1 (most of the time, $S = \text{Spec(field)}$). Let (Sch/S) be the category of S-schemes that are **separated**, and of finite type over S. Morphisms are S-morphisms. Let l be a prime in $\Gamma(S, \mathcal{O}_S^{\times})$ (i.e., invertible over S). Let $\mathcal{O}_\lambda$ be the ring of integers in some finite extension E_λ of $\mathbb{Q}_l$, with $\lambda \in \mathcal{O}_\lambda$ the uniformizer. Set $R = \mathcal{O}_\lambda/\lambda^n$ for some $n > 0$.

For each $X \in (Sch/S)$, we get $X_{ét}$ the small étale site, the abelian category $Shv(X_{ét}, R)$ which has enough injectives (and enough projectives if and only if dim $X = 0$), and a triangulated category $D(X_{ét}, R) := D(Shv(X_{ét}, R))$.

Short-term goal We want to define a triangulated subcategory $D^b_{ctf}(X_{\acute{e}t}, R) \subset D(X_{\acute{e}t}, R)$. $b =$ "bounded complexes". $c =$ "constructible cohomology". $tf =$ "of finite Tor-dimension". The last requirement is necessary for $- \overset{L}{\otimes} -$ to work nicely, and for traces to be well-defined.

We start with

$$\begin{array}{ccccc} Shv(X_{\acute{e}t}, R) & \times & Shv(X_{\acute{e}t}, R) & \to & Shv(X_{\acute{e}t}, R) \\ \mathcal{F} & & \mathcal{G} & \mapsto & \mathcal{F} \otimes \mathcal{G} \end{array}$$

where $\mathcal{F} \otimes \mathcal{G}$ is the sheaf associated to the presheaf $X_{\acute{e}t} \to (R - mod)$ given by $U \mapsto \mathcal{F}(U) \otimes \mathcal{G}(U)$.

Definition A sheaf $\mathcal{G} \in Shv(X_{\acute{e}t}, R)$ is **flat** if and only if for any short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

in $Shv(X_{\acute{e}t}, R)$, the sequence

$$0 \to \mathcal{F}' \otimes \mathcal{G} \to \mathcal{F} \otimes \mathcal{G} \to \mathcal{F}'' \otimes \mathcal{G} \to 0$$

is exact.

Proposition The collection of **flat sheaves** is adapted to $- \overset{L}{\otimes} -$.

We get a left derived functor $D^-(X_{\acute{e}t}, R) \times D^-(X_{\acute{e}t}, R) \xrightarrow{- \overset{L}{\otimes} -} D^-(X_{\acute{e}t}, R)$.

Definition An object $K \in D^-(X_{\acute{e}t}, R)$ is of **finite Tor-dimension** if and only if there exists $n \in \mathbb{Z}$ such that for any $\mathcal{F} \in Shv(X_{\acute{e}t}, R)$ and any $i < n$, $H^i(\mathcal{F} \overset{L}{\otimes} K) = 0$.

Proposition-Definition Let $K \in D^-(X_{\acute{e}t}, R)$. The following are equivalent:

- There exists $L \in D^-(X_{\acute{e}t}, R)$ such that $L^i = 0$ for all $|i| \gg 0$, L^i is **flat** and **constructible** for all $i \in \mathbb{Z}$, and $K \cong L$ in $D^-(X_{\acute{e}t}, R)$ (L is then called a **strictly perfect complex**, and K is **perfect**).
- K is of finite Tor-dimension, and $H^i(K) \in Shv(X_{\acute{e}t}, R)$ are constructible for all $i \in \mathbb{Z}$.

Let $D^b_{ctf}(X_{\acute{e}t}, R) \subset D^-(X_{\acute{e}t}, R)$ be the full subcategory whose objects are perfect complexes. This is a triangulated subcategory.

Move to 6 operations.

One $- \overset{L}{\otimes} -$.

Theorem The derived functor $D^-(X_{ét}, R) \times D^-(X_{ét}, R) \to D^-(X_{ét}, R)$ induces $D^b_{ctf}(X_{ét}, R) \times D^b_{ctf}(X_{ét}, R) \to D^b_{ctf}(X_{ét}, R)$.

Two $R\underline{Hom}$. Start from

$$\begin{array}{ccccc} Shv(X_{ét}, R)^{op} & \times & Shv(X_{ét}, R) & \to & Shv(X_{ét}, R) \\ \mathcal{F} & & \mathcal{G} & \mapsto & \underline{Hom}(\mathcal{F}, \mathcal{G}) \end{array}$$

$\underline{Hom}(\mathcal{F}, \mathcal{G})$ is actually a sheaf.

Lemma Fix $\mathcal{F} \in Shv(X_{ét}, R)$. Then $\underline{Hom}(\mathcal{F}, -)$ is **left** exact.

We get a derived functor $R\underline{Hom}(\mathcal{F}, -) : D^+(X_{ét}, R) \to D^+(X_{ét}, R)$. If we fix $\mathcal{L} \in D^+(X_{ét}, R)$, then we get a commutative diagram of induced functors:

$$\begin{array}{ccc} Shv(X_{ét}, R)^{op} & \xrightarrow{R\underline{Hom}(-,\mathcal{L})} & D(X_{ét}, R) \\ \searrow & & \nearrow \\ K(X_{ét}, R)^{op} & \longrightarrow & D(X_{ét}, R)^{op} \end{array}$$

We have a "balanced theory".

We get a bifunctor $D(X_{ét}, R)^{op} \times D(X_{ét}, R) \overset{R\underline{Hom}(-,-)}{\longrightarrow} D(X_{ét}, R)$.

Theorem This induces a bifunctor $D^b_{ctf}(X_{ét}, R)^{op} \times D^b_{ctf}(X_{ét}, R) \to D^b_{ctf}(X_{ét}, R)$.

(Hard). The fact that inclusion is fully faithful requires a third description of $D(X_{ét}, R)$, by killing the cones of quasi-isomorphisms.

Now, let $f : X \to Y$ be in $Mor(Sch/S)$.

Three f^*, "inverse image". Start with

$$\begin{array}{ccc} Shv(Y_{ét}, R) & \to & Shv(X_{ét}, R) \\ \mathcal{F} & \mapsto & f^*\mathcal{F} \end{array}$$

where $f^*\mathcal{F}$ is the sheaf associated to the presheaf $X_{ét} \to (R - mod)$ given by $U \mapsto \varinjlim_{(V,a,b)} \mathcal{F}(V)$, with the limit taked over all diagrams

$$\begin{array}{ccc} U & \xrightarrow{a} & V \\ {\scriptstyle ét}\downarrow & & \downarrow{\scriptstyle ét} \\ X & \xrightarrow{f} & Y \end{array}$$

Proposition f^* is exact.

Hence, we get $D(Y_{ét}, R) \overset{f^*}{\to} D(X_{ét}, R)$. There is no L on f^*, since no derivation takes place. [Contrast this with the situation for coherent sheaves - there, the inverse image is defined with a tensor product, so it is not exact.]

Theorem By Dévissage, this induces $D^b_{ctf}(Y_{ét}, R) \xrightarrow{f^*} D^b_{ctf}(X_{ét}, R)$.

Four Rf_*, "direct image". Start from

$$\begin{array}{ccc} Shv(X_{ét}, R) & \to & Shv(Y_{ét}, R) \\ \mathcal{F} & \mapsto & f_*\mathcal{F} \end{array}$$

where $f^*\mathcal{F}$ is the sheaf (not just a presheaf) $Y_{ét} \to (R - mod)$ given by $V \mapsto \mathcal{F}(f^{-1}(V) = X \times_{f,Y} V)$.

Lemma f_* is left exact.

So we get a right derived functor $D^+(X_{ét}, R) \xrightarrow{Rf_*} D^+(Y_{ét}, R)$.

Theorem This induces $D^b_{ctf}(X_{ét}, R) \xrightarrow{Rf_*} D^b_{ctf}(Y_{ét}, R)$.

(Very hard. This is the hardest theorem in SGA 4. First prove for f proper, use proper base change theorem. It's proved in full generality in SGA 4.5)

Example Let $a : X \to \mathrm{Spec}(\mathbb{C})$ be the structure map of a complex variety, and R the constant sheaf of a commutative ring on $\mathrm{Spec}(\mathbb{C})$. Then:

$$\begin{aligned} H^i(X^{an}, R) &\cong H^i(Rf_*f^*R) \\ H_i(X^{an}, R) &\cong H^{-i}(Rf_!f^!R) \\ H^i_c(X^{an}, R) &\cong H^i(Rf_!f^*R) \\ H^{BM}_i(X^{an}, R) &\cong H^{-i}(Rf_*f^!R) \end{aligned}$$

Thus, we can use this framework to describe ordinary homology and cohomology, along with cohomology with compact supports, and Borel-Moore homology.

February 25, 2003

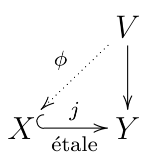
Last time, we defined $- \overset{L}{\otimes} -$, $R\underline{Hom}(-,-)$, f^*, and Rf_*.

Five $Rf_!$, "direct image with proper support". First, assume $f = j$ is étale (especially an open immersion). Define $j_! : Shv(X_{ét}, R) \to Shv(Y_{ét}, R)$ (extension by zero) as the left adjoint functor to $j^* : Shv(Y_{ét}, R) \to Shv(X_{ét}, R)$.

Proposition This exists: for $\mathcal{F} \in Shv(X_{ét}, R)$, $j_!\mathcal{F} \in Shv(X_{ét}, R)$ is the sheaf associated to the presheaf:

$$\begin{array}{ccc} Y_{ét} & \to & (R - mod) \\ V & \mapsto & \bigoplus_{\phi \in Hom_Y(V,X)} \mathcal{F}(V \xrightarrow{\phi} X) \end{array}$$

where ϕ varies over all possible ways of factoring:



$j_!$ is **exact**, so we get (term-by-term) a functor $j_! : D(X_{ét}, R) \to D(Y_{ét}, R)$ for $X \xrightarrow{j} Y$ étale.

Theorem (Nagata) Let $X \xrightarrow{f} Y$ be a morphism that is separated and of finite type, and let Y be noetherian. Then there exists a proper Y-scheme $\overline{X} \xrightarrow{\overline{f}} Y$ and an open immersion $X \overset{j}{\hookrightarrow} \overline{X}$ such that $f = \overline{f}j$.

Basically, any separated finite type morphism can be compatified. Obviously, $\overline{X}$ is not unique, for example, we could blow up a point in $\overline{X} \setminus j(X)$.

Given a morphism $f : X \to Y$ in Sch/S, apply Nagata's theorem, and get j and $\overline{f}$. Define $Rf_! : D^+(X_{ét}, R) \to D^+(Y_{ét}, R)$ as the composition

$$D^+(X_{ét}, R) \xrightarrow{j_!} D^+(\overline{X}_{ét}, R) \xrightarrow{Rf_*} D^+(Y_{ét}, R)$$

Proposition [SGA 4, Exp XVII, 7.3.5] The functor $Rf_!$ is well-defined, and independent of choice of j and $\overline{f}$ (up to canonical isomorphism). (We actually pass to a limit over all compactifications.) [doesn't seem necessary]

Theorem $Rf_!$ preserves D^b_{ctf}, i.e., it induces $D^b_{ctf}(X_{ét}, R) \xrightarrow{Rf_!} D^b_{ctf}(X_{ét}, R)$.

This actually needs to be proved to show that Rf_* preserves D^b_{ctf}.

Six $f^!$, "extraordinary inverse image".

Theorem Let $f : X \to Y$ be a morphism in (Sch/S). The functor $Rf_!$ admits a **right** adjoint functor $D^+(Y_{ét}, R) \xrightarrow{f^!} D^+(X_{ét}, R)$, i.e., for any $K \in D^+(X_{ét}, R)$ and $L \in D^+(Y_{ét}, R)$, $Hom_{D^+(Y_{ét},R)}(Rf_!K, L) \cong Hom_{D^+(X_{ét},R)}(K, f^!L)$.

This is hard to prove, and it **only** holds on the level of derived categories, **not** for sheaves.

Theorem This induces $D^b_{ctf}(Y_{ét}, R) \xrightarrow{f^!} D^b_{ctf}(X_{ét}, R)$

The proof is by dévissage.

Constant and dualizing sheaves

Define $R_S = D_S :=$ constant sheaf R on $S_{ét} \in Shv(S_{ét}, R) \underset{deg\,0}{\hookrightarrow} D^b_{ctf}(S_{ét}, R)$.

For $X \xrightarrow{a_X} S$ in (Sch/S), define $R_X := a_X^* R_S$ and $D_X := a_X^! D_S$.

Theorem (Main theorem of SGA 4 and $\frac{1}{3}$ of SGA $4\frac{1}{2}$) The formalism of 6 operations holds for $\{D^b_{ctf}(X_{ét}, R), X \in (Sch/S)\}$.

Dualizing For $X \in (Sch/S)$, define

$$\begin{array}{rcc} \mathbb{D}_X : D^b_{ctf}(X_{\acute{e}t}, R) & \stackrel{\text{contravariant}}{\to} & D^b_{ctf}(X_{\acute{e}t}, R) \\ - & \mapsto & R\underline{Hom}(-, D_X) \end{array}$$

We get $id \to \mathbb{D}_X \circ \mathbb{D}_X$, and this is an isomorphism of functors.

Note Poincaré duality pops out of this formalism. The trace formula **does not**.

$\mathcal{O}_\lambda$-sheaves

Recall that $\mathcal{O}_\lambda$ is the ring of integers of a finite extension E_λ of $\mathbb{Q}_l$, with λ a uniformizer. For X a noetherian ($\Rightarrow$ quasi-compact, quasi-separated) scheme, we get the $\mathcal{O}_\lambda$-linear abelian category $Pro(Shv(X_{\acute{e}t}, \mathcal{O}_\lambda))$: Objects are projective systems $(\mathcal{F}_n)_{n\geq 1} \in Shv(X_{\acute{e}t}, \mathcal{O}_\lambda)$ of sheaves of $\mathcal{O}_\lambda$-modules on $X_{\acute{e}t}$, and morphisms are given by:

$$Hom_{pro}((\mathcal{F}_n), (\mathcal{G}_m)) := \varprojlim_m \varinjlim_n Hom_{Shv(X_{\acute{e}t}, \mathcal{O}_\lambda)}(\mathcal{F}_n, \mathcal{G}_m)$$

["Infinity for me is on the right side"]

Definition A **standard** $\mathcal{O}_\lambda$**-sheaf** on X is an object $(\mathcal{F}_n)_{n\geq 1}$ in $Pro(Shv(X_{\acute{e}t}, \mathcal{O}_\lambda))$, such that:

1. For any $n \geq 1$, $\mathcal{F}_n$ is killed by λ^n (i.e., $\mathcal{F}_n$ "is" an $\mathcal{O}_\lambda/\lambda^n$-sheaf).
2. For any $n \geq 1$, $\mathcal{F}_n$ is constructible (i.e., $\mathcal{F}_n \in Shv_c(X_{\acute{e}t}, \mathcal{O}_\lambda)$).
3. For any $n \geq 1$, the transition morphism $\mathcal{F}_{n+1} \to \mathcal{F}_n$ in $Shv(X_{\acute{e}t}, \mathcal{O}_\lambda)$ induces an isomorphism $\mathcal{F}_{n+1} \otimes_{\mathcal{O}_\lambda/\lambda^{n+1}} \mathcal{O}_\lambda/\lambda^n \to \mathcal{F}_n$.

Definition The category $Shv_c(X, \mathcal{O}_\lambda)$ of $\mathcal{O}_\lambda$**-sheaves on** X is the full subcategory of $Pro(Shv(X_{\acute{e}t}, \mathcal{O}_\lambda))$ consisting of objects isomorphic to a standard $\mathcal{O}_\lambda$-sheaf on X.

[Note the subtle difference between $Shv_c(X, \mathcal{O}_\lambda)$, which was just defined, and $Shv_c(X_{\acute{e}t}, \mathcal{O}_\lambda)$, which was defined earlier in terms of the étale site on X.]

We get a functor $Shv_c(X, \mathcal{O}_\lambda) \stackrel{-\otimes\mathcal{O}_\lambda/\lambda^n}{\longrightarrow} Shv_c(X_{\acute{e}t}, \mathcal{O}_\lambda/\lambda^n)$.

Definition An $\mathcal{O}_\lambda$-sheaf is **lisse** if and only if for any $n \geq 1$, $\mathcal{F} \otimes \mathcal{O}_\lambda/\lambda^n$ is lisse (here, equivalent to locally constant).

Proposition

1. $Shv_c(X, \mathcal{O}_\lambda)$ is an $\mathcal{O}_\lambda$-linear abelian category. It is noetherian (i.e., increasing sequences of subobjects stabilize). We can't do good cohomology with it, as it doesn't have enough of anything.

2. Let X be a connected normal scheme, and $\overline{\eta} \to X$ a geometric point. Then the functor

$$\begin{array}{ccc} \{\text{lisse } \mathcal{O}_\lambda\text{-sheaves on } X\} & \to & \left\{ \begin{array}{c} \text{finitely generated } \mathcal{O}_\lambda\text{-modules} \\ \text{with continuous } \pi_1\text{-action} \end{array} \right\} \\ \mathcal{F} & \mapsto & \mathcal{F}_{\overline{\eta}} := \varinjlim_{n \geq 1} (\mathcal{F} \otimes \mathcal{O}_\lambda/\lambda^n)_{\overline{\eta}} \end{array}$$

is an **equivalence** of categories.

3. If $\mathcal{F} \in Shv_c(X, \mathcal{O}_\lambda)$ is an $\mathcal{O}_\lambda$-sheaf, then there exists a finite partition $X = \coprod_{i \in I} X_i$, where each X_i is locally closed, such that $\mathcal{F}|_{X_i}$ is lisse on X_i.

Let X be a complete nonsinguular curve over $\mathbb{F}_q$, and let $\eta \overset{j}{\hookrightarrow} X$ be the generic point. We get a diagram:

$$\begin{array}{ccc} LisseShv_c(\eta, \mathcal{O}_\lambda) & \xrightarrow{j_*} & Shv(X_{ét}, \mathcal{O}_\lambda) \\ & \searrow & \uparrow \\ & & Shv_c(X, \mathcal{O}_\lambda) \end{array}$$

Note that the top left category is equivalent to that of continuous $\mathcal{O}_\lambda$-representations of $Gal(\overline{\eta}/\eta)$, and the diagonal map doesn't exist unless the Galois representation of a given sheaf is unramified almost everywhere.

(How to make a Galois representation almost everywhere ramified? Take

$$\begin{pmatrix} \boldsymbol{\mu} & * \\ 0 & 1 \end{pmatrix},$$

and pass to the limit.) ??

E_λ-sheaves

Definition $Shv_c(X, E_\lambda)$ is the quotient of $Shv_c(X, \mathcal{O}_\lambda)$ by the thick full subcategory of torsion objects in $Shv_c(X, \mathcal{O}_\lambda)$. This means:

1. We have a functor
$$\begin{array}{ccc} Shv_c(X, \mathcal{O}_\lambda) & \to & Shv_c(X, E_\lambda) \\ \mathcal{F} & \mapsto & \mathcal{F} \otimes E_\lambda \end{array}$$

2. $Hom_{Shv_c(X,E_\lambda)}(\mathcal{F} \otimes E_\lambda, \mathcal{G} \otimes E_\lambda) = Hom_{Shv_c(X,\mathcal{O}_\lambda)}(\mathcal{F}, \mathcal{G}) \otimes_{\mathcal{O}_\lambda} E_\lambda$. This kills torsion.

Propositon The analogues of statements 1, 2, and 3 from the previous proposition hold.

$\overline{\mathbb{Q}_l}$-sheaves

Let $\overline{\mathbb{Q}_l}$ be an algebraic closure of $\mathbb{Q}_l$. For $\mathbb{Q}_l \underset{\text{finite}}{\subset} E_\lambda \underset{\text{finite}}{\subset} E_\lambda{}' \subset \overline{\mathbb{Q}_l}$ we get a functor

$$\begin{array}{ccc} Shv_c(X, E_\lambda) & \to & Shv_c(X, E_\lambda{}') \\ \mathcal{F} & \mapsto & \mathcal{F} \otimes E_\lambda{}' \end{array}$$

with $Hom_{Shv_c(X,E_\lambda{}')}(\mathcal{F} \otimes E_\lambda{}', \mathcal{G} \otimes E_\lambda{}') = Hom_{Shv_c(X,E_\lambda)}(\mathcal{F}, \mathcal{G}) \otimes_{E_\lambda} E_\lambda{}'$.

Definition $Shv_c(X, \overline{\mathbb{Q}_l}) := \underset{\mathbb{Q}_l \underset{\text{finite}}{\subset} E_\lambda \subset \overline{\mathbb{Q}_l}}{\text{2-}\varinjlim} Shv_c(X, E_\lambda)$

This means:

1. We have a functor
$$\begin{array}{ccc} Shv_c(X, E_\lambda) & \to & Shv_c(X, \overline{\mathbb{Q}_l}) \\ \mathcal{F} & \mapsto & \mathcal{F} \otimes \overline{\mathbb{Q}_l} \end{array}$$
   Which induces isomorphisms
$$Hom_{Shv_c(X,\overline{\mathbb{Q}_l})}(\mathcal{F} \otimes \overline{\mathbb{Q}_l}, \mathcal{G} \otimes \overline{\mathbb{Q}_l}) \xleftarrow{\sim} Hom_{Shv_c(X,E_\lambda)}(\mathcal{F}, \mathcal{G}) \otimes_{E_\lambda} \overline{\mathbb{Q}_l}.$$

2. We have a canonical isomorphism of functors:
$$- \otimes \overline{\mathbb{Q}_l} \cong (- \otimes E_\lambda{}') \otimes \overline{\mathbb{Q}_l}$$

3. Every object in $Shv_c(X, \overline{\mathbb{Q}_l})$ is of the form $\mathcal{F} \otimes \overline{\mathbb{Q}_l}$ for some $\mathcal{F} \in Shv_c(X, E_\lambda)$.

We need the 2-structure to get a representing category. Otherwise, the transition functors don't match up.

Theorem (Main theorem of SGA 5 + $\epsilon \cdot$ Weil II + Ekedahl [reference]) For each $X \in (Sch/S)$, there exist:

- A triangulated category $D_c^b(X, \mathcal{O}_\lambda)$ with t-structure $(D_c^{\leq 0}, D_c^{\geq 0})$. ["the usual one - not perverse"]
- Triangulated functors $D_c^b(X, \mathcal{O}_\lambda) \xrightarrow{-\otimes \mathcal{O}_\lambda/\lambda^n} D_{ctf}^b(X, \mathcal{O}_\lambda/\lambda^n)$ for all $n \geq 1$.
- Canonical objects $R_X, D_X \in D_c^b(X, \mathcal{O}_\lambda)$.
- Triangulated functors
  - $D_c^b(X, \mathcal{O}_\lambda) \times D_c^b(X, \mathcal{O}_\lambda) \xrightarrow{-\overset{L}{\otimes}-} D_c^b(X, \mathcal{O}_\lambda)$
  - $D_c^b(X, \mathcal{O}_\lambda)^{op} \times D_c^b(X, \mathcal{O}_\lambda) \xrightarrow{R\underline{Hom}(-,-)} D_c^b(X, \mathcal{O}_\lambda)$

For each morphism $f : X \to Y$, there exist:

- Triangulated functors
  - $D^b_c(Y, \mathcal{O}_\lambda) \xrightarrow{f^*} D^b_c(X, \mathcal{O}_\lambda)$
  - $D^b_c(X, \mathcal{O}_\lambda) \xrightarrow{Rf_*} D^b_c(Y, \mathcal{O}_\lambda)$
  - $D^b_c(X, \mathcal{O}_\lambda) \xrightarrow{Rf_!} D^b_c(Y, \mathcal{O}_\lambda)$
  - $D^b_c(Y, \mathcal{O}_\lambda) \xrightarrow{f^!} D^b_c(X, \mathcal{O}_\lambda)$

such that

1. The **core** of the t-structure $(D^{\le 0}_c, D^{\ge 0}_c)$ is equivalent to $Shv_c(X, \mathcal{O}_\lambda)$, i.e., there exist functors $Shv_c(X, \mathcal{O}_\lambda) \overset{\deg 0}{\hookrightarrow} D^b_c(X, \mathcal{O}_\lambda) \xrightarrow{H^0} Shv_c(X, \mathcal{O}_\lambda)$ composing to autoequivalence.
2. Each object of $D^b_c(X, \mathcal{O}_\lambda)$ has finite cohomological amplitude, i.e., $H^i(K) = 0$ for $|i| \gg 0$.
3. The family of functors $D^b_c(X, \mathcal{O}_\lambda) \xrightarrow{-\otimes \mathcal{O}_\lambda/\lambda^n} D^b_{ctf}(X, \mathcal{O}_\lambda/\lambda^n)$ is **conservative**, i.e., objects are isomorphic in the source if and only if they are isomorphic in the target.
4. The functors $-\overset{L}{\otimes}-$, $R\underline{Hom}(-,-)$, f^*, Rf_*, $f^!$, and $Rf_!$ and the objects R_X and D_X "commute" with $- \otimes \mathcal{O}_\lambda/\lambda^n$.
5. The formalism of 6 operations holds for $\{D^b_c(X, \mathcal{O}_\lambda)\}$.

Note Beilinson showed that this triangulated category is in fact **equivalent** to the derived category of the abelian category of perverse $\overline{\mathbb{Q}_l}$-sheaves. [reference]

(local monodromy theorem: finite index subgroup of inertia factors through a unipotent.)

March 4, 2003

Function-sheaf correspondence.

Let S be a base scheme. We get a category (Sch/S), and for each object X, we defined $D^b_c(X, \mathcal{O}_\lambda)$, $D^b_c(X, E_\lambda)$, and $D^b_c(X, \overline{\mathbb{Q}_l})$. The 6 operations work for these. (Deligne's contribution in Weil II: defined a subcategory of **mixed** complexes, for which the 6 operations hold. These have control over the behavior at archimedean valuations. Lafforgue defined an even smaller category, of **plain mixed** complexes, which also have control over non-archimedean valuations. If you want to approach a category of mixed motives, you can either try to get smaller and smaller categories satisfying the 6 operations, or you can try to construct a category from scratch and hope that it works. The problem with the second approach is that it might not work, and then you've wasted a few years of your life. Voevodsky constructed the right category for X zero dimensional.)

The vanishing cycles formalism is **not** a consequence of 6 operations. You need an additional stability hypothesis.

Supplementary results

From the definition, for $U \overset{j}{\hookrightarrow} X$ étale, we get a morphism of functors $j_! \to j_*$: $Shv(U_{ét}, R) \to Shv(X_{ét}, R)$. In general, for $X \overset{f}{\to} Y$ in (Sch/S), we get another morphism of functors $Rf_! \to Rf_* : D^b_c(X_{ét}, R) \to D^b_c(Y_{ét}, R)$, the "forget support" map.

Proposition If $f : X \to Y$ is proper, then $Rf_! \to Rf_*$ is an isomorphism.

Theorem (Dévissage) For $U \underset{\text{open}}{\overset{j}{\hookrightarrow}} X \underset{\text{closed}}{\overset{i}{\hookleftarrow}} Z$ a complementary pair, we get distinguished triangles:

$$j_!j^! \to id \to i_*i^* \to [1]$$
$$i_!i^! \to id \to (Rj_*)j^* \to [1]$$

in $D^b_c(X, \overline{\mathbb{Q}_l})$. These are dual, as duality exchanges $*$ with !. The Rs are omitted when the functors are exact.

Theorem (Poincaré duality) If $f : X \to Y$ is smooth of relative dimension d, then $f^! = f^*[2d](d)$.

Definition The **Tate twist** is given by:

$$\mathbb{Z}/l^n(1) = \boldsymbol{\mu}_{l^n} \text{ in } Shv(X_{ét}, \mathbb{Z}/l^n)$$
$$\mathbb{Z}_l(1) = (\boldsymbol{\mu}_{l^n})_{n\geq 1} \text{ with } l\text{th power transitions in } Shv(X, \mathbb{Z}_l)$$
$$\mathbb{Q}_l(1), \overline{\mathbb{Q}_l}(1) \text{ defined by tensor products}$$
$$\mathcal{F}(d) := \begin{cases} \mathcal{F} \otimes_{\overline{\mathbb{Q}_l}} \overline{\mathbb{Q}_l}(1)^{\otimes d} & d \geq 1 \\ \mathcal{F} \otimes_{\overline{\mathbb{Q}_l}} (\overline{\mathbb{Q}_l}(1)^{\vee})^{\otimes(-d)} & d < 0 \end{cases}$$

Theorem (Duality in dimension 1) Suppose X is noetherian, separated, regular, and of pure absolute dimension 1. Let $U \overset{j}{\hookrightarrow} X$ be an open dense immersion, and let $\mathcal{F} \in Shv_c(U, \overline{\mathbb{Q}_l})$ be a **lisse** sheaf on U. Then

$$\mathbb{D}_X(j_*\mathcal{F}) = j_*(\mathbb{D}_U\mathcal{F} = \mathcal{F}^{\vee}).$$

The $j_*\mathcal{F}$ lies in $Shv(X, \overline{\mathbb{Q}_l})$, which is included in $D_c^b(X, \overline{\mathbb{Q}_l})$ by the degree 0 map. The $\mathcal{F}^\vee$ is the lisse sheaf dual, which corresponds to the contragradient representation of π_1, and the j_* on the right is **not** Rj_*, but we take the sheaf j_* and view it in $D_c^b(X, \overline{\mathbb{Q}_l})$.

This is what will give us the functional equation on a curve.

Theorem (Transcendental comparison) Assume $R \in \{\mathcal{O}_\lambda/\lambda^n, \mathcal{O}_\lambda, E_\lambda, \overline{\mathbb{Q}_l}\}$, and $S = \mathrm{Spec}(\mathbb{C})$. For $X \overset{a_X}{\to} S = \mathrm{Spec}(\mathbb{C})$, $p \in \mathbb{Z}$, **define**:

$$\begin{aligned} H^p(X, R) &\cong H^p(R(a_X)_* a_X^* R_S) \\ H_p(X, R) &\cong H^{-p}(R(a_X)_! a_X^! R_S) \\ H_c^p(X, R) &\cong H^p(R(a_X)_! a_X^* R_S) \\ H_p^{BM}(X, R) &\cong H^{-p}(R(a_X)_* a_X^! R_S) \end{aligned}$$

These are canonically isomorphic to the classical topological invariants of the analytic space $X^{an} = X(\mathbb{C})$. There exists a map $\phi : X^{an} \to X$ of locally ringed spaces (this requires GAGA comparisons) such that for $\mathcal{F}$ a sheaf on X, we get $H^p(X, \mathcal{F}) \overset{\sim}{\to} H^p(X^{an}, \phi^*\mathcal{F})$, i.e., ϕ is cohomologically trivial. (first for finite coefficient sheaves $\mathcal{F}$.)

Function-sheaf correspondence "A nice function on a scheme is given by a sheaf." Let $\mathbb{F}_q$ be a finite field of characteristic p, and let $\overline{\mathbb{F}_q}$ be an algebraic closure. We get a topological isomorphism:

$$\begin{aligned} &\widehat{\mathbb{Z}} \to Gal(\overline{\mathbb{F}_q}/\mathbb{F}_q) \\ &1 \mapsto Frob_{\mathbb{F}_q} = \text{geometric Frobenius} \end{aligned}$$

Choose $l \neq p$, and fix an algebraic closure $\overline{\mathbb{Q}_l}$ of $\mathbb{Q}_l$. Let $(Sch/\mathbb{F}_q)$ be the category of separated, finite type schemes over $\mathbb{F}_q$, with $\mathbb{F}_q$-morphisms. For each $X \in (Sch/\mathbb{F}_q)$, we get categories $Shv_c(X, \overline{\mathbb{Q}_l}) \subset D_c^b(X, \overline{\mathbb{Q}_l})$.

We get the Grothendieck group $K(X, \overline{\mathbb{Q}_l})$ of $D_c^b(X, \overline{\mathbb{Q}_l})$: for $K \in D_c^b(X, \overline{\mathbb{Q}_l})$, let

$$[K] := \sum_i (-1)^i [H^i(K)] \in K(X, \overline{\mathbb{Q}_l})$$

We have 3 of the 6 operations:

$- \overset{L}{\otimes} - : D_c^b(X) \times D_c^b(X) \to D_c^b(X)$ is triangulated, associative, and commutative (need to have a field, or badness happens). This gives $D_c^b(X)$ the structure of an "acu $\otimes$-category" [in the sense of Saavedra Rivano] with R_X a unit object for $- \overset{L}{\otimes} -$, and gives $K(X, \overline{\mathbb{Q}_l})$ the structure of a commutative ring.

For $f : X \to Y$ in $(Sch/\mathbb{F}_q)$, we get $f^* : D_c^b(Y) \to D_c^b(x)$, satisfying the commutativity relation $f^*(L_1 \overset{L}{\otimes}_Y L_2) = f^*L_1 \overset{L}{\otimes}_X f^*L_2$. This gives a ring homomorphism $f^* : K(Y, \overline{\mathbb{Q}_l}) \to K(X, \overline{\mathbb{Q}_l})$.

For $f: X \to Y$ in $(Sch/\mathbb{F}_q)$, we get $Rf_!: D^b_c(X) \to D^b_c(Y)$, with commutativity relation $Rf_!(K \overset{L}{\otimes}_X f^*L) = (Rf_!K) \overset{L}{\otimes}_Y L$. This gives an additive group homomorphism $f_!: K(X, \overline{\mathbb{Q}_l}) \to K(Y, \overline{\mathbb{Q}_l})$ whose image is an ideal in $K(Y, \overline{\mathbb{Q}_l})$.

For each $X \in (Sch/\mathbb{F}_q)$, we get $Maps(X(\mathbb{F}_q), \overline{\mathbb{Q}_l}) :=$ the $\overline{\mathbb{Q}_l}$-algebra of all maps $X(\mathbb{F}_q) \to \overline{\mathbb{Q}_l}$ under pointwise addition and multiplication.

For $f: X \to Y$ in $(Sch/\mathbb{F}_q)$ we get a $\overline{\mathbb{Q}_l}$-algebra homomorphism:

$$\begin{array}{rcl} f^*: Maps(Y(\mathbb{F}_q), \overline{\mathbb{Q}_l}) & \to & Maps(X(\mathbb{F}_q), \overline{\mathbb{Q}_l}) \\ \phi & \mapsto & (x \mapsto \phi(f(x))) \end{array}$$

If X is an Artin motive (i.e., a product of field extensions of $\mathbb{F}_q$), then this is its $\overline{\mathbb{Q}_l}$-realization.

For $f: X \to Y$ in $(Sch/\mathbb{F}_q)$, we get a $\overline{\mathbb{Q}_l}$-vector space homomorphism:

$$\begin{array}{rcl} f_!: Maps(X(\mathbb{F}_q), \overline{\mathbb{Q}_l}) & \to & Maps(Y(\mathbb{F}_q), \overline{\mathbb{Q}_l}) \\ \phi & \mapsto & (y \mapsto \sum_{x \in X(\mathbb{F}_q), f(x)=y} \phi(x)) \end{array}$$

whose image is an **ideal** in $Maps(Y(\mathbb{F}_q), \overline{\mathbb{Q}_l})$.

Definition (Function-sheaf correspondence) For each $X \in (Sch/\mathbb{F}_q)$ and each $\mathcal{F} \in Shv_c(X, \overline{\mathbb{Q}_l})$, define $\langle \mathcal{F} \rangle \in Maps(X(\mathbb{F}_q), \overline{\mathbb{Q}_l})$ by:

$$\begin{array}{rcll} \langle \mathcal{F} \rangle : X(\mathbb{F}_q) & \to & \overline{\mathbb{Q}_l} & \\ x & \mapsto & \langle \mathcal{F} \rangle(x) & := Tr(Frob_x; \mathcal{F}) \\ & & & = Tr(Frob_{\mathbb{F}_q} \in Gal(\overline{\mathbb{F}_q}/\mathbb{F}_q); x^*\mathcal{F}) \end{array}$$

Properties

The correspondence induces a homomorphism of abelian groups:

$$\begin{array}{rcl} K(X, \overline{\mathbb{Q}_l}) & \to & Maps(X(\mathbb{F}_q), \overline{\mathbb{Q}_l}) \\ [K] & \mapsto & \langle K \rangle \end{array}$$

and the map $K(X, \overline{\mathbb{Q}_l}) \to \prod_n Maps(X(\mathbb{F}_{q^n}), \overline{\mathbb{Q}_l})$ is **injective**. For $K' \to K \to K'' \to [1]$ a distinguished triangle in $D^b_c(X)$, we get $\langle K \rangle = \langle K' \rangle + \langle K'' \rangle$. In particular, $\langle K \rangle = \sum_i (-1)^i \langle H^i(K) \rangle$.

For $K_1, K_2 \in D^b_c(X)$, $\langle K_1 \overset{L}{\otimes} K_2 \rangle = \langle K_1 \rangle \cdot \langle K_2 \rangle$ in $Maps(X(\mathbb{F}_q), \overline{\mathbb{Q}_l})$, i.e., the additive map $K(X, \overline{\mathbb{Q}_l}) \to Maps(X(\mathbb{F}_q), \overline{\mathbb{Q}_l})$ is a ring homomorphism.

For $f: X \to Y$ in $(Sch/\mathbb{F}_q)$ and $L \in D^b_c(Y)$, $\langle f^*L \rangle = f^*\langle L \rangle$ in $Maps(X(\mathbb{F}_q), \overline{\mathbb{Q}_l})$, i.e., we have a commutative square of ring homomorphisms:

$$\begin{array}{ccc} K(X, \overline{\mathbb{Q}_l}) & \xrightarrow{\langle - \rangle} & Maps(X(\mathbb{F}_q), \overline{\mathbb{Q}_l}) \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ K(Y, \overline{\mathbb{Q}_l}) & \xrightarrow{\langle - \rangle} & Maps(Y(\mathbb{F}_q), \overline{\mathbb{Q}_l}) \end{array}$$

Theorem (Grothendieck-Lefschetz trace formula) For $f : X \to Y$ in $(Sch/\mathbb{F}_q)$ and $K \in D_c^b(X)$, one has $\langle Rf_! K\rangle = f_!\langle K\rangle$ in $Maps(Y(\mathbb{F}_q), \overline{\mathbb{Q}_l})$, i.e., we have a commutative square of abelian group homomorphisms:

$$\begin{array}{ccc} K(X,\overline{\mathbb{Q}_l}) & \xrightarrow{\langle - \rangle} & Maps(X(\mathbb{F}_q),\overline{\mathbb{Q}_l}) \\ \downarrow f_! & & \downarrow f_! \\ K(Y,\overline{\mathbb{Q}_l}) & \xrightarrow{\langle - \rangle} & Maps(Y(\mathbb{F}_q),\overline{\mathbb{Q}_l}) \end{array}$$

Example For $\mathcal{F} \in Shv_c(X, \overline{\mathbb{Q}_l})$,

$$\sum_i (-1)^i Tr(Frob_{\mathbb{F}_q} | H_c^i(X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}, \mathcal{F})) = \sum_{x \in X(\mathbb{F}_q)} Tr(Frob_x | \mathcal{F}_{\overline{x}}).$$

Special case: take $F = \overline{\mathbb{Q}_l}$ constant sheaf. Then:

$$\begin{aligned} &\sum_i (-1)^i Tr(Frob_{\mathbb{F}_q} | H_c^i(X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}, \overline{\mathbb{Q}_l})) \\ &\quad = \#X(\mathbb{F}_q) \\ &\quad = \#\{\text{fixed points of } Frob : X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q} \to X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}\} \end{aligned}$$

Lefschetz considered the case of X a manifold, with $f : X \to X$ such that its graph intersects transversely with the diagonal.

Proof uses dévissage and 6 operations. Reduce to $\mathcal{F}$ lisse, irreducible over $X/\mathbb{F}_q$ curve (Reference: Katz. Gauss Sums, Kloosterman Sums, and Monodromy.) ?

We don't have the other three operations, because no one knows what function would give a dualizing complex.

Example over $\mathbb{Q}$

Fix l, and choose $z_1, z_2, \dots \in \mathbb{Q}$. For any $n \geq 1$, define

$$E_n := \mathbb{Q}(\text{all } l^n\text{th roots of unity, and all } l^n\text{th roots of } z_1 z_2^l \dots z_n^{l^{n-1}})$$

Choose ζ_n, a primitive l^nth root of 1, and x_n, a l^nth root of $z_1 z_2^l \dots z_n^{l^{n-1}}$, such that $\zeta_n^l = \zeta_{n-1}$ and $x_n^l = x_{n-1} \cdot z_n$.

For each $\sigma \in Gal(E_n/\mathbb{Q})$, we get maps $\zeta_n \mapsto \zeta_n^{a_n(\sigma)}$ for $a_n(\sigma) \in (\mathbb{Z}/l^n)^\times$ and $x_n \mapsto \zeta_n^{b_n(\sigma)}$ for $b_n(\sigma) \in \mathbb{Z}/l^n$. This gives us a homomorphism:

$$\begin{array}{ccc} Gal(E_n/\mathbb{Q}) & \hookrightarrow & GL_2(\mathbb{Z}/l^n) \\ \sigma & \mapsto & \begin{pmatrix} a_n(\sigma) & b_n(\sigma) \\ 0 & 1 \end{pmatrix} \end{array}$$

We get a commutative diagram of (continuous) Galois representations:

$$\begin{array}{ccc} Gal(\overline{\mathbb{Q}}/\mathbb{Q}) & \xrightarrow{\rho} & GL_2(\mathbb{Z}_l) \\ \downarrow & & \downarrow \\ Gal(E_n/\mathbb{Q}) & \xhookrightarrow{\rho_n} & GL_2(\mathbb{Z}/l^n) \\ \downarrow & & \downarrow \\ Gal(E_{n-1}/\mathbb{Q}) & \xhookrightarrow{\rho_{n-1}} & GL_2(\mathbb{Z}/l^{n-1}) \end{array}$$

If we choose z_i to be the primes, they all eventually ramify.

Let $S = \text{Spec}(\mathbb{Z}[1/l])$ and let $\eta \overset{j}{\hookrightarrow} S$ be the generic point. Each ρ_n "is" a lisse sheaf $[\rho_n]$ on η. We get constructible sheaves $\mathcal{F}_n := j_*[\rho_n]$ on S. Note that the canonical map $\mathcal{F}_n \to \mathcal{F}_{n-1}$ **does not** factor through $\mathcal{F}_n \otimes_{\mathbb{Z}/l^n} \mathbb{Z}/l^{n-1}$. Let S_n be the normalization of S in E_n, and $U_n \subset S_n$ an open subscheme, étale over S. The map $S_n \to S$ is called "generically étale" since is is étale when restricted to some open subset.

We evaluate at U_{n-1}. $\mathcal{F}_{n-1}(U_{n-1} = (j_*[\rho_{n-1}])(U_{n-1}) = [\rho_{n-1}](\eta_{n-1}) =$ trivial $(\mathbb{Z}/l^{n-1})^{\oplus 2}$, as the representation is trivial here. However,

$$\mathcal{F}_n(U_{n-1}) = [\rho_n](\eta_{n-1}) = ((\mathbb{Z}/l^n)^{\oplus 2})^{\rho_n(Gal(\overline{\mathbb{Q}}/E_{n-1}))} = ((\mathbb{Z}/l^n)^{\oplus 2})^{K_n} = l^{n-1}(\mathbb{Z}/l^n)^{\oplus 2},$$

where

$$K_n := ker(\left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right\} \to \left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right\}),$$

with the first group contained in $GL_2(\mathbb{Z}/l^n)$ and the second in $GL_2(\mathbb{Z}/l^{n-1})$.

If we evaluate stalks at ramified points $x \in S_{n-1} \setminus U_{n-1}$, we find that the stalk of $j_!$ at such a point gives inertia invariants.

March 11, 2003

[Ogus: Can you characterize the image of K in $Maps$? Cheewhye: That ... is a dream. If you could do that, it would impress ... at least 10 people in the world.]

Grothendieck's theory of L- and ϵ-functions

Let $\mathbb{F}_q$ be a finite field of characteristic p, and let $\overline{\mathbb{F}_q}$ be an algebraic closure. Choose a prime $l \neq p$, so we get $\mathbb{Q}_l$ and choose $\overline{\mathbb{Q}_l}$. Let $(Sch/\mathbb{F}_q)$ be the category of (separated finite-type) schemes over $\mathbb{F}_q$, with $\mathbb{F}_q$ morphisms. For $X \in (Sch/\mathbb{F}_q)$, we get $Shv_c(X, \overline{\mathbb{Q}_l}) \underset{deg\,0}{\subset} D^b_c(X, \overline{\mathbb{Q}_l})$, and we get $K(X, \overline{\mathbb{Q}_l})$. Recall that if $K \in D^b_c(X)$, then $[K] = \sum_i (-1)^i [H^i(K)] \in K(X, \overline{\mathbb{Q}_l})$.

Definition Let $X \in (Sch/\mathbb{F}_q)$, and $\mathcal{F} \in Shv_c(X, \overline{\mathbb{Q}_l})$. For each $x \in |X|$, we get a map $\text{Spec}\kappa(x) \overset{x}{\hookrightarrow} X$ that is a closed immersion (we use the same letter for the point and the map), so we get $x^*\mathcal{F}$, a lisse sheaf on Spec$\kappa(x)$ (note that no constructible non-lisse sheaves on $\kappa(x)$ exist). This corresponds to a representation of $Gal(\overline{\kappa(x)}/\kappa(x))$ on $\mathcal{F}_{\overline{x}}$, i.e., a $\overline{\mathbb{Q}_l}$-representation. Given a choice of isomorphsim $\overline{\kappa(x)} \to \overline{\mathbb{F}_q}$, we get an inclusion $Gal(\overline{\kappa(x)}/\kappa(x)) \hookrightarrow Gal(\overline{\mathbb{F}_q}/\mathbb{F}_q)$, taking $Frob_x := Frob_{\kappa(x)}$ to $(Frob_{\mathbb{F}_q})^{deg(\kappa(x)/\mathbb{F}_q)}$. We get $det(1 - T \cdot Frob_x; \mathcal{F}) := det(1 - T \cdot Frob_x; x^*\mathcal{F}) \in 1 + T \cdot \overline{\mathbb{Q}_l}[T]$.

This is multiplicative in $\mathcal{F}$, so we get homomorphisms of abelian groups:

$$\begin{aligned} det(1 - T \cdot Frob_x; -) &: K(X, \overline{\mathbb{Q}_l}) \to 1 + T \cdot \overline{\mathbb{Q}_l}[[T]] \\ Tr(Frob_x; -) &: K(X, \overline{\mathbb{Q}_l}) \to \overline{\mathbb{Q}_l} \qquad \text{(coefficient of } (-T)) \\ det(-T \cdot Frob_x; -) &: K(X, \overline{\mathbb{Q}_l}) \to \overline{\mathbb{Q}_l}[T^{\pm 1}]^\times \qquad \begin{pmatrix} \text{highest degree term in} \\ \text{inverse char. polynomial} \end{pmatrix} \end{aligned}$$

Definition Let $K \in D^b_c(X)$. For $x \in |X|$, the **local L-factor of K at x** is:

$$L(K; T) := det(1 - T \cdot Frob_x; K)^{(-1)} \in 1 + T\overline{\mathbb{Q}_l}[[T]]$$

The **Global L-function of K** is:

$$\begin{aligned} L(X/\mathbb{F}_q, K; T) :&= \prod_{x \in |X|} L(K; T^{deg(x)}) \\ &= \prod_{x \in |X|} det(1 - T^{deg(x)} Frob_x; K)^{(-1)} \in 1 + T \cdot \overline{\mathbb{Q}_l}[[T]] \end{aligned}$$

Theorem (Grothendieck, cohomological interpretation of L-functions)
Let $(X \overset{a_X}{\to} \text{Spec}(\mathbb{F}_q)) \in (Sch/\mathbb{F}_q)$, and $K \in D^b_c(X, \overline{\mathbb{Q}_l})$. Then:

$$L(X/\mathbb{F}_q, K; T) = L(\text{Spec}(\mathbb{F}_q)/\mathbb{F}_q, R(a_X)_! K; T) \in 1 + T \cdot \overline{\mathbb{Q}_l}[[T]]$$

Corollary (Rationality of L-functions)

$$\begin{aligned} L(X/\mathbb{F}_q, K, T) &= \prod_i det(1 - T \cdot Frob_{\mathbb{F}_q}, H^i_c(X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}, K))^{(-1)^{i+1}} \\ &\in \overline{\mathbb{Q}_l}(T)^\times \cap (1 + T \cdot \overline{\mathbb{Q}_l}[[T]]) \subset \overline{\mathbb{Q}_l}((T))^\times \end{aligned}$$

Proof of Theorem

Remark If V is a finite dimensional $\overline{\mathbb{Q}_l}$ vector space and $F \in End_{\overline{\mathbb{Q}_l}}(V)$, then:

$$T \frac{d}{dT} log(det(1 - TF)^{(-1)}) = \sum_{n \geq 1} Tr(F^n) T^n \in T \cdot \overline{\mathbb{Q}_l}[[T]]$$

For the theorem, we note that

$$T\frac{d}{dT}log(\text{left hand side}) = \sum_{n\geq 1}(\sum_{x\in X(\mathbb{F}_{q^n})} Tr(Frob_x;K))T^n$$

and

$$T\frac{d}{dT}log(\text{right hand side}) = \sum_{n\geq 1}(Tr(Frob^n_{\mathbb{F}_q};R(a_X)_!K))T^n$$

By the Grothendieck-Lefschetz trace formula, the coefficients of each T^n are equal (note that we need a characteristic zero cohomology theory to get equality from equality of logarithmic derivatives).

Hence, we get a group homomorphism:

$$\begin{array}{rcl} L(X/\mathbb{F}_q,-,T): K(X,\overline{\mathbb{Q}_l}) & \to & \overline{\mathbb{Q}_l}(T)^\times \cap (1+T\cdot\overline{\mathbb{Q}_l}[[T]]) \subset \overline{\mathbb{Q}_l}((T))^\times \\ K & \mapsto & det(1-T\cdot Frob_{\mathbb{F}_q};R(a_X)_!K)^{(-1)} \end{array}$$

Definition For $\begin{array}{c} X \\ \downarrow a_X \\ \text{Spec}(\mathbb{F}_q) \end{array} \in (Sch/\mathbb{F}_q)$, and $K \in D^b_c(X,\overline{\mathbb{Q}_l})$, the **global ϵ-function of K** is:

$$\epsilon(X/\mathbb{F}_q,K;T) := det(-T\cdot Frob_{\mathbb{F}_q};R(a_X)_!K)^{(-1)} \in \overline{\mathbb{Q}_l}[T^{\pm 1}]^\times$$

Hence, we get a group homomorphism, $\epsilon(X/\mathbb{F}_q,-;T): K(X,\overline{\mathbb{Q}_l}) \to \overline{\mathbb{Q}_l}[T^{\pm 1}]^\times$.

Corollary (Functional equation) Let $\begin{array}{c} X \\ \downarrow a_X \\ \text{Spec}(\mathbb{F}_q) \end{array} \in (Sch/\mathbb{F}_q)$ be **proper**, and let $K \in D^b_c(X,\overline{\mathbb{Q}_l})$. Then:

$$L(X/\mathbb{F}_q,K;T) = \epsilon(X/\mathbb{F}_q,K;T)L(X/\mathbb{F}_q,\mathbb{D}K;\frac{1}{T}) \in \overline{\mathbb{Q}_l}(T)^\times$$

Proof The left side can be analyzed using tools as above. By definition,

$$L(X/\mathbb{F}_q,\mathbb{D}K;T) = det(1-T\cdot Frob_{\mathbb{F}_q};R(a_X)_!\mathbb{D}K)^{(-1)},$$

and $R(a_X)_!\mathbb{D}K = \mathbb{D}R(a_X)_*K = \mathbb{D}R(a_X)_!K$. Hence, we reduce to the case of $X = \text{Spec}(\mathbb{F}_q)$, and K a lisse sheaf of rank 1 (because $\mathbb{F}_q$ has abelian absolute Galois group). Equality then amounts to

$$(1-\alpha T)^{-1} = (-\alpha T)^{-1}(1-\alpha^{-1}(\frac{1}{T}))^{-1},$$

which is true.

Note X need not be a curve.

Deligne's theory of weights: mixed complexes

Let $X \in (Sch/\mathbb{F}_q)$, and $\mathcal{F} \in Shv_c(X, \overline{\mathbb{Q}_l})$.

Definition For each closed point $x \in |X|$, the **eigenvalues of** $Frob_x$ **acting on** $\mathcal{F}$ are the reciprocals of roots of $det(1 - T \cdot Frob_x; \mathcal{F})$ (as a multiset in $\overline{\mathbb{Q}_l}$). Let $\iota : \overline{\mathbb{Q}_l} \xrightarrow{\sim} \mathbb{C}$ be an isomorphism of fields. Say $\mathcal{F}$ is **pointwise ι-pure of weight** $w \in \mathbb{R}$ if and only if for all $x \in |X|$, and any $\alpha \in \{\text{eigenvalues of } Frob_x \text{ acting on } \mathcal{F}\} \subset \overline{\mathbb{Q}_l}^\times$, one has $|\iota\alpha|_{\mathbb{C}} = (\#\kappa(x))^{w/2}$. Say $\mathcal{F}$ is **pointwise pure of weight** $w \in \mathbb{Z}$ if and only if for any isomorphism $\iota : \overline{\mathbb{Q}_l} \to \mathbb{C}$ of fields, $\mathcal{F}$ is pointwise ι-pure of weight w. Say $\mathcal{F}$ is **mixed of weight** $\leq w \in \mathbb{Z}$ if and only if there exists a finite filtration of $\mathcal{F}$ by subsheaves, such that the successive subquotients are pointwise pure of weight $\leq w$.

Pure objects are difficult to construct. Easy ones are constant sheaves and Tate twists of them, e.g. for $\mathbb{Q}_l(1)$, $Frob_x$ acts by $q^{-1} = q^{-\frac{2}{2}}$, so it has weight -2. Deligne says that the existence of interesting pure objects is one of the deepest concepts in mathematics. It's even more difficult to construct non-pure objects, because they don't exist (Lafforgue). ?

Recall (SGA 4, 4.5, 5, etc.) For $X \in (Sch/\mathbb{F}_q)$, $- \overset{L}{\otimes} -$ sends $D^b_c \times D^b_c$ into D^b_c, and $R\underline{Hom}$ sends $(D^b_c)^{op} \times D^b_c$ into D^b_c. For $f : X \to Y$ in $(Sch/\mathbb{F}_q)$, f^*, $f^!$, Rf_* and $Rf_!$ send D^b_c to D^b_c.

Definition For $X \in (Sch/\mathbb{F}_q)$ and $w \in \mathbb{Z}$, let $D^b_m(X, \overline{\mathbb{Q}_l})$ be the full triangulated subcategory of $D^b_c(X, \overline{\mathbb{Q}_l})$ consisting of all $K \in D^b_c(X, \overline{\mathbb{Q}_l})$ such that for all $i \in \mathbb{Z}$, $H^i(K) \in Shv_c(X, \overline{\mathbb{Q}_l})$ is **mixed**.

$D^b_{\leq w}(X, \overline{\mathbb{Q}_l})$ is the full triangulated subcategory of $D^b_m(X, \overline{\mathbb{Q}_l})$ consisting of $K \in D^b_m(X, \overline{\mathbb{Q}_l})$ such that for all $i \in \mathbb{Z}$, $H^i(K)$ is **mixed of weight** $\leq w + i$.

$D^b_{\geq w}(X, \overline{\mathbb{Q}_l})$ is the full triangulated subcategory of $D^b_m(X, \overline{\mathbb{Q}_l})$ consisting of $K \in D^b_m(X, \overline{\mathbb{Q}_l})$ such that $\mathbb{D}K \in D^b_c(X, \overline{\mathbb{Q}_l})$ lies in $D^b_{\leq -w}(X, \overline{\mathbb{Q}_l})$.

Note If $K \in D^b_m$, a priori $\mathbb{D}K$ need not be in D^b_m.

Theorem (Deligne, Weil II) For any $w, w' \in \mathbb{Z}$

- For any $X \in (Sch/\mathbb{F}_q)$,
  - $- \overset{L}{\otimes} -$ sents $D^b_{\leq w} \times D^b_{\leq w'}$ into $D^b_{\leq w+w'}$.
  - $R\underline{Hom}$ sends $(D^b_{\leq w})^{op} \times D^b_{\geq w'}$ into $D^b_{\geq -w+w'}$.

- For $f: X \to Y$ in $(Sch/\mathbb{F}_q)$,
  - f^* and $Rf_!$ send $D^b_{\leq w}$ into $D^b_{\leq w}$.
  - Rf_* and $f^!$ send $D^b_{\geq w}$ into $D^b_{\geq w}$.

Proof for $\overset{L}{\otimes}$ is trivial. The proof for $Rf_!$ is the main theorem of Weil II, and uses local monodromy. The proof for Rf_* follows from $Rf_!$, and is mostly dévissage.

Corollary The triangulated categories $\{D^b_m(X, \overline{\mathbb{Q}_l}) : X \in (Sch/\mathbb{F}_q)\}$ satisfy the formalism of six operations.

Corollary If $f : X \to Y$ is a **proper morphism**, and $K \in D^b_m(X, \overline{\mathbb{Q}_l})$ is **pure of weight** $w \in \mathbb{Z}$ (i.e., $K \in D^b_{\leq w} \cap D^b_{\geq w}$), then $Rf_! K = Rf_* K \in D^b_m(Y, \overline{\mathbb{Q}_l})$ is also **pure of weight** w.

Corollary (Weil's conjecture: Riemann hypothesis) Let $X/\mathbb{F}_q$ be a proper smooth variety over $\mathbb{F}_q$. Then for any $i \in \mathbb{Z}$, the polynomial $det(1 - T \cdot Frob_{\mathbb{F}_q}; H^i(X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}, \overline{\mathbb{Q}_l}))$ can be written as $\prod_i (1 - \alpha_j^{(i)} T)$, where $\alpha_j^{(i)} \in \overline{\mathbb{Q}_l}$ are algebraic numbers, and for any $|-| \in |\mathbb{Q}(\alpha_j^{(i)})|_\infty$ archimedean absolute value of $\mathbb{Q}(\alpha_j^{(i)})$, $|\alpha_j^{(i)}| = q^{i/2}$.

Why is this called the Riemann hypothesis? The zeta function is given by

$$Z(X/\mathbb{F}_q; T) = \prod_i det(1 - T \cdot Frob_{\mathbb{F}_q}; H^i_c(X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}, \overline{\mathbb{Q}_l}))^{(-1)^{i+1}}$$

For example, with curves we have the weight distribution $\frac{(1)}{(0)(2)}$. We can substitute q^{-s} for T. fix

Proof On X, we have the constant sheaf $\overline{\mathbb{Q}_l}[0] \in D^b_{\leq 0}(X, \overline{\mathbb{Q}_l})$, and $\mathbb{D}(\overline{\mathbb{Q}_l}[0]) = \overline{\mathbb{Q}_l}[2d](d) \in D^b_{\leq 0}$ by Poincaré duality, as X is smooth of relative dimension d. This means $\overline{\mathbb{Q}_l}[0]$ is pure of weight 0 on X. We get $R(a_X)_! \overline{\mathbb{Q}_l}[0] = R(a_X)_* \overline{\mathbb{Q}_l}[0]$ is also pure of weight 0 on $D^b_m(\mathrm{Spec}(\mathbb{F}_q), \overline{\mathbb{Q}_l})$, so for any $i \in \mathbb{Z}$, any $\alpha \in \{$eigenvalues of $Frob_{\mathbb{F}_q}$ acting on $H^i_c(X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}, \overline{\mathbb{Q}_l})\}$, and any field isomorphism $\iota : \overline{\mathbb{Q}_l} \to \mathbb{C}$, one has $|\iota\alpha|_{\mathbb{C}} = q^{i/2}$. Thus, α is necessarily algebraic, and the conclusion holds.

March 18, 2003

***L*- and ϵ-functions on curves** Let k be a finite field of order q and characteristic p. Let X/k be a proper smooth connected curve over k, and let $\eta \overset{j}{\hookrightarrow} X$ be its generic point (note: j is **not** necessarily of finite type). Let $F = \kappa(X) = \kappa(\eta)$ be the function field. Choose a prime $l \neq p$ and an algebraic closure $\overline{\mathbb{Q}_l}$ of $\mathbb{Q}_l$, so we get $D^b_c(X, \overline{\mathbb{Q}_l})$ and $K(X, \overline{\mathbb{Q}_l})$ as before. Choose a separable algebraic closure $\overline{F}$ of F, equivalently a geometric generic point $\overline{\eta} \to \eta \overset{j}{\hookrightarrow} X$.

We get an equivalence of categories: $LisseShv(\eta, \overline{\mathbb{Q}_l}) \overset{\overline{\eta}}{\xrightarrow{\sim}} Rep_{fin}(Gal(\overline{F}/F), \overline{\mathbb{Q}_l})$ via $\mathcal{L} \mapsto (\mathcal{L}_{\overline{\eta}}, \text{monodromy action})$. We also get a functor $j_* : LisseShv(\eta, \overline{\mathbb{Q}_l}) \to Shv(X, \overline{\mathbb{Q}_l}) := Pro(Shv(X, \mathbb{Z}/l^n)) \otimes \overline{\mathbb{Q}_l}$.

Definition A $\begin{cases} \text{lisse } \overline{\mathbb{Q}_l}\text{-sheaf } \mathcal{L} \text{ on } \eta \\ \text{continuous representation } \sigma \text{ of } Gal(\overline{F}/F) \end{cases}$ is **almost everywhere unramified** if and only if $\begin{cases} j_*\mathcal{L} \\ j_*\sigma \end{cases}$ is constructible on X, i.e., lies in $Shv_c(X, \overline{\mathbb{Q}_l}) \subset Shv(X, \overline{\mathbb{Q}_l})$.

In this case, let $S_\sigma = S_{\mathcal{L}} \subset |X|$ be the smallest subset such that $(j_*\mathcal{L})|_{X \setminus S}$ is lisse (i.e., the set of ramified places of $\begin{cases} \mathcal{L} \\ \sigma \end{cases}$).

Definition The $\begin{cases} \textbf{global } L\textbf{-function} \\ \textbf{global } \epsilon\textbf{-function} \end{cases}$ of an almost everywhere unramified Galois representation $\sigma \in Rep_{fin}(Gal(\overline{F}/F), \overline{\mathbb{Q}_l})$ is:

$$L(\sigma; T) := L(X/k, j_*\sigma; T) = \prod_{x \in |X|} det(1 - T^{deg(x)} Frob_x, x^* j_*\sigma)$$

$$\epsilon(\sigma; T) := \epsilon(X/k, j_*\sigma; T) = det(-T \cdot Frob_k; R(a_X)_!(j_*\sigma))$$

Why j_* and not Rj_*? Because of perverse sheaves. They are both artinian and noetherian. Constructible sheaves are not necessarily artinian (remove one point and iterate $j_!$). Also, perverse sheaves "span" D^b_c, in the sense that any object in D^b_c can be reached by iterated extensions of perverse sheaves. They also play nicely with duality. [ref. [BBD]] ?

If $\mathcal{L}$ is a lisse sheaf on U regular of dimension d, with $U \overset{j}{\hookrightarrow} X$ locally closed, then $\mathcal{L}[d]$ is perverse on U. For any perverse sheaf P on U, $j_{!*}(P)$ is perverse on X. This is **intermediate extension**. If X is a curve, then $j_{!*} = j_*$.

Artin already defined L-functions this way, with $x^* j_*\sigma$ written σ^{I_x}, i.e., inertia invariants.

Note that $R(a_X)_!(j_*\sigma) = R(a_X)_*(j_*\sigma)$, but since j_* is not Rj_*, we cannot compose these to $R(a_X j)_*\sigma$.

Theorem (Rationality and functional equation) Suppose $\sigma \in Rep_{fin}(Gal(\overline{F}/F))$ is almost everywhere unramified.

1. The global L-function is a rational function.

2. One has $L(\sigma;T) = \epsilon(\sigma;T)L(\sigma^\vee \frac{1}{qT})$.

For curves, this becomes

$$L(\sigma;T) = \frac{det(1 - T\cdot Frob_k; H^1(X\otimes \overline{k}, j_*\sigma))}{det(1 - T\cdot Frob_k; H^0(X\otimes \overline{k}, j_*\sigma))\cdot det(1 - T\cdot Frob_k; H^2(X\otimes \overline{k}, j_*\sigma))}$$

in $\overline{\mathbb{Q}_l}(T)^\times \cap (1 + T\overline{\mathbb{Q}_l}[[T]]) \subset \overline{\mathbb{Q}_l}((T))^\times$.

Note that H^i is pure of weight i. For α an eigenvalue of $Frob$ on H^0, $|\alpha| = q^{wt/2} = 1$. $q^{-s} = 1/\alpha$ gives $Re(s) = w/2$, so poles have real part $0, 1$ and zeroes have real part $1/2$. If σ is pure of weight 0, then $j_*\sigma \in Shv_c(X, \overline{\mathbb{Q}_l}) \subset D^b_{\le 0} \cap D^b_{\ge 0} \subset D^b_c(X, \overline{\mathbb{Q}_l})$. Dualize, get $\mathbb{D}(j_*\sigma) = j_*(\sigma^\vee) \in D^b_{\le 0}$.

Recall from last time that for X proper,

$$L(X/\mathbb{F}_q, K; T) = \epsilon(X/\mathbb{F}_q, K; T)L(X/\mathbb{F}_q, \mathbb{D}K; \frac{1}{T}).$$

Observe that by duality, we have

$$\begin{aligned} L(X/\mathbb{F}_q, \mathbb{D}(j_*\sigma); T) &= L(X/\mathbb{F}_q, j_*(\sigma^\vee)[2](1); T) \\ &= L(X/\mathbb{F}_q, j_*\sigma^\vee; \frac{1}{q}T) \end{aligned}$$

since the shift by 2 does nothing, and the Tate twist divides by q. Thus,

$$L(X/\mathbb{F}_q, \mathbb{D}(j_*\sigma); \frac{1}{T}) = L(X/\mathbb{F}_q, j_*\sigma^\vee; \frac{1}{qT}).$$

Rankin-Selberg tries to hard-wire $\boxtimes$ into L-functions. $L(\pi\boxtimes\pi'; T)$ should be $L(\pi, \pi'; T)$. These should correspond to $\sigma\otimes\sigma'$. We can restrict σ to σ_x, a representation of the local decomposition group Δ_x. [What was he getting at?] ?

Euler-Poincaré characteristics on curves Let k be a perfect field of characteristic $p \ge 0$. Let X, η, j, F, and l be as above.

For each $x \in |X|$, choose a separable algebraic geometric point $\overline{x} \to x \in X$. We get:

$$\begin{array}{lccccccc}
\text{strict henselization} & \overline{x} & \in & X_{(\overline{x})} & \ni & \eta_{\overline{x}} & \xleftarrow{\text{choose sep. geom. pt.}} & \overline{\eta_x} \\
& & & & & \searrow^{\text{max. unr. ext.}} & F_{(\overline{x})} = \kappa(\eta_{\overline{x}}) & \hookrightarrow \overline{F_x} \\
& \downarrow & & \downarrow & & \downarrow & & | \\
\text{henselization} & x & \in & X_x & \ni & \eta_x & \searrow^{\text{decomp. field}} & \\
& \| & & & & & F_{(x)} = \kappa(\eta_x) & \rightsquigarrow F_x \\
& \| & & \downarrow & & \downarrow & \downarrow & \\
& x & \in & X & \ni & \eta & F &
\end{array}$$

From the Galois extension

$$\begin{array}{ccc} \overline{\eta_x} & \xrightarrow{I_x} & \eta_{\overline{x}} \\ & \searrow^{D_x} & \downarrow \\ & & \eta_x \end{array}$$

we get the exact sequnce

$$\begin{array}{ccccccccc} 1 & \longrightarrow & I_x & \longrightarrow & G_x & \longrightarrow & Gal(\kappa(\overline{x})/\kappa(x)) & \longrightarrow & 1 \\ & & \| & & \| & & \downarrow \cong & & \\ & & \pi_1(\eta_{\overline{x}}, \overline{\eta_x}) & & \pi_1(\eta_x, \overline{\eta_x}) & & \widehat{\mathbb{Z}} & & \end{array}$$

We also get an exact sequence for inertia:

$$1 \longrightarrow P_x \longrightarrow I_x \xrightarrow{t} \widehat{\mathbb{Z}}(1)'(k) \longrightarrow 1$$

where P_x is the **wild inertia group**, which is the pro-p-Sylow subgroup of I_x. $\widehat{\mathbb{Z}}(1)'(k)$ is the **tame quotient**, defined as $\varprojlim_N \boldsymbol{\mu}_N(\overline{k})$ with the limit taken over all N satisfying $(N,p) = 1$. Here, $\overline{k} = \Gamma(\overline{x}, \mathcal{O}_{\overline{x}})$. Incidentally, the map t has a nice formula.

For $x \in |X|$ and $\mathcal{F} \in Shv_c(X_{(x)}, \overline{\mathbb{Q}_l})$, we get the **local monodromy of** $\mathcal{F}$, defined as $\mathcal{F}_{\eta_x} \in LisseShv(\eta_x) \xrightarrow{\sim} Rep(G_x, \overline{\mathbb{Q}_l})$, and we get the **fiber of** $\mathcal{F}$ **at** x, defined as $\mathcal{F}_x \in LisseShv(x) \xrightarrow{\sim} Rep(Gal(\kappa(\overline{x})/\kappa(x)))$.

This gives us the **generic rank** $r_{\eta_x}(\mathcal{F}) :=$ rank of $\mathcal{F}_{\eta_x}$, and the **fiber rank** $r_x(\mathcal{F}) :=$ rank of $\mathcal{F}_x$. These are additive in $\mathcal{F}$, so we get homomorphisms $r_{\eta_x}, r_x : K(X_{(x)}, \overline{\mathbb{Q}_l}) \to \mathbb{Z}$, since we can describe ranks of complexes via alternating sum.

If $\mathcal{F} \in Shv_c(X, \overline{\mathbb{Q}_l})$, then for any $x \in |X|$, $r_{\eta_x}(\mathcal{F}|_{X_{(x)}})$ is constant, so we define this to be $r_\eta(\mathcal{F})$, the rank of $\mathcal{F}_\eta$. This gives us a homomorphism $r_\eta : K(X, \overline{\mathbb{Q}_l}) \to \mathbb{Z}$.

Fix $x \in |X|$. We have an upper ramification filtration on I_x:

$$I_x = I_x^{(0)} \supset I_x^{(\lambda)} \supset I_x^{(\lambda')}$$

for $0 \le \lambda \le \lambda'$ in $\mathbb{R}_{\ge 0}$ (ref. Serre, Local Fields). For each $\lambda \ge 0$, we define

$$I_x^{(\lambda+)} := \overline{\bigcup_{\epsilon > 0} I_x^{(\lambda+\epsilon)}} \subset I$$

where the overline indicates closure in I. We have the following properties:

- Each I_x^λ is a closed normal subgroup of I_x.
- $I_x^{(\lambda+)} \subset I_x^{(\lambda)}$.

- $I_x^{(0+)} = P_x$
- $\bigcap_{\epsilon>0} I_x^{(\lambda-\epsilon)} = I_x^{(\lambda)}$ for all $\lambda > 0$ (left continuity).
- $\bigcap_{\lambda>0} I_x^{(\lambda)} = \{1\}$.

Lemma Let $W \in Rep(P_x, \overline{\mathbb{Q}_l})$. Then the action of P_x factors through a finite quotient.

This is because the automorphism group of a $\overline{\mathbb{Q}_l}$ vector space has a pro-l open subgroup. As a consequence, $I_x^{(\lambda)} \subset P_x$ acts trivially for $\lambda \gg 0$, and W is semisimple as a P_x representation.

Definition Let $W \in Rep(P_x, \overline{\mathbb{Q}_l})$ be an **irreducible** representation. Then there exists a unique $\lambda \in \mathbb{R}_{\geq 0}$ such that $W^{I_x^{(\lambda)}} = 0$ and $W^{I_x^{(\lambda+)}} = W$. This λ is called the **slope**, or **break** of W.

In general, if W is not irreducible, the **slopes of** W is the set

$$\bigwedge(W) = \{\lambda \in \mathbb{R}_{\geq 0} : \lambda \text{ is a slope of an irreducible piece of } W\}$$

so we get the **slope decomposition for** W:

$$W = \bigoplus_{\lambda \in \bigwedge(W)} W_\lambda \in Rep(P_x, \overline{\mathbb{Q}_l})$$

where W_λ is the sum of all irreducible parts of W with slope λ. This smells like an isotypic decomposition.

Definition For $W \in Rep(P_x, \overline{\mathbb{Q}_l})$, the **Swan conductor of** W is:

$$Swan(W) := \sum_{\lambda \in \bigwedge(W)} \lambda \cdot rank(W_\lambda) \in \mathbb{R}_{\geq 0}$$

For $W \in Rep(I_x, \overline{\mathbb{Q}_l})$, $Swan(W) := Swan(W|_{P_x})$.

Example Let $W \in Rep(I_x)$. Then $Swan(W) = 0$ if and only if P_x acts trivially if and only if W is **tame**.

Theorem (Hasse-Arf) Let $W \in Rep(I_x, \overline{\mathbb{Q}_l})$. Then the slopes of W are all rational numbers, and for any $\lambda \in \bigwedge(W)$, $\lambda \cdot rank(W_\lambda)$ is an integer. In particular, $Swan(W) \in \mathbb{Z}_{\geq 0}$.

Definition For $x \in |X|$ and $\mathcal{F} \in Shv(X_{(x)}, \overline{\mathbb{Q}_l})$, the **Swan conductor of** $\mathcal{F}$ **at** x is:

$$s_x(\mathcal{F}) := Swan(\mathcal{F}_{\eta_x}) \in \mathbb{Z}_{\geq 0}.$$

We get $s_x : K(X_{(x)}, \overline{\mathbb{Q}_l}) \to \mathbb{Z}$.

Definition Fix $x \in |X|$ and $\mathcal{F} \in Shv_c(X_{(x)}, \overline{\mathbb{Q}_l})$. The **Artin conductor of $\mathcal{F}$ at x** is:

$$a_x(\mathcal{F}) = r_{\eta_x}(\mathcal{F}) - r_x(\mathcal{F}) + s_x(\mathcal{F}) \in \mathbb{Z}.$$

We get $a_x : K(X_{(x)}, \overline{\mathbb{Q}_l}) \to \mathbb{Z}$. For example, if $\mathcal{F}$ is lisse on $X_{(x)}$, then $a_x(\mathcal{F}) = 0$.

Definition For $K \in D_c^b(X, \overline{\mathbb{Q}_l})$, the **Euler-Poincaré characteristic of K** is:

$$\begin{aligned}\chi(K) &:= rank(R(a_X)_! K \in D_c^b(\mathrm{Spec}(k), \overline{\mathbb{Q}_l})) \\ &= \sum_i (-1)^i dim_{\overline{\mathbb{Q}_l}} H_c^i(X \otimes_k \overline{k}, K)\end{aligned}$$

We get $\chi : K(X, \overline{\mathbb{Q}_l}) \to \mathbb{Z}$.

Note that a_X above denotes the structure morphism to Spec(k), not the Artin conductor.

Theorem (Grothendieck-Néron-Ogg-Shafarevich Euler-Poincaré characteristic formula – Raynaud says that Grothendieck's name is the only one that really belongs)

Let X/k be a proper smooth connected curve over a perfect field k. Let $K \in D_c^b(X, \overline{\mathbb{Q}_l})$. Then:

$$\chi(K) = \chi(\overline{\mathbb{Q}_l}) \cdot r_\eta(K) - \sum_{x \in |X|} deg(x) a_x(K)$$

Here, $\overline{\mathbb{Q}_l}$ is the constant sheaf, and we can write $\chi(\overline{\mathbb{Q}_l}) = c(2 - 2g)$, since $c = dim_{\overline{k}}(H^0(X \otimes_k \overline{k}, \mathcal{O}_{X \otimes_k \overline{k}})$ and $cg = dim_{\overline{k}} H^1(X \otimes_k \overline{k}, \mathcal{O}_{X \otimes_k \overline{k}})$.

March 25, 2003

No notes (spring break)

April 1, 2003

Local ϵ-factors on curves

Let k be a finite field of order q and characteristic p. Let X/k be a proper smooth connected curve over k, and let $F = \kappa(X)$ be its function field. Fix a nontrivial additive character $\psi_0 : k \to \mathbb{R}/\mathbb{Z}$. [We use $\mathbb{R}/\mathbb{Z}$ instead of $U(1)$ so we don't have to choose a square root of minus one.] Recall [Oct 11 lecture] that for any $x \in |X|$ the following diagram commutes:

$$\begin{array}{ccc} Hom_{\text{top. ab. gp.}}(\mathbb{A}_F/F, \mathbb{R}/\mathbb{Z}) & \longrightarrow & Hom_{\text{top. ab. gp.}}(F_x, \mathbb{R}/\mathbb{Z}) \\ \cong \uparrow & & \uparrow \cong \\ \Omega^1_{F/k} & \longrightarrow & \Omega^1_{F_x/\kappa(x)} \end{array}$$

The bottom left term is a one dimensional F vector space, with the discrete topology. The maps are defined as follows:

$$\begin{array}{ccc} \psi := (a \mapsto \psi_0(\sum_{x\in|X|} Tr_{\kappa(x)/k}(res_x(a_x \cdot \omega_x)))) & \longmapsto & \psi_x := (a_x \mapsto \psi_0(Tr_{\kappa(x)/k}(res_x(a_x \cdot \omega_x)))) \\ \uparrow & & \uparrow \\ \omega & \longmapsto & \omega_x \end{array}$$

Souped up residue theorem: If $\psi_x \in Hom(F_x, \mathbb{R}/\mathbb{Z})$ corresponds to $\omega_x \in \Omega^1_{F_x/\kappa(x)}$, then the local conductor $c(\psi_x) := \max\{c \in \mathbb{Z} : \psi_X|_{\mathfrak{m}_x^{-c}} \text{ is trivial}\} \in \mathbb{Z}$ is **equal** to $v_x(\omega_x) \in \mathbb{Z}$, where $v_x(f d\pi_x) = v_x(f)$ when π_x is a uniformizer and $f \in F_x^\times$. If $\psi \in Hom(\mathbb{A}_F/F, \mathbb{R}/\mathbb{Z})$ is nontrivial, then (here's the assertion)

$$\sum_{x\in|X|} deg(x) \cdot c(\psi_x) = \sum_{x\in|X|} deg(x) v_x(\omega_x)$$

is a constant, equal to $c(2g-2)$, where $c = dim_{\overline{k}} H^0(X \otimes \overline{k}, \mathcal{O}_X)$ and $c \cdot g = dim_{\overline{k}} H^1(X \otimes \overline{k}, \mathcal{O}_X)$ (Riemann-Roch).

Choose a prime $l \neq p$. We get $Shv_c(X, \overline{\mathbb{Q}_l}) \subset D^b_c(X, \overline{\mathbb{Q}_l})$ and $K(X, \overline{\mathbb{Q}_l})$. Let η be the generic point of X. For each $x \in |X|$, we have the henselization: $x \in X_{(x)} \ni \eta_x$. We get $Shv_c(X_{(x)}, \overline{\mathbb{Q}_l})$, $D^b_c(X_{(x)}, \overline{\mathbb{Q}_l})$, and $K(X_{(x)}, \overline{\mathbb{Q}_l})$.

Any nonzero $\omega_x \in \Omega^1_{F_x/\kappa(x)}$ corresponds to a nonzero $\psi_x \in Hom(F_x, \mathbb{R}/\mathbb{Z})$.

We want to define homomorphisms $\epsilon(X_{(x)}, -, \omega_x; T) : K(X_{(x)}, \overline{\mathbb{Q}_l}) \to \overline{\mathbb{Q}_l}[T^{\pm 1}]^\times$ called the **local ϵ-factor of $-$ with respect to** $\begin{cases} \psi_x \\ \omega_x \end{cases}$ such that one can prove the global statement (for any $K \in D^b_c(X, \overline{\mathbb{Q}_l})$):

$$\epsilon(X/k, K; T) := det(-T \cdot Frob_k, R(a_X)_! K)^{(-1)}$$

is equal to

$$\prod_{x\in|X|} \epsilon(X_{(x)}, K|_{X_{(x)}}, \omega_x; T^{deg(x)})$$

in $\overline{\mathbb{Q}_l}[T^{\pm 1}]^\times$.

Local conditions We already have homomorphisms for all $x \in |X|$:

$v_\eta(-) = v_{\eta_x}(-) : K(X, \overline{\mathbb{Q}_l}) \to \mathbb{Z}$	generic rank
$v_x(-) : K(X, \overline{\mathbb{Q}_l}) \to \mathbb{Z}$	fiber rank at x
$s_x(-) : K(X, \overline{\mathbb{Q}_l}) \to \mathbb{Z}$	Swan conductor at x
$a_x(-) : K(X, \overline{\mathbb{Q}_l}) \to \mathbb{Z}$	Artin conductor (= total rank drop)

Definition The **local conductor of x with respect to ω_x** is:

$$a(X_{(x)}, -, \omega_x) := a_x(-) + r_\eta(-) v_x(\omega_x)$$

This is a homomorphism $K(X_{(x)}, \overline{\mathbb{Q}_l}) \to \mathbb{Z}$.

Local constant

Let F_x be the completion of F at x, and let $\mathbb{C}$ be an algebraic closure of $\mathbb{R}$. Recall Tate's thesis [October 4]: He associated to every pair (χ_x, ψ_x), where $\chi_x : F_x^\times \to \mathbb{C}^\times$ is a quasi-character and $\psi_x : F_x \to \mathbb{R}/\mathbb{Z}$ is a nontrivial additive character, an ϵ-factor $\epsilon(\chi_x, \psi_x; T) \in \mathbb{C}[T^{\pm 1}]^\times$ which has the form:

$$\epsilon(\chi_x, \psi_x; T) =: q_x^{c(\psi_x)/2} \cdot b(\chi_x, \psi_x) \cdot T^{a(\chi_x, \psi_x)}$$

where $q_x = \#\kappa(x)$, $a(\chi_x, \psi_x) \in \mathbb{Z}$, and $b(\chi_x, \psi_x) \in \mathbb{C}^\times$.

Definition-Theorem (Deligne-Langlands) Let $\mathcal{H}$ be the set of pairs (S, ω), where $S = \{\eta, s\}$ is a henselian discrete valuation scheme (i.e., a "trait", French for "dash" or "hyphen") of equal characteristic $p > 0$ with a finite residue field over k, and $\omega \in \Omega^1_{\widehat{\kappa(\eta)}} \setminus \{0\}$. Fix an isomorphism $\iota : \overline{\mathbb{Q}_l} \xrightarrow{\sim} \mathbb{C}$. Then there exists a unique collection of homomorphisms $\{b(S, -, \omega) : K(S, \overline{\mathbb{Q}_l}) \to \overline{\mathbb{Q}_l}^\times\}_{(S,\omega)\in\mathcal{H}}\}$ satisfying:

1. If $K \in D_c^b(S, \overline{\mathbb{Q}_l})$ is supported on closed points in S (i.e., $K_\eta = 0$), then $b(S, K, \omega) = det(-Frob_{\kappa(S)}; K)^{(-1)}$.

2. If η'/η is a finite separable extension, and $S' \xrightarrow{f} S$ is the normalization of S in η', and $K' \in D_c^b(S', \overline{\mathbb{Q}_l})$ is such that $v_{\eta'}(K') = 0 \in \mathbb{Z}$, then $b(S, Rf_! K', \omega) = b(S', K', f^*\omega) \in \overline{\mathbb{Q}_l}$.

3. If $\mathcal{L} \in LisseShv(\eta, \overline{\mathbb{Q}_l})$ is of rank one (i.e., it is a multiplicative character by class field theory), and $\chi : \widehat{\kappa(\eta)}^\times \to \overline{\mathbb{Q}_l}^\times$ is the (unique) quasi-character of $\widehat{\kappa(\eta)}^\times$ such that for any geometric point $\overline{\eta} \to \eta$ the following composition is χ:

$$\begin{array}{ccccccc} \widehat{\kappa(\eta)}^\times & \xrightarrow[\text{LCFT}]{\cong} & W(\overline{\eta}/\eta)^{ab} & \hookrightarrow & Gal(\overline{\eta}/\eta)^{ab} & \longrightarrow & GL(\mathcal{L}_{\overline{\eta}}) & \xrightarrow{\cong} & \overline{\mathbb{Q}_l}^\times \\ & & & & & & \uparrow & \nearrow_{[\mathcal{L}]\ \text{monodromy rep.}} & \\ & & & & & & Gal(\overline{\eta}/\eta) & & \end{array}$$

   and if $\psi : \widehat{\kappa(\eta)} \to \mathbb{R}/\mathbb{Z}$ is the nontrivial additive character of $\widehat{\kappa(\eta)}$ corresponding to ω, **then**

$$b(S, j_*\mathcal{L}, \omega) = \iota^{-1} b(\iota \circ \chi, \psi) \in \overline{\mathbb{Q}_l}^\times$$

   where the b on the right is given by Tate's thesis.

Definition The **local ϵ-factor with respect to** ω_x is:

$$\epsilon(X_{(x)}, -, \omega_x; T) := q_x^{r\eta_x(-)\cdot v_x(\omega_x)/2} \cdot b(X_{(x)}, -, \omega_x) \cdot T^{a(X_{(x)}, -, \omega_x)}$$

This is a homomorphism: $K(X_{(x)}, \overline{\mathbb{Q}_l}) \to \overline{\mathbb{Q}_l}[T^{\pm 1}]^\times$.

Theorem (Laumon's product formula) Let X/k be a proper smooth connected curve over a finite field k, and let F be its function field. Let $\omega \in \Omega^1_{F/k}$ be a nonzero meromorphic 1-form on F. Then for any $K \in D^b_c(X)$,

$$\begin{aligned}\epsilon(X/k, K; T) &:= det(-T \cdot Frob_k; R(a_X)_! K)^{(-1)} \\ &= \prod_{x \in |X|} \epsilon(X_{(x)}, K|_{X_{(x)}}, \omega_x; T^{deg(x)})\end{aligned}$$

This gives an equality of homomorphisms $K(X, \overline{\mathbb{Q}_l}) \to \overline{\mathbb{Q}_l}[T^{\pm 1}]^\times$. The values have the form $q^c \cdot b \cdot T^a$. Equating a amounts to the GNOSEP characteristic formula [March 18], and equating c gives Riemann-Roch [Today].

$\ast$ Now we have enough to state the structure of Lafforgue's proof (Deligne's induction principle).

Let k be a finite field of order q and characteristic p. Let X/k be a proper smooth connected curve over k (for the proof, without loss of generality, we assume X/k is geometrically connected). Let $F = \kappa(X)$ be the function field of X, and $\mathbb{A}$ its ring of adèles. Let $\mathbb{C}$ be an algebraic closure of $\mathbb{R}$. For each $r \geq 1$,

$$\mathcal{A}^r(F, \mathbb{C}) := \left\{ \begin{array}{c} \text{isomorphism classes of cuspidal automorphic} \\ \text{irreducible complex representations of } GL_r(\mathbb{A}) \\ \text{whose central quasi-character is of finite order} \end{array} \right\}$$

For each $\pi \in \mathcal{A}^r(F, \mathbb{C})$, let $S_\pi \subset |X|$ be the finite set of places ramified for π, and for all $x \in |X| \setminus S_\pi$, let $\{z_1(\pi_x), \ldots, z_r(\pi_x)\} \subset \mathbb{C}^\times$ be the **Satake parameters of** π_x (also called **Hecke eigenvalues of** π_x).

Choose a prime $l \neq p$, and let $\overline{\mathbb{Q}_l}$ be an algebraic closure of $\mathbb{Q}_l$. Choose a separable algebraic closure $\overline{F}$ of F, and get $Gal(\overline{F}/F)$. For $r \geq 1$,

$$\mathcal{G}^r(F, \overline{\mathbb{Q}_l}) := \left\{ \begin{array}{c} \text{isomorphism classes of irreducible continuous representations} \\ \text{of } Gal(\overline{F}/F) \text{ on a } \overline{\mathbb{Q}_l}\text{-vector space of rank } r\text{, almost} \\ \text{everywhere unramified, whose determinant is of finite order} \end{array} \right\}$$

For each $\sigma \in \mathcal{G}^r(F, \overline{\mathbb{Q}_l})$, let $S_\sigma \subset |X|$ be the finite set of ramified places for σ. For any $x \in |X| \setminus S_\sigma$, let $\{z_1(\sigma_x), \ldots, z_r(\sigma_x)\} \subset \overline{\mathbb{Q}_l}^\times$ be the **Frobenius eigenvalues of** σ_x.

Fix a nonzero $\psi \in Hom_{\text{top. ab. gp.}}(\mathbb{A}/F, \mathbb{R}/\mathbb{Z})$ corresponding to a nonzero $\omega \in \Omega^1_{F/k}$. For each pair $(\pi, \pi') \in \mathcal{A}^r(F, \mathbb{C}) \times \mathcal{A}^{r'}(F, \mathbb{C})$ for $r, r' \geq 1$ and for each $x \in |X|$, we have local L- and ϵ-factors:

- $L(\pi_x \times \pi'_x; T) \in \mathbb{C}(T)^\times \cap (1 + T \cdot \mathbb{C}[[T]]) \subset \mathbb{C}((T))^\times$
- $\epsilon(\pi_x \times \pi'_x, \psi_x; T) \in \mathbb{C}[T^{\pm 1}]^\times \subset \mathbb{C}((T))^\times$

These are associated to π_x, π'_x, and ψ_x via Rankin-Selberg convolutions (Jacquet, Piatetski-Shapiro, and Shalika). For each pair $(\sigma, \sigma') \in \mathcal{G}^r(F, \overline{\mathbb{Q}_l}) \times \mathcal{G}^{r'}(F, \overline{\mathbb{Q}_l})$ for $r, r' \geq 1$, and for each $x \in |X|$, we have local L- and ϵ-factors:

- $L(\sigma_x \otimes \sigma'_x; T) \in \overline{\mathbb{Q}_l}(T)^\times \cap (1 + T \cdot \overline{\mathbb{Q}_l}[[T]]) \subset \overline{\mathbb{Q}_l}((T))^\times$
- $\epsilon(\sigma_x \otimes \sigma'_x; \omega_x; T) \in \overline{\mathbb{Q}_l}[T^{\pm 1}]^\times \subset \overline{\mathbb{Q}_l}((T))^\times$

associated to $\sigma_x \otimes \sigma'_x$ and ω_x according to Grothendieck and Deligne-Langlands.

Definition Let $\iota : \overline{\mathbb{Q}_l} \xrightarrow{\sim} \mathbb{C}$ be an isomorphism of fields. One says that $\pi \in \mathcal{A}^r(F, \mathbb{C})$ and $\sigma \in \mathcal{G}^r(F, \overline{\mathbb{Q}_l})$ are in **Langlands correspondence with respect to** ι if and only if for any $x \in |X| \setminus S_\pi \setminus S_\sigma$, $\{z_1(\pi_x), \ldots, z_r(\pi_x)\} = \iota\{z_1(\sigma(x), \ldots, z_r(\sigma_x)\} \subset \mathbb{C}$.

Definition For $r \geq 1$ and $\iota : \overline{\mathbb{Q}_l} \xrightarrow{\sim} \mathbb{C}$, define the following assertions:

- $(\mathcal{A} \leftarrow \mathcal{G})^r_\iota$: For any $\sigma \in \mathcal{G}^r(F, \overline{\mathbb{Q}_l})$, there exists a unique $\pi \in \mathcal{A}^r(F, \mathbb{C})$ such that π and σ are in Langlands correspondence with respect to ι, and $S_\pi \subset S_\sigma$.
- $(\mathcal{A} \rightarrow \mathcal{G})^r_\iota$: For any $\pi \in \mathcal{A}^r(F, \mathbb{C})$, there exists a unique $\sigma \in \mathcal{G}^r(F, \overline{\mathbb{Q}_l})$ such that π and σ are in Langlands correspondence with respect to ι, and $S_\sigma \subset S_\pi$.
- $(L\epsilon)^r_\iota$: For any $\pi \in \mathcal{A}^r(F, \mathbb{C})$ and $\sigma \in \mathcal{G}^r(F, \overline{\mathbb{Q}_l})$ in Langlands correspondence with respect to ι, any r' satisfying $1 \leq r' \leq r$, any $\pi' \in \mathcal{A}^{r'}(F, \mathbb{C})$ and $\sigma' \in \mathcal{G}^{r'}(F, \overline{\mathbb{Q}_l})$ in Langlands correspondence with respect to ι, and any $x \in |X|$, one has $L(\pi_x \times \pi'_x; T) = \iota L(\sigma_x \otimes \sigma'_x; T)$ in $\mathbb{C}(T)^\times \cap (1 + T \cdot \mathbb{C}[[T]])$, and $\epsilon(\pi_x \times \pi'_x, \psi_x; T) = \iota\epsilon(\sigma_x \otimes \sigma'_x, \omega_x; T)$ in $\mathbb{C}[T^{\pm 1}]^\times$.
- $(RP)^r$: For any $\pi \in \mathcal{A}^r(F, \mathbb{C})$ and any $x \in |X|$, $\pi_x \in Rep_{adm}(GL_r(F_x), \mathbb{C})$ is **tempered**.
- $(P)^r_\iota$: For any $\sigma \in \mathcal{G}^r(F, \overline{\mathbb{Q}_l})$ and $x \in |X| \setminus S_\sigma$, $\sigma_x \in Rep_{\text{fin. cts.}}(Gal(\overline{F_x}/F_x), \overline{\mathbb{Q}_l})$ is pointwise ι-pure of weight 0 (i.e., $|\iota z_i(\sigma_x)|_{\mathbb{C}} = 1$ for all $i = 1, \ldots, r$).

Proposition For any $\iota : \overline{\mathbb{Q}_l} \xrightarrow{\sim} \mathbb{C}$, $(RP)^1$ and $(P)^1_\iota$ hold.

Theorem (Class field theory for X/k) For any $\iota : \overline{\mathbb{Q}_l} \xrightarrow{\sim} \mathbb{C}$, $(\mathcal{A} \leftarrow \mathcal{G})^1_\iota$, $(\mathcal{A} \rightarrow \mathcal{G})^1_\iota$, and $(L\epsilon)^1_\iota$ hold.

The first two were covered on October 4. $(L\epsilon)^1_\iota$ needs treatment of the ramified places (next time).

Theorem (Part 1 of Deligne's principle of induction) Fix $r \geq 2$ and $\iota : \overline{\mathbb{Q}_l} \xrightarrow{\sim} \mathbb{C}$. Assume for all r' with $1 \leq r' < r$, $(\mathcal{A} \rightarrow \mathcal{G})^{r'}_\iota$ and $(L\epsilon)^{r'}_\iota$ hold. Then $(\mathcal{A} \leftarrow \mathcal{G})^r)_\iota$ holds.

Theorem (Lafforgue) Fix $r \geq 2$ and $\iota : \overline{\mathbb{Q}_l} \xrightarrow{\sim} \mathbb{C}$. Assume for any r' with $1 \leq r' < r$, $(\mathcal{A} \leftarrow \mathcal{G})^{r'}_\iota$, $(\mathcal{A} \rightarrow \mathcal{G})^{r'}_\iota$, and $(P)^{r'}_\iota$ hold. Then $(\mathcal{A} \rightarrow \mathcal{G})^r_\iota$ holds. Assume furthermore that $(\mathcal{A} \leftarrow \mathcal{G})^r_\iota$ holds. Then $(P)^r_\iota$ holds.

Theorem (Part 2 of Deligne's principle of induction) Fix $r \geq 2$ and $\iota : \overline{\mathbb{Q}_l} \xrightarrow{\sim} \mathbb{C}$. Assume for all r' with $1 \leq r' < r$, $(\mathcal{A} \rightarrow \mathcal{G})^{r'}_\iota$ and $(RP)^{r'}_\iota$ hold. Assume furthermore that $(\mathcal{A} \rightarrow \mathcal{G})^r_\iota$ and $(P)^r_\iota$ hold. Then $(RP)^r$ and $(L\epsilon)^r_\iota$ hold. (Actually, we get $(RP)^r$, and then $(RP)^r$ implies $(L\epsilon)^r_\iota$.)

Note that $(\mathcal{A} \rightarrow \mathcal{G})^r_\iota$ and $(P)^r_\iota$ together imply $(RP)^r$ at all unramified places.

April 8, 2003

[Q. What about number fields? A. Only the one dimensional case is fully known, and not much else. Shimura-Taniyama is a special case of $(\mathcal{A} \leftarrow \mathcal{G})^2_\iota$.]

Today we cover Deligne part 1.

Assume for all r' with $1 \leq r' < r$, $(\mathcal{A} \rightarrow \mathcal{G})^{r'}_\iota$ and $(L\epsilon)^{r'}_\iota$ hold. Let $\sigma \in \mathcal{G}^r(F, \overline{\mathbb{Q}_l})$ be given, so we get $S_\sigma \subset |X|$, the finite set of ramified places. We want $\pi \in \mathcal{A}^r(F, \mathbb{C})$ such that $S_\pi \subset S_\sigma$, and in Langlands correspondence with respect to ι with σ.

We use the converse theorem to construct a candidate representation.

For each $x \in |X| \setminus S_\sigma$, $\sigma_x \in Rep(Gal(\overline{F}/F), \overline{\mathbb{Q}_l})$ is unramified, so we get Frobenius eigenvalues $\{z_1(\sigma_x), \ldots, z_r(\sigma_x)\} \subset \overline{\mathbb{Q}_l}^\times$. Define $\pi_x \in Rep_{adm}(GL_r(F_x), \mathbb{C})$ to be **the** irreducible admissible unramified representation of $GL_r(F_x)$ with Satake parameters $\{z_1(\pi_x), \ldots, z_r(\pi_x)\} = \iota\{z_1(\sigma_x), \ldots, z_r(\sigma_x)\} \subset \mathbb{C}^\times$.

Remark These need not be generic, but are "induced of Whittaker type", so we can apply Rankin-Selberg theory and the converse theorem. Recall that "generic" means "has a Whittaker model". All permutations of $q^{\frac{r-1}{2}} z_1(\pi_x), q^{\frac{r-3}{2}} z_2(\pi_x), \ldots$ give distinct values. why?

What about ramified places? Throw in some garbage (nice garbage, though).

For $x \in S_\sigma$, **choose** $\pi_x \in Rep_{adm}(GL_r(F_x), \mathbb{C})$ to be **any irreducible admissible generic representation** of $GL_r(F_x)$, whose central character $\chi_{\pi_x} \in Hom(F_x^\times, \mathbb{C}^\times)$ is in local CFT correspondence with respect to ι with $det(\sigma_x) \in Hom(Gal(\overline{F_x}/F_x), \overline{\mathbb{Q}_l}^\times)$.

We provisionally define

$$\pi = \bigotimes_{x \in |X|}^{res} \pi_x \in Rep_{adm}(GL_r(\mathbb{A}), \mathbb{C}).$$

This is irreducible, admissible, and its central character $\chi_\pi = \bigotimes_{x\in|X|} \chi_{\pi_x}$ is in global CFT correspondence with respect to ι with $det(\sigma)$.

Let $\chi = \bigotimes_{x\in|X|} \chi_x \in \begin{cases} \mathcal{G}^1(F, \overline{\mathbb{Q}_l}) \\ \mathcal{A}^1(F, \mathbb{C}) \end{cases}$. (Why twist by ramified characters? You get more information.)

Key lemma Suppose χ is **sufficiently ramified** at all of the places $x \in S_\sigma$. then for any $r' > r$ and any $\pi' \in \mathcal{A}^{r'}(F, \mathbb{C})$ with $S_{\pi'}$ **disjoint** from S_σ, one has: $\begin{cases} L(\chi\pi \times \pi'; T) \\ L(\chi^{-1}\pi^\vee \times \pi'^\vee; T) \end{cases}$ lie in $1 + T\mathbb{C}[T]$, i.e., they are polynomials, and satisfy the functional equation:

$$L(\chi\pi \times \pi'; T) = \epsilon(\chi\pi \times \pi'; T) L(\chi^{-1}\pi^\vee \times \pi'^\vee; T) \in \mathbb{C}[T^{\pm 1}]$$

Proof Later.

Assume the key lemma for now. By the converse theorem, after **adjusting** the factors π_x only for $x \in S_\sigma$, we get $\widetilde{\pi}$ (the central character of $\chi\pi$ is $\chi \cdot \chi_\pi$). The representation $\widetilde{\chi_\pi}$ of $GL_r(\mathbb{A})$ is automorphic, so $\widetilde{\pi}$ is also automorphic (untwist). By construction, $S_\pi \subset S_\sigma$.

It remains to show that $\widetilde{\pi}$ is cuspidal. Suppose it is not (but is still automorphic). By Langlands' classification, there exists a nontrivial partition $r = r_1 + \cdots + r_k$, and there exist cuspidal automorphic representations $\pi^1, \ldots, \pi^k$ of $GL_{r_1}(\mathbb{A}), \ldots, GL_{r_k}(\mathbb{A})$ respectively, all with $S_{\pi^i} \subset S_\pi \subset S_\sigma$, such that for all $x \in |X| \setminus S_\pi$,

$$\{z_1(\pi_x), \ldots, z_r(\pi_x)\} = \coprod_{i=1}^{k} \{z_1(\pi^i), \ldots, z_{r_i}(\pi^i)\} \subset \mathbb{C}$$

By $(\mathcal{A} \to \mathcal{G})_\iota^{r_i}$, for each $i = 1, \ldots, k$, there exists $\sigma^i \in Rep(Gal(\overline{F}/F), \overline{\mathbb{Q}_l})$ irreducible of rank r_i, in Langlands correspondence with respect to ι with π^i. Then $\bigoplus_{i=1}^k \sigma^i$ is rank r, reducible, and has the same set of Frobenius eigenvalues at every $x \in |X| \setminus S_\sigma$ as our given irreducible σ. By Čebotarev density, they are isomorphic, and this is a contradiction.

Proof of key lemma We have $r' < r$ and $\pi' \in \mathcal{A}^{r'}(F, \mathbb{C})$ with $S_{\pi'}$ disjoint from S_σ. By $(\mathcal{A} \to \mathcal{G})_\iota^{r'}$, there exists $\sigma' \in \mathcal{G}^{r'}(F, \overline{\mathbb{Q}_l})$ in Langlands correspondence with respect to ι with π', satisfying $S_{\sigma'} = S_{\pi'}$. We have χ, which will be fixed at the end of the proof. $\pi'\chi \in \mathcal{A}^{r'}(F, \mathbb{C})$ and $\sigma' \otimes \chi \in \mathcal{G}^{r'}(F, \overline{\mathbb{Q}_l})$ are in Langlands correspondence with respect to ι. By $(L\epsilon)_\iota^{r'}$, for any $x \in |X| \setminus S_\sigma$, we get product formulas:

$$
\begin{aligned}
L(\pi_x\chi_x \times \pi'_x; T) &= \iota L(\sigma_x \otimes \chi \otimes \sigma'_x; T) \\
L(\pi_x^\vee\chi_x^{-1} \times \pi_x'^\vee; T) &= \iota L(\sigma_x^\vee \otimes \chi_x^{-1} \otimes \sigma_x'^\vee; T) \\
\epsilon(\pi_x\chi_x \times \pi'_x, \psi_x; T) &= \iota\epsilon(\sigma_x \otimes \chi_x^{-1} \otimes \sigma_x'^\vee, \omega_x; T)
\end{aligned}
$$

We want similar equalities at $x \in S_\sigma$. We achieve this by making χ very ramified at S_σ with respect to $\begin{cases} \sigma, \omega \\ \pi, \psi \end{cases}$.

Digression

Theorem (Twist of Galois representations) Let F be a nonarchimedean local field.

1. Let $\sigma \in Rep(Gal(\overline{F}/F), \overline{\mathbb{Q}_l})$ be of rank r, and let $\chi \in Hom(Gal(\overline{F}/F), \overline{\mathbb{Q}_l}^\times)$ be a quasi-character.
   (a) If σ and χ are unramified, then $L(\chi \otimes \sigma; T) = \frac{1}{\prod_{1 \le i \le r}(1 - z_i(\sigma) \cdot z(\chi) \cdot T)}$.
   (b) If σ is unramified and χ is ramified, then $L(\chi \otimes \sigma; T) = 1$.
   (c) If σ is arbitrary, and χ is **sufficiently ramified** with respect to σ (e.g., $Swan(\chi) > Swan(\sigma)$. Other sufficient conditions also tend to involve conductors.), then $L(\chi \otimes \sigma; T) = 1$.
2. In addition, let $\omega \in \Omega^1_{F/k}$ be a nonzero meromorphic 1-form (corresponding to an additive character).
   (a) If σ is unramified, χ is arbitrary, and ω is arbitrary, then $\epsilon(\chi \otimes \sigma, \omega; T) = \epsilon(\chi, \omega; T)^{r-1}\epsilon(\chi \otimes det(\sigma), \omega; T)$.
   (b) If σ is unramified, χ is unramified, and ω is unramified, then $\epsilon(\chi \otimes \sigma, \omega; T) = 1$.
   (c) If σ is arbitrary, ω is arbitrary, and χ is sufficiently ramified with respect to σ and ω, then $\epsilon(\chi \otimes \sigma, \omega; T) = \epsilon(\chi, \omega; T)^{r-1}\epsilon(\chi \otimes det(\sigma), \omega; T)$ (Deligne-Henniart).

Theorem (Twist of local representations of) GL_r. Let F be as above, and $1 \le r' \le r$.

1. Let π and π' be as above. Turn $\overline{\mathbb{Q}_l}$ into $\mathbb{C}$. Let $\chi \in Hom(GL_1(F), \mathbb{C})$ be a quasi-character.

(a) If π, π' and χ are unramified, then

$$L(\pi\chi \times \pi'; T) = \frac{1}{\prod_{1 \le i \le r, 1 \le j \le r'} (1 - z_i(\pi) z_j(\pi') z(\chi) T)}.$$

(b) similar to above

(c) similar to above

2. In addition, let $\psi \in Hom(F, \mathbb{R}/\mathbb{Z})$ be a nontrivial additive character.

(a) If π and π' are unramified, χ is arbitrary and ψ is arbitrary, then $\epsilon(\pi\chi \times \pi', \psi; T) = \epsilon(\chi, \psi; T)^{rr'-1} \epsilon(\chi \cdot \chi_\pi^{r'} \chi_{\pi'}^r, \psi; T)$. Note the switched exponents.

(b) Same as above. Replace "ω unramified" with "$c(\psi) = 0$". (Henniart)

(c) Same as above.

Back to proof of key lemma By choosing χ sufficiently ramified at S_σ, we get for all $x \in S_\sigma$:

$$L(\pi_x \chi_x \times \pi'_x; T) = \iota L(\sigma_x \otimes \chi_x \otimes \sigma'_x; T)$$
$$L(\pi_x^\vee \chi_x^{-1} \times \pi_x'^\vee; T) = \iota L(\sigma_x^\vee \otimes \chi_x^{-1} \otimes \sigma_x'^\vee; T)$$

For all $x \in |X|$,

$$\begin{aligned} \epsilon(\pi_x \chi_x \times \pi'_x; T) &= \epsilon(\chi_x, \psi_x; T)^{rr'-1} \epsilon(\chi_x \chi_{\pi_x}^{r'} \times \chi_{\pi'_x}^r, \psi_x; T) \\ &= \iota\epsilon(\chi_x, \omega_x; T)^{rr'-1} \epsilon(\chi_x det(\sigma_x)^{r'} det(\sigma'_x)^r, \omega_x; T) \qquad \text{local CFT} \\ &= \iota\epsilon(\sigma_x \otimes \chi_x \otimes \sigma'_x, \omega_x; T) \end{aligned}$$

Combine what we had 1? page ago with the above:

$$L(\pi\chi \times \pi'; T) = \iota L(\sigma \otimes \chi \otimes \sigma'; T) \in 1 + T \cdot \mathbb{C}[[T]]$$
$$L(\pi^\vee \chi^{-1} \times \pi'^\vee; T) = \iota L(\sigma^\vee \otimes \chi^{-1} \otimes \sigma'^\vee; T) \in 1 + T \cdot \mathbb{C}[[T]]$$

and by Laumon's ϵ-product formula,

$$\epsilon(\pi\chi \times \pi'; T) = \iota\epsilon(\sigma \otimes \chi \otimes \sigma'; T) \in \mathbb{C}[T^{\pm 1}]^\times.$$

By Grothendieck's cohomological interpretation of L-functions, $L(\sigma \otimes \chi \otimes \sigma'; T)$ and $L(\sigma^\vee \otimes \chi^{-1} \otimes \sigma'^\vee; T)$ are rational functions and satisfy a functional equation with $\epsilon(\sigma \otimes \chi \otimes \sigma'; T)$.

It remains to show that the L-functions are polynomials. Since $\begin{cases} \chi \otimes \sigma \\ \chi^{-1} \otimes \sigma^\vee \end{cases}$ are irreducible of rank r, while $\begin{cases} \sigma'^\vee \\ \sigma' \end{cases}$ are irreducible of rank $r' < r$, the representations

$\begin{cases} \chi \otimes \sigma \otimes \sigma' \cong \underline{Hom}(\chi \otimes \sigma, \sigma'^{\vee}) \\ \chi^{-1} \otimes \sigma^{\vee} \otimes \sigma'^{\vee} \cong \underline{Hom}(\chi^{-1} \otimes \sigma^{\vee}, \sigma') \end{cases}$ have no $\pi_1^{geom} - \begin{cases} \text{invariants} \\ \text{coinvariants} \end{cases}$ (Recall that $L = \frac{H^1}{H^0 H^2}$). Thus, the L-functions have no poles, and hence lie in $1 + T \cdot \mathbb{C}[T]$.

April 15, 2003

Last time, we proved Deligne's principle of induction, part 1. For $r \geq 2$ and $\iota : \overline{\mathbb{Q}_l} \xrightarrow{\sim} \mathbb{C}$, assume for all r' with $1 \leq r' < r$, $(\mathcal{A} \to \mathcal{G})_\iota^{r'}$ and $(L\epsilon)_\iota^{r'}$ hold. Then $(\mathcal{A} \leftarrow \mathcal{G})^r)_\iota$ holds.

Next goal: Deligne's principle of induction, part 2. For $r \geq 2$ and $\iota : \overline{\mathbb{Q}_l} \xrightarrow{\sim} \mathbb{C}$, assume for all r' with $1 \leq r' < r$, $(\mathcal{A} \to \mathcal{G})_\iota^{r'}$ and $(RP)_\iota^{r'}$ hold. Assume furthermore that $(\mathcal{A} \to \mathcal{G})_\iota^r$ and $(P)_\iota^r$ hold. Then $(RP)^r$ and $(L\epsilon)_\iota^r$ hold.

Eventually: Lafforgue's theorem. For $r \geq 2$ and $\iota : \overline{\mathbb{Q}_l} \xrightarrow{\sim} \mathbb{C}$, assume for any r' with $1 \leq r' < r$, $(\mathcal{A} \leftarrow \mathcal{G})_\iota^{r'}$, $(\mathcal{A} \to \mathcal{G})_\iota^{r'}$, and $(P)_\iota^{r'}$ hold. Then $(\mathcal{A} \to \mathcal{G})_\iota^r$ holds. Assume furthermore that $(\mathcal{A} \leftarrow \mathcal{G})_\iota^r$ holds. Then $(P)_\iota^r$ holds.

Classification of irreducible representations of GL_r over a local field
(Bernstein-Zelevinsky)

Let F be a nonarchimedean local field, k its residue field, p the characteristic of k, and q the order of k. Fix $r \geq 1$, so we get $Rep_{adm}(GL_r(F), \mathbb{C})$.

Recall that (V, π) smooth means that for all $v \in V$, $Stab_G(v) \subset G$ is open. For (V, π) to be admissible, means that it is smooth and that for any open compact subgroup $K \subset G$, V^K is finite dimensional.

We have the following heirarchy of representations:

$$\{\text{supercuspidals}\} \subset \left\{ \begin{array}{c} \text{essentially} \\ \text{square-} \\ \text{integrables} \end{array} \right\} \subset \left\{ \begin{array}{c} \text{essentially} \\ \text{tempered} \end{array} \right\} \subset \left\{ \begin{array}{c} \text{all irreducible} \\ \text{admissibles} \end{array} \right\}$$

Bernstein-Zelevinsky says: Each of the above sets can be classified in terms of the previous, so it suffices to study supercuspidals.

Definition Let $\pi \in Rep_{adm}(GL_r(F))$, so $\pi^\vee \in Rep_{adm}(GL_r(F))$. For each $v \in V_\pi$ and $v^\vee \in V_{\pi^\vee}$, the function $\phi_{v^\vee, v} : GL_r(F) \to \mathbb{C}$ given by $g \mapsto \langle v^\vee, \pi(g)v \rangle = \langle \pi^\vee(g^{-1})v^\vee, v \rangle$ is called a **matrix coefficient of** π.

Properties of matrix coefficients

- They are locally constant functions on $GL_r(F)$.
- If π admits a central quasi-character $\chi_\pi : Z(F) \to \mathbb{C}^\times$ (e.g., π is irreducible), then $\phi_{v^\vee, v}(zg) = \chi_\pi(z) \cdot \phi_{v^\vee, v}(g)$ for all $z \in Z(F)$, $g \in GL_r(F)$.

- If χ_π is **unitary** (i.e., $|\chi_\pi(z)| = 1$ for all $z \in Z(F)$), then the function $GL_(F) \to \mathbb{R}_{\geq 0}$ defined by $g \mapsto |\phi_{v^\vee,v}(g)|$ is $Z(F)$-invariant under left or right translation, so it defines function on $Z(F)\backslash GL_r(F)$.

Definition An irreducible admissible unitarizable (i.e., whose central character χ_π is unitary) representation $(V_\pi, \pi) \in Rep_{adm}(GL_r(F))$ is $\begin{cases} \text{supercuspidal unitarizable} \\ \text{square integrable} \\ \text{tempered} \end{cases}$

if and only if for any $v \in V_\pi$ and any $v^\vee \in V_{\pi^\vee}$, the matrix coefficient $\phi_{v^\vee,v}$

$\begin{cases} \text{has compact support modulo center} \\ \text{is } L^2 \text{ modulo center} \\ \text{is } L^{2+\epsilon} \text{ modulo center for all } \epsilon > 0 \end{cases}$, i.e.,

$$\begin{cases} \text{there exists a compact} K_{v^\vee,v} \subset GL_r(F) \text{ such that } Supp(\phi_{v^\vee,v}) \subset Z(F)K_{v^\vee,v}. \\ \int_{Z(F)\backslash GL_r(F)} |\phi_{v^\vee,v}(g)|^2 \frac{dg}{dz} < \infty \forall dg, dz \text{ Haar measure on } GL_r(F), Z(F). \\ \int_{Z(F)\backslash GL_r(F)} |\phi_{v^\vee,v}(g)|^{2+\epsilon} \frac{dg}{dz} < \infty \forall dg, dz \text{ Haar measure on } GL_r(F), Z(F) \forall \epsilon > 0. \end{cases}$$

(V, π) is $\begin{cases} \text{supercuspidal} \\ \text{essentially square integrable} \\ \text{essentially tempered} \end{cases}$ if and only if there exists a quasi-character $\chi : Z(F) \to \mathbb{C}^\times$ such that $(V, \pi \otimes \chi)$ given by $g \mapsto \pi(g)(\chi(det(g)|_v)$ is

$\begin{cases} \text{supercuspidal unitarizable.} \\ \text{square integrable.} \\ \text{tempered.} \end{cases}$

Definition Consider the following subgroups: blocks

$$\begin{array}{lcccc} N & = & \text{unipotent radical of } P & = & \begin{pmatrix} I & & * \\ & I & \\ 0 & & I \end{pmatrix} \\ \cap & & & & \\ P & = & \text{parabolic} \subset GL_r & = & \begin{pmatrix} * & & * \\ & * & \\ 0 & & * \end{pmatrix} \\ \uparrow\downarrow & & & & \\ M & = & \text{Levi quotient/subgroup} & = & \begin{pmatrix} * & & 0 \\ & * & \\ 0 & & * \end{pmatrix} \end{array}$$

and let fix

$$\begin{array}{ccc} \delta_{P(F)}: & P(F) \longrightarrow \mathbb{R}^{\times}_{>0} & \text{be the modular quasi-character.} \\ & \downarrow \qquad\quad \uparrow & \\ & M(F) \longrightarrow q^{\mathbb{Z}} & \end{array}$$

$d(pnp^{-1}) = \delta_{P(F)}(p)dn$ for any left Haar measure dn on $N(F)$. $S_{P(F)}(p) = |det(Ad_p), Lie(N(F))|_F$ for all $p \in P(F)$. S stands for Satake.

Parabolic induction functor

We define

$$\begin{aligned} ind_{P,M}^{GL_r} : Rep_{smooth}(M(F)) &\to Rep_{smooth}(GL_r(F)) \\ (V_\pi, \pi) &\mapsto (I_\pi, ind(\pi)) \end{aligned}$$

where

- $I_\pi :=$ the complex vector space of functions $\phi : GL_r(F) \to V_\pi$ such that ϕ is uniformly locally constant (i.e., invariant under right translation by some open subgroup K), and such that for any $m \in M(F)$, $n \in N(F)$, and $x\ inGL_r(F)$, $\phi(nmx) = S_{P(F)}(nm)^{1/2}\pi(m)\phi(x)$. (The S factor preserves unitarity.)
- $ind(\pi) : GL_r(F) \to GL(I_\pi)$ is given by $g \mapsto (\phi \mapsto (x \mapsto \phi(xg)))$.

Jacquet restriction functor

We define

$$\begin{aligned} res_{P,M}^{GL_r} : Rep_{smooth}(GL_r(F)) &\to Rep_{smooth}(M(F)) \\ (V_\pi, \pi) &\mapsto (R_\pi, res(\pi)) \end{aligned}$$

where

$$\begin{aligned} R_\pi &= V_\pi / span_{\mathbb{C}}(\{\pi(n)v - v : n \in N(F), v \in V_\pi\}) \\ &= N(F)\text{-coinvariants of } V_\pi \end{aligned}$$

and $res(\pi) : M(F) \to GL(R_\pi)$ is given by $m \mapsto \delta_{P(F)}^{1/2}(m)(\pi(m))$.

Theorem $(res_{P,M}^{GL_r}, ind_{P,M}^{GL_r})$ is a pair of **exact** functors that are left-right adjoint to each other, and send admissibles to admissibles (Frobenius reciprocity).

Theorem (Bernstein-Zelevinsky) Let $\pi \in Rep_{adm}(GL_r(F))$ be an irreducible admissible representation. Then π is supercuspidal if and only if for any proper parabolic $P \subsetneq G$, one has $res_{P,M}^{GL_r}(\pi) = 0$ in $Rep_{adm}(M(F))$, i.e., supercuspidal representations are precisely those irreducible representations that do **not** occur as a $\begin{cases}\text{quotient} \\ \text{constituent}\end{cases}$ i.e., they do not occur as an irreducible subquotient for any $ind_{P,M}^{GL_r}$.

Roughly, if we regroup our partition of r into bigger chunks (for defining M, N, and P), the supercuspidals are the new representations we get.

Definition (Tate twist) For any $s \in \mathbb{R}$, we get $GL_r(F) \to \mathbb{C}^\times$, an unramified quasi-character of $GL_r(F)$ given by $g \mapsto q^{-v_F(det(g))\cdot s} = |det(g)|_F^s$. Denote this by $\mathbf{1}(s)$.

If π is in a representation of $GL_r(F)$, let $\pi(s) := \pi \otimes \mathbf{1}(s)$.

Definition A **segment** is a set of supercuspidals of **some** GL_s of the form $\Delta = \{\rho, \rho(1), \dots \rho(t-1)\}$, $t \geq 1$ an integer. [The numbers in parentheses refer to Tate twists. For some reason, that wasn't obvious to me.]

We can view segments as representations of $(GL_s(F))^t \subset GL_r(F)$.

Theorem-Definition (Bernstein-Zelevinsky) For any segment $\Delta = \{\rho, \rho(1), \dots \rho(t-1)\}$, where $r = st$ and ρ is supercuspidal of $GL_s(F)$, we get the representation $ind_{P,(GL_s(F))^t}^{GL_r(F)}(\rho \times \rho(1) \times \cdots \times \rho(t-1))$.

1. This induced representation has a **unique** irreducible quotient, denoted $Q(\Delta)$, called the **Langlands quotient** of Δ. (Incidentally, Langlands didn't come up with this. He did the global theory ten years earlier.) [Drawing of 10 boxes, stacked in a triangle]

2. $Q(\Delta)$ is essentially square integrable.

3. Every $\begin{cases} \text{essentially square integrable} \\ \text{square integrable} \end{cases}$ irreducible representation is isomorphic to some $\begin{cases} Q(\Delta) \\ Q(\Delta). \text{ where } \rho(\frac{t-1}{2}) \text{ is unitarizable.} \end{cases}$

We have to deal with the ambiguity in partitions of r.

Definition Two segments Δ_1 and Δ_2 are **linked** if and only if $\Delta_1 \not\subseteq \Delta_2$, $\Delta_2 \not\subseteq \Delta_1$, and $\Delta_1 \cup \Delta_2$ is a segment.

Definition If $\Delta_1 = \{\rho_1, \dots, \rho_1(t_1 - 1)\}$ and $\Delta_2 = \{\rho_2, \dots, \rho_2(t_2 - 1)\}$ are two segments, then one says that Δ_1 **precedes** Δ_2 if and only if Δ_1 and Δ_2 are linked, and there exists an integer $t \geq 1$ such that $\rho_2 = \rho_1(t)$.

Theorem-Defintion (Bernstein-Zelevinsky)

1. Let $\Delta_1, \dots, \Delta_k$ be segments, such that for any $i < j$, Δ_i does **not** precede Δ_j. Then the parabolic induced representation from $Q(\Delta_1) \times \cdots \times Q(\Delta_k)$ has a **unique** irreducible quotient, denoted $Q(\Delta_1, \dots, \Delta_k)$.

2. If $\Delta_1, \ldots, \Delta_k$ and $\Delta'_1, \ldots, \Delta'_{k'}$ both satisfy the "does not precede" condition, then $Q(\Delta_1, \ldots, \Delta_k) \cong Q(\Delta'_1, \ldots, \Delta'_{k'})$ if and only if the sequences are the same up to reordering.

3. Every irreducible admissible representation $\pi \in Rep_{adm}(GL_r(F))$ is of the form $Q(\Delta_1, \ldots, \Delta_k)$ for some segments $\Delta_1, \ldots, \Delta_k$ satisfying the "does not precede" condition.

Theorem (Jacquet) Let $\Delta_1, \ldots, \Delta_k$ be segments satisfying the "does not precede" condition.

1. Suppose each $Q(\Delta_i)$ is square integrable (not just essentially). Then the representation $Q(\Delta_1, \ldots, \Delta_k)$ is **tempered**.

2. Every tempered irreducible $\pi \in Rep_{adm}(GL_r(F))$ is isomorphic to some representation of the form $Q(\Delta_1, \ldots, \Delta_k)$, where each $Q(\Delta_i)$ is square integrable.

Example Take $\Delta_1 = \{\mathbf{1}(\frac{r-1}{2})\}, \Delta_2 = \{\mathbf{1}(\frac{r-3}{2})\}, \ldots, \Delta_r = \{\mathbf{1}(\frac{1-r}{2})\}$. Then Δ_i and Δ_{i+1} are linked, but for all $i < j$, Δ_i does not precede Δ_j, and $Q(\Delta_1, \ldots, \Delta_r)$ is the trivial one dimensional representation of $GL_r(F)$. By uniqueness, these are the only segments that work.

Example Take $\Delta = \{\mathbf{1}(\frac{1-r}{2}), \ldots, \mathbf{1}(\frac{r-1}{2})\}$. Then $Q(\Delta)$ is the **generalized Steinberg representation**. It is very highly ramified.

Example Take $\Delta_1, \ldots, \Delta_r$ such that each is a singleton, made up of a single unramified quasi-character of $GL_1(F)$. Assume that $\Delta_1, \ldots, \Delta_r$ are **pairwise unlinked**. Then $Q(\Delta_1, \ldots, \Delta_r)$ is the unramified principal series representation of $GL_r(F)$, whose Satake parameters are $z(\Delta_1), \ldots, z(\Delta_r)$.

April 22, 2003

Definition Let $\pi \in Rep_{adm}(GL_r(F))$ be an admissible representation, which admits a central quasicharacter $\chi_\pi : Z(F) \to \mathbb{C}^\times$ (e.g., π irreducible). Define $|\pi| : \mathbb{G}_m(F) \to \mathbb{R}^\times_{>0} \hookrightarrow \mathbb{C}^\times$ to be the unique **unramified** quasicharacter of $\mathbb{G}_m$ (which extends to $|\pi| : GL_r(F) \stackrel{det}{\to} \mathbb{G}_m(F) \to \mathbb{R}^\times_{>0} \hookrightarrow \mathbb{C}^\times$) such that the central quasicharacter of $\pi \otimes |\pi|^{-1}$ is unitary. Thus, $|\pi| = |\chi_\pi|^{1/r} : \mathbb{G}_m(F) = Z(F) \to \mathbb{R}^\times_{>0}$.

Definition Let $\Delta = \{\rho, \ldots, \rho(t-1)\}$ be a segment (ρ is supercuspidal irreducible). Define $|\Delta| : \mathbb{G}_m(F) \to \mathbb{R}^\times_{>0}$ an unramified quasicharacter of $\mathbb{G}_m(F)$ by $|\Delta| := |Q(\Delta)| = |\rho(\frac{t-1}{2})|$.

Note By Jacquet's theorem, for any segments $\Delta_1, \ldots, \Delta_k$, such that Δ_i does not precede Δ_j for any $i < j$, $Q(\Delta_1, \ldots, \Delta_k)$ is tempered if and only if $|Delta_1| =$

$\cdots = |Delta_k| = \mathbf{1}$, the trivial quasicharacter of $\mathbb{G}_m(F)$. In particular, if $\Delta_1, \ldots, \Delta_r$ are unramified quasicharacters of $GL_1(F)$ whose Satake parameters $z(\Delta_i)$ satisfy $|z(\Delta_1)| = \cdots = |z(\Delta_r)| = 1$ in $\mathbb{C}$, then the unramified principal series representation $Q(\Delta_1, \ldots, \Delta_r)$ of $GL_r(F)$ is **tempered**.

Theorem (Jacquet, Shalika) Let $\pi \in Rep_{adm}(GL_r(F), \mathbb{C})$ be a irreducible admissible **generic unramified unitarizable** representation. Then its Satake parameters $z_1(\pi), \ldots, z_r(\pi)$ satisfy $q^{-1/2} < |z_i(\pi)| < z^{1/2}$, i.e., the $z_i(\pi) \in \mathbb{C}^\times$ have weights in $(-1, 1)$.

Theorem (Tadič) Let $\pi \in Rep_{adm}(GL_r(F), \mathbb{C})$ be an irreducible admissible **generic unitarizable** representaion, so $\pi = Q(\Delta_1, \ldots, \Delta_k)$ for some segments $\Delta_1, \ldots, \Delta_k$ (satisfying: Δ_i does not precede Δ_j for $i < j$, unique up to ordering). The Satake parameters $z(|\Delta_1|), \ldots, z(|\Delta_r|)$ of the unramified quasicharacters $|\Delta_1|, \ldots, |\Delta_r|$ of $\mathbb{G}_m(F)$ satisfy $q^{-1/2} < |z(|\Delta_i|)| < q^{1/2}$.

Theorem (Jacquet, Piatetski-Shapiro, Shalika)

1. Let $\pi = Q(\Delta_1, \ldots, \Delta_k)$ and $\pi' = Q(\Delta'_1, \ldots, \Delta'_l)$ be arbitrary irreducible admissible representations. Then

$$L(\pi \times \pi'; T) = \prod_{1 \le i \le k, 1 \le j \le l} L(Q(\Delta_i) \times Q(\Delta'_j); T) \in 1 + T\mathbb{C}[[T]]$$

2. Let $\Delta = \{\rho, \ldots, \rho(t-1)\}$ and $\Delta' = \{\rho', \ldots, \rho'(t'-1)\}$ be segments with $t' \le t$ (so ρ and ρ' are supercuspidal irreducibles). Then

$$L(Q(\Delta) \times Q(\Delta'); T) = \prod_{0 \le i \le t'-1} L(\rho(t-1) \times \rho'(i); T) \in 1 + T\mathbb{C}[[T]]$$

3. Let $\rho \in Rep_{adm}(GL_r(F))$ and $\rho' \in Rep_{adm}(GL_{r'}(F))$ be supercuspidal irreducible representations. Then $L(\rho \times \rho'; T) = \prod_z (1 - z^{-1}T)^{-1}$ in $1 + T\mathbb{C}[[T]]$, where z runs over $\{\alpha \in \mathbb{C}^\times : (\rho')^\vee \cong \rho \otimes \alpha^{v(det(-))}\}$. Thus, $\{$poles of $L(\rho \times \rho'^\vee; T)\} = \{\alpha \in \mathbb{C}^\times : \rho' \cong \rho \otimes \alpha^{v(det(-))}\}$ is empty if $r \neq r'$.

Note For $s \in \mathbb{R}$, $q^s \times \{$poles of $L(\rho \times \rho'; T)\} = \{$poles of $L(\rho(s) \times \rho'; T)\} = \{$poles of $L(\rho \times \rho'(s); T)\}$.

Proposition (Location of poles of local L-factors) Let $\pi \in Rep_{adm}(GL_r(F), \mathbb{C})$ and $\pi' \in Rep_{adm}(GL_{r'}(F), \mathbb{C})$ be irreducible admissible **generic** representations.

1. If both π and π' are **tempered**, then for any poles $z \in \mathbb{C}^\times$ of $L(\pi \times \pi'; T)$, one has $|z| \ge 1$.

2. If one of π or π' is **tempered** and the other is **unitarizable**, then for any poles $z \in \mathbb{C}^\times$ pf $L(\pi \times \pi'; T)$, one has $|z| > q^{-1/2}$ (this is strict). In particular, $L(\pi \times \pi'; T)$ and $L(\pi^\vee \times \pi'^\vee; \frac{1}{qT})$ have **no poles in common**. (The poles of $L(\pi^\vee \times \pi'^\vee; \frac{1}{qT})$ would satisfy $|z| < q^{1/2}/q = q^{-1/2}$.)

Proof Suppose $\pi = Q(\Delta_1, \ldots, \Delta_k)$ is tempered/unitarizable, and $\pi' = Q(\Delta'_1, \ldots, \Delta'_l)$ is tempered.

- By part 1 of J-PS-S,

$$\{\text{poles of } L(\pi \times \pi'; T)\} = \bigcup_{i,j} \{\text{poles of } L(Q(\Delta_i) \times Q(\Delta'_j); T)\}.$$

- The problem is reduced to

$$\begin{aligned} \pi &= Q(\Delta), & \Delta &= \{\rho, \ldots, \rho(t-1)\} \\ \pi' &= Q(\Delta'), & \Delta' &= \{\rho', \ldots, \rho'(t'-1)\} \end{aligned}$$

  with

$$\begin{aligned} |z(|\rho(\frac{t-1}{2})|)| &= 1 \text{ or } \in (q^{-1/2}, q^{1/2}) \text{ by Tadič} \\ |z(|\rho'(\frac{t'-1}{2})|)| &= 1 \end{aligned}$$

  By part 2 of J-PS-S,

$$\begin{aligned} &\{\text{poles of } L(Q(\Delta) \times Q(\Delta'); T)\} = \\ &= \{q^{\frac{t_{max}-1}{2}}\} \cdot \{q^{\frac{1-t_{min}}{2}}, q^{\frac{3-t_{min}}{2}}, \ldots, q^{\frac{t_{min}-3}{2}}, q^{\frac{t_{min}-1}{2}}\} \cdot \\ &\quad \cdot \{\text{poles of } L(\rho(\frac{t-1}{2}) \times \rho'(\frac{t'-1}{2}); T)\} \end{aligned}$$

- By part 3 of J-PS-S, if $z \in \{\text{poles of } L(\rho(\frac{t-1}{2}) \times \rho'(\frac{t'-1}{2}); T)\}$, then

$$\begin{cases} |z| = 1 \text{ if } \pi \text{ is tempered.} \\ |z| \in (q^{-1/2}, q^{1/2}) \text{ if } \pi \text{ is unitarizable.} \end{cases}$$

- So if $z \in \{\text{poles of } L(Q(\Delta) \times Q(\Delta'); T)\}$, then

$$\begin{cases} |z| \geq q^{\frac{t_{max}-1}{2}} \cdot q^{\frac{1-t_{min}}{2}} \cdot 1 \geq 1 \text{ if } \pi \text{ is tempered.} \\ |z| > q^{\frac{t_{max}-1}{2}} \cdot q^{\frac{1-t_{min}}{2}} \cdot q^{-1/2} \geq q^{-1/2} \text{ if } \pi \text{ is unitarizable.} \end{cases}$$

Proposition (Analytic criterion for temperedness) Let $\pi = Q(\Delta_1 = (\rho_1, \ldots, \rho_1(t_1 - 1)), \ldots, \Delta_k = (\rho_k, \ldots, \rho_k(t_k - 1))) \in Rep_{adm}(GL_r(F), \mathbb{C})$ be an irreducible admissible **generic unitarizable** representation. Suppose for each $j \in \{1, \ldots, k\}$ the following holds:

$$(*)_j \begin{cases} \text{Let } \rho \text{ be any unramified twist of } \rho_j \\ \text{such that } |\rho| = 1 \text{ (trivial quasi-character of } \mathbb{G}_m(F)) \\ \text{(so } \rho \text{ is supercuspidal unitarizable, hence tempered).} \\ \text{Then for any } z \in \{\text{poles of } L(\pi \times \rho^\vee; T)\}, \\ \text{one has } |z| \in (q^{1/2})^{\mathbb{Z}} \subset \mathbb{C}^\times \end{cases}$$

Then π is **tempered**.

Proof By Jacquet's theorem, it suffices to show:

$$\text{For all } j \in \{1, \ldots, k\}, z(|\Delta_j|) = 1 \in \mathbb{C}^\times.$$

- Fix j, pick ρ to be any unramified twist of ρ_j with $|\rho| = 1$.
- By part 1 of J-PS-S, $L(\pi \times \rho^\vee; T) = \prod_{i=1}^k L(Q(\Delta_i) \times \rho^\vee; T)$.
- By part 2 of J-PS-S, this equals $\prod_{i=1}^k L(\rho_i(t_i - 1) \times \rho^\vee; T)$.
- By part 3 of J-PS-S, the jth factor of this product gives a pole $z \in \mathbb{C}^\times$ of $L(\pi \times \rho^\vee; T)$ with $|z| = z(|\rho_j|) \cdot q^{t_i - 1} \in \mathbb{R}^\times_{>0}$.
- So hypothesis $(*)_j$ implies $z(|\rho_j|) \in (q^{1/2})^{\mathbb{Z}}$, so
$$z(|\Delta_j|) = z(|\rho_j(\frac{t_j - 1}{2})|) \in (q^{1/2})^{\mathbb{Z}}$$
- By Tadič, if π is unitarizable, then $z(|\Delta_j|) \in (q^{-1/2}, q^{1/2})$, so $z(|\Delta_j|) = 1$.

April 29, 2003

Results from last time

1. **Location of poles of local L-factors** Let $\pi_x \in Rep_{adm}(GL_r(F_x))$ and $\pi'_x \in Rep_{adm}(GL_r(F_x))$ be irreducible admissible generic representations (local). If one of π_x, π'_x is tempered and the other is unitarizable, then $L(\pi_x \times \pi'_x; T)$ and $L(\pi_x^\vee \times \pi_x'^\vee; \frac{1}{qT})$ have **no poles in common**. (consequence of a stronger result)

2. **Analytic criterion for temperedness** (local analogue of converse theorem) Let $\pi_x \in Rep_{adm}(GL_r(F_x))$ be an irreducible admissible generic unitarizable representation. Suppose for any $r' \leq r$, any $\rho \in Rep_{adm}(GL_{r'}(F_x))$ supercuspidal unitarizable, and any $z \in \{\text{poles of } L(\pi_x \times \rho^\vee; T)\}$, one has $|z| = |q_x^{1/2}|^{\mathbb{Z}} \subset \mathbb{C}^\times$. Then π_x is tempered.

Note that if π_x is unramified, then temperedness is equivalent to $|z_i(\pi_x)| = 1$, by Jacquet.

Today Proof of Deligne's principle of induction, part 2.

Theorem Let $r \geq 2$, $\iota : \overline{\mathbb{Q}_l} \xrightarrow{\sim} \mathbb{C}$. Assume for all r' with $1 \leq r' < r$, that $(\mathcal{A} \to \mathcal{G})_\iota^{r'}$ and $(RP)_\iota^{r'}$ hold. Assume further that $(\mathcal{A} \to \mathcal{G})_\iota^r$ and $(P)_\iota^r$ hold. Then $(RP)^r$ and $(L\epsilon)_\iota^r$ hold.

Proof (Beginning) Let $\pi \in \mathcal{A}^r(F, \mathbb{C})$ be given. By $(\mathcal{A} \to \mathcal{G})_\iota^r$, we get $\sigma \in \mathcal{G}^r(F, \overline{\mathbb{Q}_l})$ in Langlands correspondence with respect to ι with π, such that $S_\sigma \subseteq S_\pi$. Let $\psi \in Hom_{\text{top. ab. gp.}}(F\backslash\mathbb{A}, \mathbb{R}/\mathbb{Z}) \setminus \{0\}$. By class field theory, we get a corresponding meromorphic differential $\omega \in \Omega^1_{F/k} \setminus \{0\}$.

Key lemma Let $r' \leq r$ and $\pi' \in \mathcal{A}^{r'}(F, \mathbb{C})$ be given. By $(\mathcal{A} \to \mathcal{G})_\iota^{r'}$, we get $\sigma' \in \mathcal{G}^{r'}(F, \overline{\mathbb{Q}_l})$ in Langlands correspondence with respect to ι with π', such that $S_{\sigma'} \subseteq S_{\pi'}$. Then for all $x \in |X|$,

$$\frac{L(\pi_x \times \pi'_x; T)}{\epsilon(\pi_x \times \pi'_x, \psi_x; T) \cdot L(\pi_x^\vee \times \pi_x'^\vee; \frac{1}{q_x T})} = \iota \frac{L(\sigma_x \otimes \sigma'_x; T)}{\epsilon(\sigma_x \otimes \sigma'_x, \omega_x; T) \cdot L(\sigma_x^\vee \otimes \sigma_x'^\vee; \frac{1}{q_x T})} \in \mathbb{C}(T)^\times.$$

Call this equation $\#$. Assume this for now (the proof is essentially computation).

Theorem (Deligne, local monodromy analysis) Suppose $\sigma \in Rep_{cts}(Gal(\overline{F}/F), \overline{\mathbb{Q}_l})$ is almost everywhere unramified, and is **pure** of some weight $w \in \mathbb{Z}$. Then $j_*\sigma \in Shv_c(X, \overline{\mathbb{Q}_l})$ is mixed of weight $\leq w$, i.e., for any $x \in |X|$, $x^* j_* \sigma \in Shv_m(x, \overline{\mathbb{Q}_l})$ has weights $\leq w$.

$$\begin{array}{ccc} \sigma & & j_*\sigma \\ | & & | \\ \mathrm{Spec}(F) = \eta & \overset{j}{\hookrightarrow} & X \end{array}$$

Corollary For any $x \in |X|$, any $z \in \{$poles of $L(\sigma_x; T) = L(x^* j_* \sigma; T) = \frac{1}{\prod(1-\alpha T)}$ in $\overline{\mathbb{Q}_l}\}$, and any $\iota : \overline{\mathbb{Q}_l} \xrightarrow{\sim} \mathbb{C}$, one has $|\iota z|_\mathbb{C} \in (q_x^{1/2})^{\mathbb{Z}_{\geq -w}}$.

In particular, $L(\sigma_x \otimes \sigma'_x; T)$ has poles satisfying $|\iota z| \in (q^{1/2})^{\mathbb{Z}_{\geq 0}}$, and $L(\sigma_x^\vee \otimes \sigma_x'^\vee; \frac{1}{q_x T})$ has poles satisfying $|\iota z| \in (q^{1/2})^{\mathbb{Z}_{\leq -2}}$, so there is **no cancellation**.

Lemma (Local-to-global extension of supercuspidals) Let $x \in |F|$ and $r \geq 1$ be given, and let $\rho \in Rep_{adm}(GL_r(F_x))$ be a supercuspidal unitarizable irreducible representation with χ_ρ of finite order. Then there exist $\pi = \bigotimes^{res}_{x \in |X|} \pi_x \in \mathcal{A}^r(F, \mathbb{C})$, i.e., cuspidal automorphic irreducible representations of $GL_r(\mathbb{A})$ with finite order χ_π, such that $\pi_x \cong \rho$ in $Rep_{adm}(GL_r(F_x))$.

Proof Lafforgue gave a one paragraph proof using the Arthur-Selberg trace formula.

$(RP)^r$: Fix $x \in |X|$. We want to show that π_x is tempered. By the analytic criterion of temperedness, it suffices to show that for any $r' \leq r$, and $\rho \in Rep_{adm}(GL_{r'}(F_x))$ a supercuspidal unitarizable irreducible representation with χ_ρ of finite order, and any $z \in \{\text{poles of } L(\pi_x \times \rho^\vee; T)\}$, we have $|z| \in (q_x^{1/2})^{\mathbb{Z}}$. Pick r' and ρ as above. Apply the local-to-global extension lemma, and get $\pi' \in \mathcal{A}^{r'}(F, \mathbb{C})$ such that $\pi'_x \cong \rho$. Apply $(\mathcal{A} \to \mathcal{G})_\iota^{r'}$ to get $\sigma' \in \mathcal{G}^{r'}(F, \overline{\mathbb{Q}_l})$ in Langlands correspondence with respect to ι with π', such that $S_{\sigma'} \subseteq S_{\pi'}$.

Our situation

- We have σ pure of weight 0 by $(P)_\iota^r$.
- We have σ' pure of weight 0 by $(RP)^{r'}$ if $r' < r$ and by $(P)_\iota^r$ if $r' = r$.
- π_x is **unitarizable** (hence the central character has finite order).
- $\pi'_x \cong \rho^\vee$ is supercuspidal and unitarizable, hence **tempered**.

Steps:

1. The key lemma says:
$$\frac{L(\pi_x \times \pi'_x; T)}{\epsilon(\pi_x \times \pi'_x, \psi_x; T) \cdot L(\pi_x^\vee \times \pi_x'^\vee; \frac{1}{q_x T})} = \iota \frac{L(\sigma_x \otimes \sigma'_x; T)}{\epsilon(\sigma_x \otimes \sigma'_x, \omega_x; T) \cdot L(\sigma_x^\vee \otimes \sigma_x'^\vee; \frac{1}{q_x T})}.$$
2. The "locations of poles" result implies $L(\pi_x \times \pi'_x; T)$ and $L(\pi_x^\vee \times \pi_x'^\vee; \frac{1}{q_x T})$ have **no** cancellation.
3. Deligne's monodromy analysis implies $L(\sigma_x \otimes \sigma'_x; T)$ and $L(\sigma_x^\vee \otimes \sigma_x'^\vee; \frac{1}{q_x T})$ have no cancellation.
4. ϵ-factors are monomials, and local L-factors have constant term 1, so
$$L(\pi_x \times \pi'_x; T) = \iota L(\sigma_x \times \sigma'_x; T)$$
$$\epsilon(\pi_x \times \pi'_x, \psi_x; T) = \iota \epsilon(\sigma_x \otimes \sigma'_x, \omega_x; T)$$

For $(RP)^r$, observe that $L(\pi_x \times \rho; T) = L(\pi_x \times \pi'_x; T) = \iota L(\sigma_x \otimes \sigma'_x; T)$, which implies $|\iota z| \in (q^{1/2})^{\mathbb{Z}_{\geq 0}}$, so we win.

$(L\epsilon)_\iota^r$. Fix $r' \leq r$, $\pi' \in \mathcal{A}^{r'}(F, \mathbb{C})$, and $\sigma' \in \mathcal{G}^{r'}(F, \overline{\mathbb{Q}_l})$ in Langlands correspondence with respect to ι. Fix $x \in |X|$. We want to show that the result of step 4 holds. Our situation is as above: σ is pure of weight 0 by $(P)_\iota^r$, σ' is pure of weight 0 by $(RP)^{r'}$ if $r' < r$ and by $(P)_\iota^r$ if $r' = r$, π_x is unitarizable, and π'_x is tempered - $(RP)^{r'}$ was just proved for $r' = r$. We do steps 1-4, and get the final statement.

Proof of key lemma We have π and σ in Langlands correspondence with respect to ι. Also π' and σ'. We want the big equality of fractions $\#$ from about two pages ago. Let $S := S_\pi \cup S_{\pi'} \cup \{\text{ramified places of } \psi \text{ and } \omega\}$. This is still finite. From the twisting theorems of Henniart and Deligne-Henniart, one knows that for any $\chi \in \mathcal{A}^1(F,\mathbb{C})$ (equivalently, in $\mathcal{G}^1(F,\overline{\mathbb{Q}_l})$), either for all $x \in |X| \setminus S$ or for all $x \in |X|$ with χ_x sufficiently ramified with respect to $\begin{cases}\pi_x \\ \sigma_x\end{cases}, \begin{cases}\pi'_x \\ \sigma'_x\end{cases}, \begin{cases}\psi_x \\ \omega_x\end{cases}$:

$$\begin{aligned} L(\pi_x \cdot \chi_x \times \pi'_x; T) &= \iota L(\sigma_x \otimes \chi_x \otimes \sigma'_x; T) \\ L(\pi_x^\vee \cdot \chi_x^{-1} \times \pi_x'^\vee; T) &= \iota L(\sigma_x \otimes \chi_x^{-1} \otimes \sigma_x'^\vee; T) \\ \epsilon(\pi_x \cdot \chi_x \times \pi'_x, \psi_x; T) &= \iota\epsilon(\sigma_x \otimes \chi_x \otimes \sigma'_x, \omega_x; T) \end{aligned}$$

Clearly, $\#$ holds when $x \in |X| \setminus S$. It remains to treat $x \in S$.

Fix $x \in S$. Choose $\chi \in \begin{cases}\mathcal{A}^1(F,\mathbb{C}) \\ \mathcal{G}^1(F,\overline{\mathbb{Q}_l})\end{cases}$ such that

- χ_x is trivial.
- For any $y \in S \setminus \{x\}$, χ_y is sufficiently highly ramified with respect to $\begin{cases}\pi_y \\ \sigma_y\end{cases}$ $\begin{cases}\pi'_y \\ \sigma'_y\end{cases}$ $\begin{cases}\psi_y \\ \omega_y\end{cases}$.

(Use Artin-Schrier cover and gemoetric reduction to $\mathbb{P}^1$, or apply the Grunwald-Wang theorem from class field theory, viz., one can interpolate any finite set of local characters by a global one.) We substitute $T \mapsto T^{deg(x)}$. Then the left side of $\#$ is:

$$\frac{L(\pi_x\chi_x \times \pi'_x; T^{deg(x)})}{\epsilon(\pi_x\chi_x \times \pi'_x, \psi_x; T^{deg(x)}) \cdot L(\pi_x^\vee \chi_x^{-1} \times \pi_x'^\vee; \frac{1}{(qT)^{deg(x)}})} =$$

$$= \frac{\left[\prod_{y \in |X| \setminus \{x\}} \epsilon(\pi_y\chi_y \times \pi'_y, \psi_y; T^{deg(y)})\right]\left[\prod_{y \in |X| \setminus \{x\}} L(\pi_y^\vee \chi_y^{-1} \times \pi_y'^\vee; \frac{1}{(qT)^{deg(y)}})\right]}{\left[\prod_{y \in |X| \setminus \{x\}} L(\pi_y\chi_y \times \pi'_y; T^{deg(y)})\right]} \text{ in } \mathbb{C}^\times$$

by

- definition of L- and ϵ-functions as Euler products.
- rationality of global L-functions.
- convergence of glocal ϵ-functions.
- global functional equation.

We continue:

$$= \iota \frac{\left[\prod_{y\in|X|\setminus\{x\}} \epsilon(\sigma_y\chi_y \times \sigma'_y, \omega_y; T^{deg(y)})\right] \left[\prod_{y\in|X|\setminus\{x\}} L(\sigma_y^\vee \chi_y^{-1} \times \sigma_y'^\vee; \frac{1}{(qT)^{deg(y)}})\right]}{\left[\prod_{y\in|X|\setminus\{x\}} L(\sigma_y\chi_y \times \sigma'_y; T^{deg(y)})\right]} \text{ in } \mathbb{C}^\times$$

by twisting theorems.

$$= \iota \frac{L(\sigma_x \otimes \chi_x \otimes \sigma'_x; T^{deg(x)})}{\epsilon(\sigma_x \otimes \chi_x \otimes \sigma'_x, \omega_x; T^{deg(x)}) \cdot L(\sigma_x^\vee \otimes \chi_x^{-1} \otimes \sigma_x'^\vee; \frac{1}{(qT)^{deg(x)}})}$$

by

- product formulas of global L- and ϵ- functions of Grothendieck and Laumon.
- rationality of global L-functions.
- convergence of global ϵ-functions.
- global functional equation.

This gives us the right side of $\#$.

May 6, 2003

We have shown: For $r \geq 2$, $\iota : \overline{\mathbb{Q}_l} \xrightarrow{\sim} \mathbb{C}$,

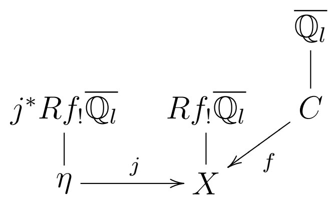
1. Assume for all r' with $1 \leq r' < r$, $(\mathcal{A} \to \mathcal{G})_\iota^{r'}$ and $(L\epsilon)_\iota^{r'}$ hold. Then $(\mathcal{A} \leftarrow \mathcal{G})^r)_\iota$ holds (converse theorem).
2. Assume for all r' with $1 \leq r' < r$, that $(\mathcal{A} \to \mathcal{G})_\iota^{r'}$ and $(RP)_\iota^{r'}$ hold, and assume that $(\mathcal{A} \to \mathcal{G})_\iota^r$ and $(P)_\iota^r$ hold. Then $(RP)^r$ and $(L\epsilon)_\iota^r$ hold (analytic criterion for temperedness).

What is left

Theorem (Lafforgue) Assume for any r' with $1 \leq r' < r$, $(\mathcal{A} \leftarrow \mathcal{G})_\iota^{r'}$, $(\mathcal{A} \to \mathcal{G})_\iota^{r'}$, and $(P)_\iota^{r'}$ hold. Then $(\mathcal{A} \to \mathcal{G})_\iota^r$ holds. If furthermore $(\mathcal{A} \leftarrow \mathcal{G})_\iota^r$ holds, then $(P)_\iota^r$ holds.

Theorem (Zelevinsky) $\pi = Q(\Delta_1, \ldots, \Delta_k)$ is **generic** if and only if no two segments are linked.

Suppose you have an X-scheme C/X with an action of $GL_r(\mathbb{A})$:



Hope: $j^* Rf_! \overline{\mathbb{Q}_l} \cong \bigoplus^{finite} (\sigma \otimes \pi)$ as a representation of $Gal(\overline{F}/F) \times GL_r(\mathbb{A})$. We need another point of view for $GL_r(\mathbb{A})$ representations.

Hecke Algebras Let G be a topological group that is locally compact, Hausdorff, and totally disconnected. We get an abelian category $Rep_{adm}(G, \mathbb{C})$. Objects are typically written (V, π). Assume further that G is **unimodular** (i.e., that one/every left Haar measure is right G-invariant, e.g., G is a reductive or unipotent group over a local field or $\mathbb{A}$). If dg is a left Haar measure, and I_1 and I_2 are open compact subgroups of G, then $\frac{vol(I_1,dg)}{vol(I_2,dg)} \in \mathbb{Q}^\times_{>0}$.

Definition A Haar measure dg on G is **rational** if and only if $vol(I, dg) \in \mathbb{Q}^\times_{>0}$ for one/every open compact subgroup $I \subseteq G$.

Fix such a rational Haar measure dg. e.g., if $G = GL_r(\mathbb{A})$, then $K := GL_r(\mathcal{O}_\mathbb{A}) = \prod_{x \in |X|} GL_r(\mathcal{O}_{F_x})$ is a maximal open compact subgroup. We want $vol(K, dg) = 1$.

Definition The **Hecke algebra of** G **with respect to** dg is $\mathcal{H} = \mathcal{H}(G, dg) := C_c^\infty(G, \mathbb{Q})$. This is the $\mathbb{Q}$-vector space of $\mathbb{Q}$-valued locally constant compactly supported functions on G, endowed with the **convolution product**:

$$\begin{aligned} &\mathcal{H} \times \mathcal{H} \to \mathcal{H} \\ &f_1 \quad f_2 \mapsto f_1 * f_2 := (x \mapsto \int_G f_1(xg) f_2(g^{-1}) dg = \int_G f_1(g) f_2(g^{-1}x) dg \end{aligned}$$

This is an associative algebra, but it has a unit if and only if G is finite (need a point distribution at identity).

Definition For any open compact subgroup $I \subset G$, let $e_I := \frac{1}{vol(I,dg)}$(characteristic function of I in G) $\in \mathcal{H}$. This is an idempotent element: $e_I * e_I = e_I$. We get:

$$\begin{aligned} e_I * \mathcal{H} * e_I \subset \mathcal{H} \qquad & \mathbb{Q}\text{-subalgebra of } I\text{-bi-invariant functions on } G \\ e_I * \mathcal{H} \subset \mathcal{H} \qquad & \text{left } e_I * \mathcal{H} * e_I\text{-module, right } \mathcal{H}\text{-ideal inside } \mathcal{H} \\ H * e_I \subset \mathcal{H} \qquad & \text{right } e_I * \mathcal{H} * e_I\text{-module, left } \mathcal{H}\text{-ideal inside } \mathcal{H} \end{aligned}$$

If $f \in \mathcal{H}$, then

$$\begin{aligned} e_I * f &= (x \mapsto \int_G e_I(g) f(g^{-1}x) dg = \frac{1}{vol(I, dg)} \int_I f(gx) dg) \\ f * e_I &= (x \mapsto \frac{1}{vol(I, dg)} \int_I f(xg) dg) \end{aligned}$$

If E is any field of characteristic 0, we get $\mathcal{H}_E := \mathcal{H} \otimes_\mathbb{Q} E$, and we define $e_I * \mathcal{H}_E * e_I$, etc. similarly.

Definition A left $\mathcal{H}$-module V is called **smooth** (or **nondegenerate**) if and only if for any $v \in V$ there exists $f \in \mathcal{H}$ such that $f * v = v$.

We get $Mod_{smooth}(\mathcal{H}) \subseteq Mod(\mathcal{H})$ the full subcategory of smooth modules. This is stable under subquotients, finite direct limits, and finite inverse limits. $Mod_{smooth}(\mathcal{H})$ is an abelian category.

Now, replace $\mathcal{H}$ with $\mathcal{H}_{\mathbb{C}}$.

Define a functor $Rep_{smooth}(G, \mathbb{C}) \to Mod_{smooth}(\mathcal{H})$ by $(V, \pi) \mapsto V$, where V on the right side is equipped with a left $\mathcal{H}$ module structure by $\mathcal{H} \times V \to V$ given by $(f, v) \mapsto f * v := \pi(f)v = \int_G f(g)(\pi(g)v)dg$.

Define a functor $Mod_{smooth}(\mathcal{H}) \to Rep_{smooth}(G, \mathbb{C})$ by $M \mapsto (M, \pi_M)$, where for any $m \in M$ and $g \in G$, $\pi(g)m$ is defined as follows:

- Choose $f \in \mathcal{H}$ such that $f * v = v$.
- Choose $I \subset G$ and open compact subgroup such that $e_I * f = f$. Then $e_I * v = e_I * f * v = f * v = v$.
- Set $(\delta_g * e_I) := (x \mapsto e_I(g^{-1}x)) \in \mathcal{H}$.
- Set $\pi_M(g)v := (\delta_g * e_I) * v \in M$. This is well-defined.

Lemma The above functors $Rep_{smooth}(G, \mathbb{C}) \leftrightarrows Mod_{smooth}(\mathcal{H})$ are equivalences of categories, quasi-inverse to each other.

Fix $I \subset G$ an open compact subgroup. We get a functor

$$\begin{aligned} Mod_{smooth}(\mathcal{H}) &\to Mod(e_I * \mathcal{H} * e_I) \\ (V, *) &\mapsto (e_I * V, *) \end{aligned}$$

Note that $e_I * V = V^I$, the left I-invariant vectors.

Lemma This functor is **exact**,

$$\begin{aligned} \text{with left adjoint} \quad Mod(e_I * \mathcal{H} * e_I) &\to Mod_{smooth}(\mathcal{H}) \\ M &\mapsto (\mathcal{H} * e_I) \otimes_{e_I * \mathcal{H} * e_I} M \\ \text{and right adjoint} \quad M &\mapsto (Hom_{\text{left } e_I * \mathcal{H} * e_I}(e_I * \mathcal{H}, M))^{smooth} \end{aligned}$$

and adjunction morphisms:

$$\begin{aligned} M &\mapsto e_I * ((\mathcal{H} * e_I) \otimes_{e_I * \mathcal{H} * e_I} M) \\ e_I * (Hom_{e_I * \mathcal{H} * e_I}(e_I * \mathcal{H}, M))^{smooth} &\mapsto M \end{aligned}$$

Both are isomorphisms.

Proposition The functor

$$\begin{aligned} Mod_{smooth}(\mathcal{H}) &\to Mod(e_I * \mathcal{H} * e_I) \\ (V, *) &\mapsto (e_I * \mathcal{H} * e_I) \end{aligned}$$

induces a bijection

$$\left\{ \begin{array}{c} \text{isomorphism classes} \\ \text{of irreducible nonzero} \\ (V,*) \in Mod_{smooth}(\mathcal{H}) \\ \text{with } e_I * V \neq 0 \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{isomorphism classes} \\ \text{of irreducible nonzero} \\ (M,*) \in Mod(e_I * \mathcal{H} * e_I) \end{array} \right\}$$

Recall:

$$\begin{array}{ccccc} Rep_{adm}(G,\mathbb{C}) & \leftrightarrows & Mod_{adm}(\mathcal{H}) & = & \left\{ \begin{array}{c} \text{smooth } V \text{ such that} \\ dim(e_I * V = V^I) < \infty \end{array} \right\} \\ \cap & & \cap & & \\ Rep_{smooth}(G,\mathbb{C}) & \leftrightarrows & Mod_{smooth}(\mathcal{H}) & & \end{array}$$

Key lemma/construction Let $(V,*) \in Mod_{smooth}(\mathcal{H})$ be a smooth left $\mathcal{H}$-module, and let $M \subseteq e_I * V$ be any subobject in $Mod(e_I * \mathcal{H} * e_I)$. Define $\widetilde{M} := (\mathcal{H} * e_I) \otimes_{e_I * \mathcal{H} * e_I} M \in Mod_{smooth}(\mathcal{H})$. This has adjunction morphism $\widetilde{M} \to V$ given by $f \otimes m \mapsto f * m$. Define $\overline{M} := image(\widetilde{M} \to V) \hookrightarrow V$ in $Mod_{smooth}(\mathcal{H})$. Then (here's the assertion) we have isomorphisms $M \xrightarrow{\sim} e_I * \widetilde{M} \xrightarrow{\sim} e_I * \overline{M}$ in $Mod(e_I * \mathcal{H} * e_I)$.

Proof Apply the **exact** functor $e_I * -$ to $\widetilde{M} \twoheadrightarrow \overline{M} \hookrightarrow V$. We get a diagram:

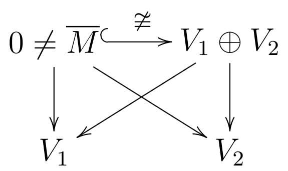
$$\begin{array}{ccccc} e_I * \widetilde{M} & \twoheadrightarrow & e_I * \overline{M} & \hookrightarrow & e_I * V \\ \text{adjunction} \uparrow & & & & \uparrow \text{subobject} \\ M & & \xrightarrow{\cong} & & M \end{array}$$

and we need to check that it commutes.

Three things to prove for the proposition: Well-defined, injective, and surjective.

1. Suppose $(V,*) \in Mod_{smooth}$ is irreducible, such that $e_I * V \neq 0$. We want $e_I * V$ irreducible as a $e_I * \mathcal{H} * e_I$-module. If M is a nonzero subobject of $e_I * V$ in $Mod(e_I * \mathcal{H} * e_I)$, we get $\overline{M} \subseteq V$ in $Mod(\mathcal{H})$, nonzero by the key lemma. If V is irreducible, then $\overline{M} = V$, so $M \cong e_I * \overline{M} = e_I * V$. Thus, $e_I * V$ is irreducible.

2. **Injectivity** Suppose $(V_1,*)$ and $(V_2,*)$ are both irreducible smooth $\mathcal{H}$-modules with $e_I * V_1 \neq 0 \neq e_I * V_2$. Let $\alpha : e_I * V_1 \xrightarrow{\sim} e_I * V_2$ be an isomorphism in $Mod(e_I * \mathcal{H} * e_I)$. We want an isomorphism in $Mod_{smooth}(\mathcal{H})$. Set $M = image(e_I * V_1 \to e_I * V_1 \oplus e_I * V_2)$, where the map is given by $x \mapsto x \oplus \alpha x$. M is nonzero, and irreducible in $Mod(e_I * \mathcal{H} * e_I)$. We get



in $Mod_{smooth}(\mathcal{H})$. The arrows coming from $\overline{M}$ are nonzero, hence surjective. Goursat's lemma implies $\overline{M}$ is the graph of an isomorphism $V_1 \xrightarrow{\sim} V_2$.

3. **Surjectivity** Suppose $(M, *) \in Mod(e_I * \mathcal{H} * e_I)$ is irreducible. We get $\widetilde{M} = (\mathcal{H} * e_I) \otimes_{e_I * \mathcal{H} * e_I} M$ in $Mod_{smooth}(\mathcal{H})$, and $\widetilde{M_0} := \{m \in \widetilde{M} : e_I * \mathcal{H} * m = 0 \text{ in } \widetilde{M}\}$. We get an exact sequence in $Mod_{smooth}(\mathcal{H})$:

$$0 \to \widetilde{M_0} \to \widetilde{M} \to \widetilde{M}/\widetilde{M_0} \to 0$$

Claim: For any sub-$\mathcal{H}$-module $V \subseteq \widetilde{M}$, either $V \subseteq \widetilde{M_0}$ or $V = \widetilde{M}$ (this implies $\widetilde{M}/\widetilde{M_0}$ is irreducible). Assuming this, we apply the exact functor $e_I * -$ to the above exact sequence to get

$$0 \to e_I * \widetilde{M_0} \to e_I * \widetilde{M} \to e_I * (\widetilde{M}/\widetilde{M_0}) \to 0$$

$e_I * \widetilde{M_0} = 0$ by definition, so the next two terms are isomorphic. Thus, we have $M \xrightarrow{\sim} e_I * \widetilde{M} \xrightarrow{\sim} e_I * (\widetilde{M}/\widetilde{M_0}$ irreducible)

Proof of claim If $e_I * V = 0$, then $V \subseteq \widetilde{M_0}$. Suppose $e_I * V \neq 0$. Then the morphism $e_I * V \hookrightarrow e_I * \widetilde{M}$ ($\cong M$ by key Lemma) is an isomorphism by irreducibility. Then $V \supseteq e_I * V = e_I * \widetilde{M}$ (the first containment is as a vector space), and $e_I * \widetilde{M}$ generates $\widetilde{M}$ as an $\mathcal{H}$-module, so $V = \widetilde{M}$. check

May 13, 2003

Let k be a finite field of order q and characteristic p, and let X be a proper smooth geometrically connected curve over k. Let F be the function field of X, and let $\mathbb{A}$ be its ring of adèles. Let $r \geq 1$ be an integer, $G = GL_r(\mathbb{A})$, and $K = GL_r(\mathcal{O}_{\mathbb{A}}) = \prod'_{x \in |X|} GL_r(\mathcal{O}_x)$ the maximal compact subgroup of G. Let $\mathcal{H} = \mathcal{H}(G, dg$ such that $vol(K, dg) = 1) \otimes \mathbb{C}$ be the Hecke algebra.

Definition A **level** is a closed subscheme $N \hookrightarrow X$ that is finitely supported.

For each level $N \hookrightarrow X$, let $K_N := ker(K \twoheadrightarrow GL_r(\mathcal{O}_N))$, which is an open compact subgroup of G. We get idempotents $e_N := [K : K_N] \cdot$ (characteristic function of K_N in G) $\in \mathcal{H}$.

Lemma $\mathcal{H} = \varinjlim_N e_N * \mathcal{H} * e_N$.

Recall from last time:

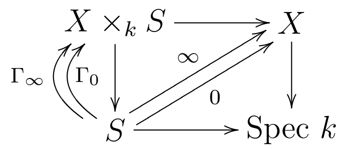
$$\left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{irreducible nonzero} \\ \mathcal{H}\text{-modules } (V, *) \\ \text{with } V^{K_N} \neq 0 \end{array} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{irreducible nonzero} \\ e_N * \mathcal{H} * e_N\text{-modules } (M, *) \end{array} \right\}$$

Notation For any k-scheme S, let $Frob_S$ denote the Frobenius k-endomorphism given by the qth power map. This is **not** the geometric Frobenius. Lafforgue systematically uses $Frob^{-1}$ in his L-function work.

Definition (Drinfel'd) A **(right) shtuka of rank** r **on a** k**-scheme** S is the following data:

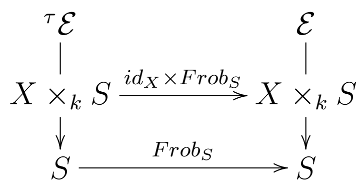
1. A vector bundle $\mathcal{E}$ on $X \times_k S$ of rank r (i.e., a locally free $\mathcal{O}_{X \times_k S}$-module of rank r).

2. A **(right) modification** of $\mathcal{E}$: This is a diagram $\mathcal{E} \overset{j}{\hookrightarrow} \mathcal{E}' \overset{t}{\hookleftarrow} \mathcal{E}''$, where $\mathcal{E}'$ and $\mathcal{E}''$ are vector bundles of rank r on $X \times_k S$, and j and t are **injective** homomorphisms of **coherent** $\mathcal{O}_{X \times_k S}$-modules whose cokernels (which are **invertible** $\mathcal{O}_S$-modules) are coherent $\mathcal{O}_{X \times_k S}$-modules supported on the graphs of morphisms $\infty, 0 : S \to X$.



The morphism ∞ is called the **pole** of the shtuka, and the morphism 0 is called the **zero** of the shtuka. $\Gamma_0 \supset supp(t)$ and $\Gamma_\infty \supset supp(j)$ are defined by the universal property of the fiber product $X \times_k S$.

3. An isomorphism $({}^\tau\mathcal{E} := (id_X \times Frob_S)^*\mathcal{E}) \xrightarrow{\sim} \mathcal{E}''$.



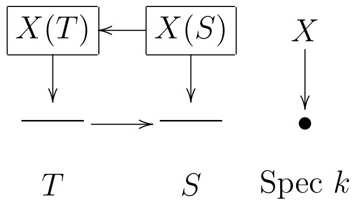
Let Cht^r denote the moduli stack of shtukas of rank r.

What is a stack? The idea came from SGA 1, on moduli of curves, and was crystallized in Deligne-Mumford, on stable cuves. Consider Yoneda's fully faithful embedding:

$$Sch/k \to Hom(Sch^{op}, Sets)$$
$$X \mapsto X(-) := Hom(-, X)$$

Suppose for example the X classifies genus 2 curves. The collections of points

$X(S), X(T)$ and maps between them gives all of the information on X.



The problem is, $X(S) = \{ \begin{pmatrix} C \\ \downarrow \\ S \end{pmatrix}, genus(C) = 2\}/\cong$ fails to give any automorphism structure. The solution is that instead of having a **set** $X(S)$, we let $X(S)$ be the **category** of all genus 2 curves over S.

We want a 2-functor which for every base scheme as input returns a **category** which classifies the functor. When the only morphisms are *id* on objects, then we have a set.

$$Hom(Sch_k^{op}, Sets) \subset 2 - Hom(Sch_k^{op}, Cat)$$

A stack is a 2-functor satsfying certain conditions. Morphisms are natural transformations satisfying a cartesian diagram of the form:

$$\begin{array}{ccc} X \times S & \longrightarrow & X(-) \\ \downarrow & & \downarrow \\ S & \longrightarrow & Y(-) \end{array}$$

where we have schemes on the left side, and stacks on the right.

Definition X is called smooth, proper, étale, etc. if and only if $X \times S$ is.

$Cht^r(S) :=$ category of shtukas of rank r over S.

Proposition Cht^r is a Deligne-Mumford algebraic stack, and the morphism $(\infty, 0) : Cht^r \to X \times X$ is **smooth** of relative dimension $2r - 2$.

Notes

- We won't define "Deligne-Mumford" but it is good news. It means we can pretend it is a scheme.
- $\widetilde{\mathcal{E}} = (\mathcal{E} \hookrightarrow \mathcal{E}' \hookleftarrow \mathcal{E}'') \in Cht^r(S)$ gives rise to two morphisms $\infty, 0 \in X(S)$, and these define the morphism in question.
- Cht^r is **not** quasi-compact, and **not** separated, e.g., fo $r = 2$, take $\mathbb{P}^2$, blow it up at a point, pick a point on the exceptional divisor, blow it up, and repeat infinitely many times.

Definition

- Let $N \hookrightarrow X$ be a **level**, and $\widetilde{\mathcal{E}} = (\mathcal{E} \overset{j}{\hookrightarrow} \mathcal{E}' \overset{t}{\hookleftarrow} \mathcal{E}'' \overset{\sim}{\leftarrow} {}^\tau\mathcal{E})$ a shtuka af rank r over S. Assume the pole and zero of $\widetilde{\mathcal{E}}$ avoids N. Then a **level** N **structure on** $\widetilde{\mathcal{E}}$ is an isomorphism

$$\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_N = \mathcal{E} \otimes_{\mathcal{O}_{X \times_k S}} \mathcal{O}_{N \times_k S} \xrightarrow{\sim} \mathcal{O}_{N \times_k S}^{\oplus r}$$

  of $\mathcal{O}_{N \times_k S}$-modules such that the diagram

$$\begin{array}{ccccccc} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_N & \overset{j}{\hookrightarrow} & \mathcal{E}' \otimes_{\mathcal{O}_X} \mathcal{O}_N & \overset{t}{\hookleftarrow} & \mathcal{E}'' \otimes_{\mathcal{O}_X} \mathcal{O}_N & \overset{\cong}{\longleftarrow} & {}^\tau\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_N \\ & {\scriptstyle u,\,\cong} \searrow & & & & \swarrow {\scriptstyle {}^\tau u,\,\cong} & \\ & & \mathcal{O}_{N \times_k S}^{\oplus r} & & & & \end{array}$$

  commutes.

- Let Cht_N^r denote the moduli stack of shtukas of rank r with level N structure.

We get a canonical "forget level structure" morphism:

$$Cht_N^r \xrightarrow{f} Cht^r \times_{X \times X} (X - N) \times (X - N).$$

Proposition f is a finite étale Galois cover with Galois group $GL_r(\mathcal{O}_N)$. Hence, Cht_N^r is smooth of relative dimension $2r - 2$ over $(X - N) \times (X - N)$.

We need to truncate Cht^r to get finite cohomology $\to$ trace formula $\to$ Galois representation. Morally, Cht^r encodes all of the Galois representations we want, but we can't see them as they are simultaneously realized. [pQFT analogy?]

Definiton A **polygon (of rank** r**)** is a function $p : [0, r] \to \mathbb{R}$ that is continuous, piecewise linear with integer break points, such that $p(0) = p(n) = 0$. It is **sufficiently convex** if and only if there exists $\lambda \gg 0$ such that for any r' with $1 \leq r' < r$, the difference of slopes $(p(r') - p(r' - 1)) - (p(r' + 1) - p(r'))$ is at least λ. check

Definition Let $\widetilde{\mathcal{E}} = (\mathcal{E} \overset{j}{\hookrightarrow} \mathcal{E}' \overset{t}{\hookleftarrow} \mathcal{E}'' \overset{\sim}{\leftarrow} {}^\tau\mathcal{E})$ be a shtuka over $S = \mathrm{Spec}$(algebraically closed field). Then the **degree** of $\widetilde{\mathcal{E}}$ is $deg(\widetilde{\mathcal{E}}) := deg(det(\widetilde{\mathcal{E}})) \in \mathbb{Z}$. The **slope** of $\widetilde{\mathcal{E}}$ is $\mu(\widetilde{\mathcal{E}}) := deg(\widetilde{\mathcal{E}})/rank(\widetilde{\mathcal{E}})$.

Let $\widetilde{\mathcal{E}}_\bullet = (0 = \widetilde{\mathcal{E}}_0 \subsetneq \widetilde{\mathcal{E}}_1 \subsetneq \cdots \subsetneq \widetilde{\mathcal{E}}_k = \widetilde{\mathcal{E}})$ be a strict filtration of $\widetilde{\mathcal{E}}$ by subobjects. The polygon $p^{\widetilde{\mathcal{E}}_\bullet} : [0, r] \to \mathbb{R}$ is the polygon with breakpoints at $rank(\widetilde{\mathcal{E}}_i)$, $i = 0, \ldots, k$ given by:

$$p^{\widetilde{\mathcal{E}}_\bullet}(rank(\widetilde{\mathcal{E}}_i)) := deg(\widetilde{\mathcal{E}}) - \mu(\widetilde{\mathcal{E}}) \cdot rank(\widetilde{\mathcal{E}}_i)$$

Proposition-Definition Among the polygons $p^{\widetilde{\mathcal{E}}_i}$ as $\widetilde{\mathcal{E}}_\bullet$ ranges over **all** filtrations of $\widetilde{\mathcal{E}}$, there exists a unique largest polygon $\overline{p}^{\widetilde{\mathcal{E}}_\bullet}$, called the **Harder-Narasimhan canonical polygon of $\widetilde{\mathcal{E}}_\bullet$.**

Note There is a finest filtration giving $\overline{p}$, called the **Harder-Narasimhan canonical filtration**.

Fix a level $N \hookrightarrow X$ and a polygon $p : [0, r] \to \mathbb{R}$. Let $Cht_N^{r,\overline{p} \leq p}$ be the substack of Cht_N^r whose geometric points S have their Harder-Narasimhan polygon $\overline{p}$ dominated by p.

$$\begin{array}{ccccc} \mathcal{E} & & & & \\ | & & & & \\ X \times S & \to & X & \hookleftarrow & N \\ \downarrow & & \downarrow & & \\ S & \longrightarrow & \bullet & & \end{array}$$

Proposition $Cht_N^{r,\overline{p} \leq p}$ is **open** in Cht_N^r, i.e.,

$$\begin{array}{ccc} \square & \longrightarrow & Cht_N^{r,\overline{p}\leq p} \\ {\scriptstyle \text{open}} \big\downarrow & & \big\downarrow \\ S & \longrightarrow & Cht_N^r \end{array}$$

with the leftmost arrow denoting the inclusion of an open subscheme. Even better, we have a stratification by degree:

$$Cht_N^{r,\overline{p} \leq p} = \coprod_{d \in \mathbb{Z}} Cht_N^{r,d,\overline{p} \leq p}$$

and the pieces are of finite type over $(X - N) \times (X - N)$. Thus, we can take cohomology.

Lemma There is a canonical action of $\mathbb{A}^\times$ on Cht^r and Cht_N^r that stabilizes $Cht_N^{r,\overline{p} \leq p}$.

Out of any idèle element $a \in \mathbb{A}^\times$, we can make a line bundle $\mathcal{L}_a : U \mapsto \{f \in F = \kappa(X) : a_x f_x \in \mathcal{O}_x \subset F_x, \forall x \in U\}$. The action comes from tensoring with $\mathcal{L}_a$. If $deg(a) \neq 0$, then the action is free.

Theorem (Lafforgue) Let $r \geq 1$, $N \hookrightarrow X$, $p : [0, r] \to \mathbb{R}$ be as before. Assume p is sufficiently convex, and $deg(a) \neq 0$.

1. There exists a moduli compactification $\overline{Cht_N^{r,\overline{p} \leq p}}/a^{\mathbb{Z}}$ of $Cht_N^{r,\overline{p} \leq p}/a^{\mathbb{Z}}$ (by "iterated shtukas" - relax the isomorphism condition on $\mathcal{E}'' \leftarrow {}^\tau\mathcal{E}$), i.e. bar?

$$\begin{array}{ccc} Cht_N^{r,\overline{p}\le p}/a^{\mathbb{Z}} & \hookrightarrow & \overline{Cht_N^{r,\overline{p}\le p}}/a^{\mathbb{Z}} \\ & {\scriptstyle f,\ \text{proper}} \searrow \quad \swarrow {\scriptstyle \text{fin. type, smooth}} & \\ & (X-N)\times(X-N) & \end{array}$$

Note: any scheme over a noetherian base admits a compactification, by Nagata's theorem. However, it is often difficult to characterize the extra points. The whole purpose of the (ref?) JAMS paper is to construct this compactifiation.

2. For any prime $l \neq p$, the cohomology sheaves

$$H_c^i(\overline{Cht_N^{r,\overline{p}\le p}}/a^{\mathbb{Z}}) := R^i f_! \overline{\mathbb{Q}_l}$$

where $\overline{\mathbb{Q}_l}$ denotes the constant sheaf on $\overline{Cht_N^{r,\overline{p}\le p}}/a^{\mathbb{Z}}$, are **lisse** sheaves on $(X - N) \times (X - N)$.

Definition (Lafforgue) Let X' and X'' be smooth curves over k (a finite field), and let $r \geq 1$ be an integer. A $\overline{\mathbb{Q}_l}$-lisse sheaf on $X' \times X''$ is called r**-negligible** if and only if **each** of its irreducible subquotients is a **direct factor** of some $q'^*\sigma' \otimes q''^*\sigma''$, where σ' and σ'' are irreducible of rank $< r$ on X' and X'', respectively.

$$\begin{array}{ccccc} \sigma' & & X'\times X'' & & \sigma'' \\ | & \swarrow {\scriptstyle q'} & & {\scriptstyle q''} \searrow & | \\ X' & & & & X'' \end{array}$$

A $\overline{\mathbb{Q}_l}$-lisse sheaf on $X' \times X''$ is called r**-essential** if and only if **none** of its irreducible subquotients is r-negligible.

Since the fiber dimension of Cht_N^r over $(X-N)\times(X-N)$ is $2r-2$, the cohomology only appears in dimensions $0, \ldots, 4r-4$. The most interesting is $2r-2$.

Let $\mathcal{L}_N = H_c^{2r-2}(\overline{Cht_N^{r,\overline{p}\le p}}/a^{\mathbb{Z}})$. This is a lisse $\overline{\mathbb{Q}_l}$-sheaf on $(X - N) \times (X - N)$. Set $F^0\mathcal{L}_N = 0$. Given $F^{2i}\mathcal{L}_N (i \geq 0)$, define:

- $F^{2i+1}\mathcal{L}_N$ such that $\frac{F^{2i+1}\mathcal{L}_N}{F^{2i}\mathcal{L}_N}$ is the sum of all lisse subsheaves of $\frac{\mathcal{L}_N}{F^{2i}\mathcal{L}_N}$ that are r-negligible.
- $F^{2i+2}\mathcal{L}_N$ such that $\frac{F^{2i+2}\mathcal{L}_N}{F^{2i+1}\mathcal{L}_N}$ is the sum of all lisse subsheaves of $\frac{\mathcal{L}_N}{F^{2i+1}\mathcal{L}_N}$ that are r-essential.

Theorem (Lafforgue) Assume $r \geq 2$. Given $\iota : \overline{\mathbb{Q}_l} \to \mathbb{C}$, pick a geometric generic point:

$$\begin{array}{ccccc} & \overline{\eta} \longrightarrow & (X-N)\times(X-N) & & \\ & \swarrow {\scriptstyle q'} & & {\scriptstyle q''} \searrow & \\ X-N & & & & X-N \end{array}$$

Assume for all r' with $1 \le r' < r$ that $(\mathcal{A} \to \mathcal{G})_\iota^{r'}$, $(\mathcal{A} \leftarrow \mathcal{G})_\iota^{r'}$, and $(P)_\iota^{r'}$ hold.

1. Each subquotient $\frac{F^{i+1}\mathcal{L}_N}{F^i\mathcal{L}_N}$ (for all $i \ge 0$) can be given with an action of

   $$(W(X-N,\overline{\eta}) \times W(X-N,\overline{\eta}) \times (e_N * \mathcal{H} * e_N \otimes_{\mathbb{C},\iota^{-1}} \overline{\mathbb{Q}_l})$$

   Note: W denotes Weil group. $\mathcal{H}$ moves between polygons, so we don't get an algebraic action on cohomology. Let ?

   $$\mathcal{L}_{N,ess} := \left(\bigoplus_{i\ge 0} \frac{F^{2i+2}\mathcal{L}_N}{F^{2i+1}\mathcal{L}_N}\right)^{ss},$$

   where ss denotes semisimplification. This is a semisimple representation of $(W(X-N,\overline{\eta}) \times W(X-N,\overline{\eta}) \times (e_N * \mathcal{H} * e_N \otimes_{\mathbb{C},\iota^{-1}} \overline{\mathbb{Q}_l})$. Let

   $$\mathcal{A}^r_{N,a}(F,\mathbb{C}) := \{\pi \in \mathcal{A}^r(F,\mathbb{C}) : \chi_F(a) = 1 \in \mathbb{C}^\times, V_\pi^{K_N} = e_N * V_\pi \ne 0\}.$$

2. $$\mathcal{L}_{N,ess} \cong \bigoplus_{\pi \in \mathcal{A}^r_{N,a}(F,\mathbb{C})} (\mathcal{L}_\pi \boxtimes \iota^{-1}(e_N * V_\pi))(1-r),$$

   where the $(1-r)$ denotes Tate twist, $\mathcal{L}_\pi$ is **pure of weight** 0, of the form $q'^*\sigma_\pi \otimes q''^*\sigma_\pi^\vee$, and σ_π is irreducible of rank r on $X-N$, and is in Langlands correspondence with respect to ι with π.

The proof involves heavy use of the Arthur-Selberg trace formula. [Did he really stare at the trace formula for six years?]

This gives $(\mathcal{A} \to \mathcal{G})_\iota^r$. By the converse theorem, we have $(\mathcal{G} \to \mathcal{A})_\iota^r$. Loop around, and get $(P)_\iota^r$.

I'll try to write up notes this summer and give them to you. [ha ha, that never happened]

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