Recent advances in moonshine

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JMS Special Lecture

What is moonshine?

Strange connections between finite groups and modular forms

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Strange connections between finite groups and modular forms

The connections should be “very special”

Infinitely many cases $\Rightarrow$ not moonshine!

Monstrous Moonshine (1978-1992)

Classification of finite simple groups (1982-2004)

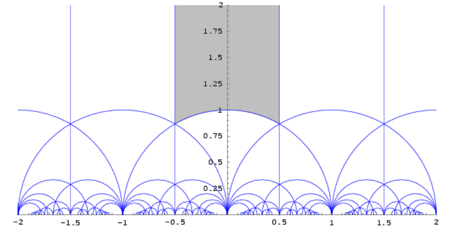
Any finite simple group is one of the following

- A cyclic group of prime order
- An alternating group A_n $(n \geq 5)$
- A group of Lie type (16 infinite families)
- One of 26 sporadic simple groups

Largest sporadic: Monster, about $8 \cdot 10^{53}$ elements (Griess 1982).
194 irred. repres. of dim $1, 196883, 21296876, \ldots$

$SL_2(\mathbb{Z})$ action on complex upper half-plane $\mathfrak{H}$

Generators: $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) : z \mapsto z + 1$, $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) : z \mapsto -1/z$



(Wikipedia)

J-function as Hauptmodul

The quotient space $SL_2(\mathbb{Z})\backslash\mathfrak{H}$ has genus zero. J generates the function field. Fourier expansion: $q^{-1} + 196884q + 21493760q^2 + \cdots (q = e^{2\pi i z})$

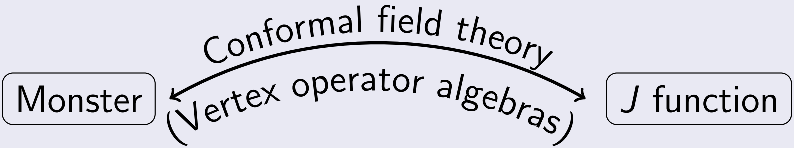
Coefficients of J and Irreducible Monster reps

$$\begin{aligned} 196884 &= 1 + 196883 \text{ (McKay, 1978)} \\ 21493760 &= 1+196883+21296876 \text{ (Thompson, 1979)} \\ 864299970 &= 2\times 1+2\times 196883+21296876+842609326 \\ \vdots & \quad \vdots \end{aligned}$$

How to continue this sequence?

McKay-Thompson conjecture: Natural graded rep $\bigoplus_{n=0}^{\infty} V_n$ of $\mathbb{M}$ such that $\sum \dim V_n q^{n-1} = J$.

Idea: Physics forms a bridge



Solution: Frenkel, Lepowsky, Meurman 1988

Constructed a vertex operator algebra $V^\natural = \bigoplus_{n \geq 0} V_n^\natural$ (the Moonshine Module), such that $\sum_{n \geq 0} (\dim V_n^\natural) q^{n-1} = J$ and $\mathrm{Aut}\, V^\natural = \mathbb{M}$.

Refined correspondence

Thompson's suggestion: replace graded dimension with graded trace of non-identity elements.

Monstrous Moonshine Conjecture (Conway, Norton 1979)

There is a faithful graded representation $V = \bigoplus_{n \geq 0} V_n$ of the monster $\mathbb{M}$ such that for all $g \in \mathbb{M}$, the series $T_g(\tau) = \sum_{n \geq 0} \mathrm{Tr}(g|V_n) q^{n-1}$ is the q-expansion of a congruence Hauptmodul (= "generates function field of genus 0 $\mathfrak{H}$-quotient").

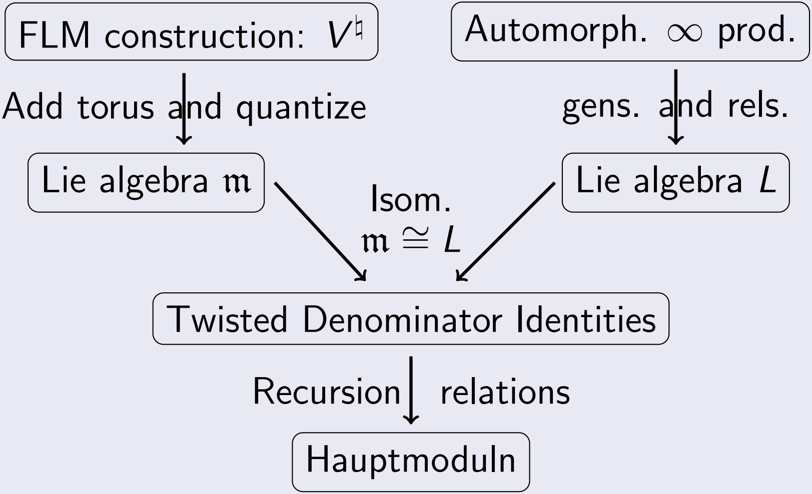
First proof (Atkin, Fong, Smith 1980)

Theorem: A virtual representation of $\mathbb{M}$ exists yielding the desired functions.
No construction.

Second proof (Borcherds 1992)

Theorem: The Conway-Norton conjecture holds for $V^\natural$.

Outline of Borcherds's proof



Add a torus

Functor: tensor with lattice VA $V^\natural \mapsto V^\natural \otimes V_{II_{1,1}}$.
Central charge increases by 2 (from 24 to 26).

Quantize (need central charge 26 = critical dim)

Old canonical quantization: Primary mod spurious.
Equivalent functor: H^1_{BRST}.
Get Lie algebra $\mathfrak{m}$ with monster action.

Oscillator cancellation (no-ghost theorem)

$\mathfrak{m}_{m,n} \cong V^\natural_{1+mn}$ when $(m,n) \neq (0,0)$.

Infinite product identity (Koike-Norton-Zagier)

$$J(\sigma) - J(\tau) = p^{-1} \prod_{m>0, n\in\mathbb{Z}} (1 - p^m q^n)^{c(mn)}$$

where $J(\tau) = \sum_{n\geq -1} c(n)q^n$, $p = e^{2\pi i\sigma}$, $q = e^{2\pi i\tau}$.

Remarkable property

Left side is pure in p and q.
Vanishing of pq^2 term $\Rightarrow c(4) = c(3) + \binom{c(1)}{2}$.
Get isom $V_5^\natural \cong V_4^\natural \oplus \Lambda^2(V_2^\natural)$ of monster reps.

End of Borcherds's proof

- All $\mathbb{M}$-reps $V_n^\natural$ are determined by $(V_n^\natural)_{n=0}^6$.
- Same for coefficients of McKay-Thompson series $T_g(\tau) = \sum_{n\geq 0} \mathrm{Tr}(g|V_n)q^{n-1}$.
- Theorem (Koike): Conway-Norton's candidate functions satisfy the same recursion relations.
- suffices to check first 7 terms.

Theorem (Cummins, Gannon 1997)

The recursion relations alone are sufficient to get $\Gamma_0(N)$-invariant Hauptmodul property.

Generalized Monstrous Moonshine (1987-2016)

Moonshine away from the monster?

Suggested by Conway-Norton 1979.
Computations by Queen 1980.
Example: Baby monster irreps $1, 4371, 96255, \ldots$
$q^{-1} + 4372q + 96256q^2 + \cdots$ is Hauptmodul for $\Gamma_0(2)^+$.

Strange observation (Norton)

Only groups "inside" the monster are interesting.
(central extensions of centralizers of elements)

The Conjecture (Norton 1987):

- $g \in \mathbb{M} \Rightarrow V(g)$ graded proj. rep. of $C_{\mathbb{M}}(g)$
- $(g, h), gh = hg \Rightarrow Z(g, h; \tau)$ holomorphic on $\mathfrak{H}$

1. q-expansion of $Z(g, h; \tau)$ is graded trace of (a lift of) h on $V(g)$.
2. Z is invariant under simultaneous conjugation of the pair (g, h) up to scalars.
3. $Z(g, h; \tau)$ constant or a Hauptmodul.
4. $Z(g, h; \frac{a\tau+b}{c\tau+d})$ proportional to $Z(g^a h^c, g^b h^d; \tau)$.
5. $Z(g, h; \tau) = J(\tau)$ if and only if $g = h = 1$.

Brute force solution (like Atkin-Fong-Smith)?

This is a finite problem:

- Finitely many conjugacy classes of commuting pairs, and possible levels are bounded.
- Central extensions of centralizers "can be computed".

Not finite enough for 2017

- We still haven't classified the commuting pairs.
- We still don't know character tables of all centralizers, let alone central extensions.

Physics Language (Dixon, Ginsparg, Harvey 1988)

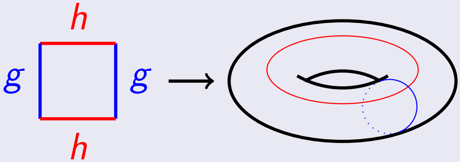
$V(g)$ - twisted sectors of a monster CFT.
$Z(g, h; \tau)$ - genus 1 partition functions (with twisted boundary conditions).
All except Hauptmodul claim (3) "follow" from conformal field theory considerations.

Algebraic Interpretation

$V(g) =$ irreducible g-twisted $V^\natural$-module $V^\natural(g)$
$Z(g, h; \tau) = \mathrm{Tr}(\tilde{h} q^{L(0)-1} | V(g))$.

Geometric interpretation of Z

Physicists draw boundary conditions as colorings.



Commuting pair (g, h) describes hom $\pi_1(E) \to \mathbb{M}$.
$SL_2(\mathbb{Z})$ action changes generating pair.
Ignoring scalar ambiguities, claims (2) and (4) say that Z is a function on the moduli space of elliptic curves with principal $\mathbb{M}$-bundles.

First Breakthrough (Dong, Li, Mason 1997)

- Existence and uniqueness (up to isom.) of $V^\natural(g)$.
- Convergence of power series defining Z.
- Settles claims (1), (2), (5).
- Reduces $SL_2(\mathbb{Z})$ claim (4) to g-rationality.

Theorem (C, Miyamoto 2016)

Category of g-twisted $V^\natural$-modules is semisimple.
This resolves the $SL_2(\mathbb{Z})$-compatibility claim (4).

g-rationality is really a corollary

Main theorem of [C-Miyamoto] is: If V is strongly regular, then so is the fixed-point subVOA V^g.

Here, "strongly regular" means roughly "module category is a modular tensor category".

This gives modular functions for traces of automorphisms of VOAs in infinitely many cases (therefore not really moonshine).

Main steps of proof

1. V a C_2-cofinite VOA, CFT type, σ finite order aut, $\Rightarrow V^\sigma$ is C_2-cofinite (Miyamoto 2013)
2. If V is also regular, then V^σ is a projective V^σ-module (uses Huang-Lepowsky-Zhang 2007-2011).
3. Any irreducible V^σ-module W is rigid, i.e., get isom. $W \boxtimes V^\sigma \to W \boxtimes (W^\vee \boxtimes W) \to (W \boxtimes W^\vee) \boxtimes W \to V^\sigma \boxtimes W$ (uses Huang's genus 1 fcns + Verlinde + Miyamoto's pseudo-trace).

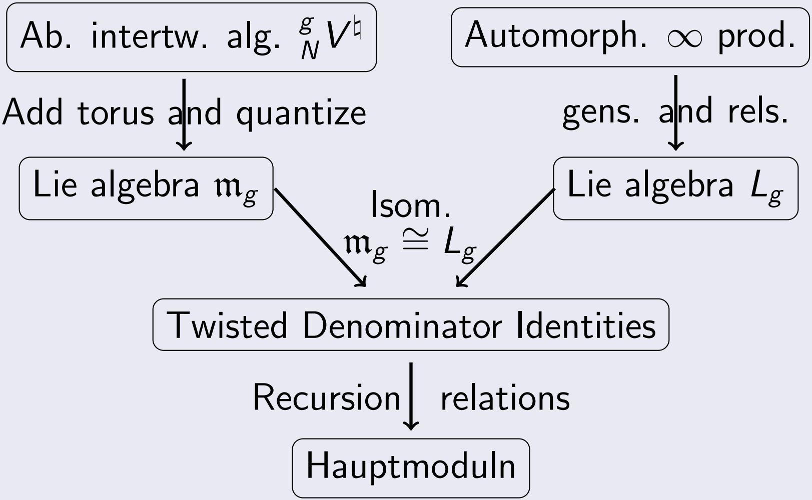
On to claim (3)

We now need to show that all $Z(g, h; \tau)$ are Hauptmoduln or constant.

Second Breakthrough (Höhn 2003)

Generalized Moonshine for 2A (Baby monster case).
- Gives outline for proving Hauptmodul claim (3).

Borcherds-Höhn program for Hauptmoduln



Right side (C 2009)

Borcherds products of the form:

$$T_g(\sigma) - T_g(-1/\tau) = p^{-1} \prod_{m>0, n \in \frac{1}{N}\mathbb{Z}} (1 - p^m q^n)^{c^g_{m,n}(mn)}$$

- Exponent $c^g_{m,n}(mn)$ is q^{mn}-coefficient of a v.v. modular function formed from $\{T_{g^i}(\tau)\}_{i=0}^{N-1}$.
- L_g is a $\mathbb{Z} \oplus \frac{1}{N}\mathbb{Z}$-graded BKM Lie algebra.
- Simple roots of multiplicity $c^g_{1,n}(n)$ in degree $(1, n)$.

Third Breakthrough (van Ekeren, Möller, Scheithauer 2015)

- There exists an abelian intertwining algebra structure on

$$ {}^{g}V^{\natural} := \bigoplus_{i=0}^{|g|-1} V^{\natural}(g^{i}) $$

- Dimensions of eigenspaces match coefficients $c^{g}_{m,n}(k)$ of v.v. modular function.

Add a torus and quantize

- Take a graded tensor product with a lattice abelian intertwining algebra $V_{II_{1,1}(-1/N)}$
- Get conformal VA, $c = 26$, graded by 2d lattice, has invariant form.
- Apply a bosonic string quantization functor.
- For Fricke g (i.e., $T_g(\tau) = T_g(-1/N\tau)$), get a BKM Lie algebra $\mathfrak{m}_g$ with real simple root.
- graded by $II_{1,1}(-1/N) \cong \mathbb{Z} \oplus \frac{1}{N}\mathbb{Z}$.

Comparison

Borcherds-Kac-Moody Lie algebras:

- $\mathfrak{m}_g$ has canonical projective action of $C_{\mathbb{M}}(g)$.
- L_g has "nice shape": known simple roots, good homology.

Isomorphism from matching root multiplicities:
$\dim(L_g)_{m,n} = (\mathfrak{m}_g)_{m,n} = c^g_{m,n}(mn)$.

Transport de structure $\Rightarrow L_g$ gets $\widetilde{C_{\mathbb{M}}(g)}$ action.

End of proof (C 2016)

Virtual $\widetilde{C_{\mathbb{M}}(g)}$-module isom $H_*(E_g, \mathbb{C}) \cong \bigwedge^* E_g$ implies equivariant Hecke operators $n\hat{T}_n$ given by

$$n\hat{T}_n Z(g,h,\tau) = \sum_{ad=n, 0\le b<d} Z(g^d, g^{-b}h^a, \tfrac{a\tau+b}{d})$$

act by monic polynomials on $Z(g,h,\tau)$.

- Hauptmodul condition follows (C 2008).
- Constants come from (g,h) such that all $g^a h^c$ are non-Fricke when $(a,c) = 1$, using claim (4).

This resolves the final claim (3).

Stronger version of conjecture?

Folklore: constant ambiguities are precisely controlled by a "Moonshine element" $\gamma^\natural \in H^3(\mathbb{M}, \mathbb{C}^\times)$.

- $H^3(\mathbb{M}, \mathbb{C}^\times)$ not known to be nontrivial.
- G. Mason says $|\gamma^\natural| \in 24\mathbb{Z}$ if $\gamma^\natural$ exists.
- Existence of canonical element $\gamma^\natural$ follows from non-abelian twisted fusion (in progress).
- $\mathbb{M}$ is enhanced to "categorical group" $\tilde{\mathbb{M}}$.
- Z naturally lives on space $\mathcal{M}_{1,1}^{\tilde{\mathbb{M}}}$

Connections to elliptic cohomology and tmf?

- Segal and Stolz-Teichner: interpretation of tmf in terms of CFTs (hence VOAs).
- Dependence on commuting pairs looks like Hopkins-Kuhn-Ravenel "higher character" theory at height 2.
- Claims (1), (2), (4) suggest $V^\natural \in tmf(B\tilde{\mathbb{M}})$
- Equivariant Hecke operators $\hat{T}_n$, used in Hauptmodul proof, first appeared as cohomology operations for $\mathcal{E}\ell\ell(BG)$. Explicit formula given in (Ganter 2007) .

Rademacher sums and quantum gravity (2009-)

Rademacher's sum, 1938

- Try to make an $SL_2(\mathbb{Z})$-invariant function from the $B(\mathbb{Z}) = \{\pm \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)\}$-invariant function q^{-1}.
- Problem: the sum $\sum_{\gamma \in B(\mathbb{Z})\backslash SL_2(\mathbb{Z})} e(-\gamma\tau)$ diverges everywhere.
- Regularize: subtract constants at infinity.

$$e(-\tau) + \lim_{K\to\infty} \sum_{\substack{0<c<K \\ -K^2<d<K^2 \\ (c,d)=1}} e\left(-\frac{a\tau+b}{c\tau+d}\right) - e\left(-\frac{a}{c}\right)$$

converges conditionally to $J(\tau) + 12$.

Generalization by Duncan-Frenkel 2009

- Allow large class of groups Γ in $SL_2(\mathbb{R})$.
- Allow arbitrary poles at distinguished cusp.
- Arbitrary non-positive weight.
- For weight 0, adjustments to constant terms.

Connection to Hauptmodul

Weight 0 sum is Γ-Hauptmodul $\Leftrightarrow$ Γ is genus 0.
If not, modular function plus weight 2 cusp form.

Rademacher sums are natural in quantum gravity

- Cosets $B(\mathbb{Z})\backslash SL_2(\mathbb{Z})$ enumerate asymptotically AdS_3 spacetimes with torus boundary.
- Non-trivial cosets correspond to BTZ black hole solutions of Einstein's equations.
- This gives a semiclassical "sum over histories" when computing quantum gravity partition function.

(Dijkgraaf-Maldacena-Moore-Verlinde: "A black hole Farey tale", Manschot-Moore: "A modern Farey tale")

Moonshine-gravity proposal (Duncan, Frenkel 2009)

- Generalized moonshine is connected to "second-quantized twisted chiral gravity" through AdS/CFT.
- Denominator formulas for Monstrous Lie algebras $\mathfrak{m}_g$ come from totalized Rademacher sums, which also describe gravity Fock spaces.

Warning: Quantization of 2+1 dimensional gravity is still far from rigorous.

K3 Mathieu Moonshine (2010-)

K3 Mathieu moonshine

An experimental mathematical observation motivated by physics.

K3 surfaces

A K3 surface is a compact complex surface that is simply connected and has trivial holomorphic canonical class.

Examples

Fermat quartic: $V(x^4 + y^4 + z^4 + w^4) \subset \mathbb{P}^3_{\mathbb{C}}$
Kummer: Blow up orbifold points in $(\mathbb{C}^2/\Lambda)/\{\pm 1\}$.

Theorem (Kodaira 1964)

The underlying smooth 4-manifolds of any two K3 surfaces are diffeomorphic.

Moduli space of complex structures

The moduli space of K3 surfaces is a connected complex 20-manifold. Algebrizable part is 19-dimensional.

Elliptic genus (Landweber-Stong, Ochanine 1980s)

- Homomorphism $\Omega^{SO} \to \mathcal{M}(\Gamma_0(2))$.
- {closed oriented mfds} $\to$ {modular forms}
- Enhancements by Witten, Hirzebruch, Krichever.

2-variable Elliptic genus

M a complex d-manifold. Define $Ell(M) \in y^{d/2}\mathbb{Z}[y, y^{-1}][[q]]$ as holom. Euler char. of $y^{-d/2} \bigoplus_{n\geq 1} (\Lambda_{-yq^{n-1}} \bar{T}_M \otimes \Lambda_{-y^{-1}q^n} T_M \otimes S_{q^n} \bar{T}_M \otimes S_{q^n} T_M)$

Theorem (Borisov, Libgober 1999)

If M is Calabi-Yau, then $Ell(M)$ is a Jacobi form of weight 0 and index $d/2$. In particular, $Ell(K3)$ has index 1.

Uniqueness

The space of Jacobi forms of weight 0 and index 1 is one-dimensional, spanned by $Ell(K3) = 2\phi_{0,1}$. $2y + 20 + 2y^{-1} + q(20y^2 - 128y + 216 - 128y^{-1} + 20y^{-2}) + O(q^2)$

Superconformal elliptic genus (Witten)

For a representation $\mathcal{H}$ of $\mathcal{N} = 2$ superconformal algebra, one defines the elliptic genus as $Ell(\mathcal{H}) = \mathrm{Tr}_{\mathcal{H}_{RR}}(q^{L_0 - c/24} y^{J_0} (-1)^F \bar{q}^{\bar{L}_0 - \bar{c}/24} (-1)^{\bar{F}})$

Physics conjecture (Witten)

Given a sigma model CFT with target Calabi-Yau X and Hilbert space $\mathcal{H}$, $Ell(\mathcal{H}) = Ell(X)$.

Enhanced supersymmetry for K3

K3 surfaces have hyperKähler structure, so their CFTs have action of $\mathcal{N} = 4$ superconformal algebra.

Natural question

Decompose $Ell(K3)$ into elliptic genera for irreducible $\mathcal{N} = 4$ representations?
- the genera are linearly independent

Eguchi, Ooguri, Tachikawa 2010

Decomposition into $\mathcal{N} = 4$ characters:

$$Ell(K3)(\tau, z) = 20\chi_{1/4,0} - 2\chi_{1/4,1/2} + \sum_{n \geq 1} A_n \chi_{1/4+n,1/2}$$

where $A_1 = 2 \times 45$, $A_2 = 2 \times 231$, $A_3 = 2 \times 770$.

Surprising observation

The numbers 45, 231, 770 are dimensions of irreducible reps of the sporadic group M_{24}

Theorem (Gannon 2012) - like Atkin-Fong-Smith

There is a $\mathcal{N} = 4$-representation with faithful commuting action of M_{24}, whose elliptic genus is $Ell(K3)$, such that taking traces of elements of M_{24} yields Jacobi forms of small level.

Additional suggestive evidence

- Setting $A_n = \dim H_n$, the series $\sum A_n q^n$ is a mock modular form, and so is $\sum \mathrm{Tr}(g|H_n)q^n$.
- Analogue of Hauptmodul property (Cheng, Duncan 2012): The trace forms are weight $1/2$ Rademacher sums.

Big mystery: where do we get M_{24} symmetry?

- No M_{24}-symmetry on K3 surfaces (Mukai 1988, Kondo 1998). Only get subgroups of M_{23} with ≥ 5 orbits.
- No M_{24}-symmetry of K3 CFTs (Gaberdiel, Hohenegger, Volpato 2011). Moduli space is $\mathrm{Aut}(II_{4,20})\backslash O_{4,20}(\mathbb{R})/(O_4(\mathbb{R}) \times O_{20}(\mathbb{R}))$. Stabilizers fix 4-dim subspace - naturally live in Co_1, but too small.

How much structure do we need?

$\mathcal{N} = 4$ rep.	???	$\mathcal{N} = 4$ superCFT

less structure ⟷ more structure

more symmetry ⟷ less symmetry

∞-dim	M_{24}	small groups

Holomorphic vertex operator superalgebras?

- Generalized Mathieu Moonshine (Gaberdiel, Persson, Ronellenfitsch, Volpato 2012) suggests good orbifold behavior.
- Chiral de Rham constructions proposed, but few computations.
- Conway moonshine module $V^{s\natural}$ (Duncan, Mack-Crane 2015) may be manipulated to produce some K3-like characters.

Symmetry surfing (Taormina, Wendland 2013)

Moduli space of K3 CFTs is 80-dimensional, with scattered symmetries. Thus, try gluing symmetries from different points.

- Works well for Kummer surfaces - get max subgroup $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes A_8 \subset M_{24}$
- Recent progress on connections with $V^{s\natural}$

Newer Moonshines (2012-, 2014-, 2017-)

Umbral moonshine (Cheng, Duncan, Harvey 2012)

For each Niemeier (even unimod. pos. def. rank 24) lattice N, get:

- the umbral group $G^N = \mathrm{Aut}\, N / Weyl(N)$
- graded representations K^N of G^N, such that
- graded traces are vector valued mock modular forms (vector rep. tied to Coxeter # of N)
- shadows are specific theta functions.

K3 Mathieu moonshine is the case $N = A_1^{24}$.

Theorem (Duncan, Griffin, Ono 2015)

Umbral moonshine modules exist.

- Only $N = E_8^3$ case has a known construction.
- Many umbral functions come from $V^{s\natural}$.
- Connections to physics and geometry are still speculative, and the subject of active research.
- Example (Cheng, Harrison 2014): Niemeier lattices $\leftrightarrow$ duVal degenerations of marked K3. $Ell(K3) =$ sum of umbral genus and singular local genus.

Thompson moonshine observation (Piezas 2014)

Coefficients of the weight 1/2 modular form $f_3 = q^{-3} - 248q + 26752q^4 - \cdots$ "come from" sporadic group Th.

Partial (Generalized Monstrous) explanation

For g in class 3C, $Z(g, 1; \tau) = \sqrt[3]{j(\tau/3)}$ $= q^{-1/9} + 248q^{2/9} + 4124q^{5/9} + \cdots$, and $C_{\mathbb{M}}(g) = \mathbb{Z}/3\mathbb{Z} \times Th$. Coeffs give reps of Th, and chars are Hauptmoduln. $\sqrt[3]{j(\tau/3)} \sim$ theta lift of f_3.

Problem:

This only explains Th representations for coefficients of q^{n^2} in f_3.

Refined observation (Harvey, Rayhoun 2015)

There is a $\frac{1}{2}\mathbb{Z}$-graded Th-module whose graded super-dimension is the weight $1/2$ form $2f_3 + 248\theta$. Graded traces are also "nice" weight $1/2$ forms.

Theorem (Griffin, Mertens 2016)

A Thompson moonshine module exists.

No construction or natural explanation.

Skew-holomorphic moonshine (Duncan, Harvey, Rayhoun $\geq$ 2017)

- Thompson moonshine appears to be the level 1 case of a more general phenomenon involving weight $1/2$ forms that lift to Hauptmoduln for Fricke-containing genus zero groups.
- Calculations are still underway.
- Physics is still quite unclear.

Summary

- Generalized Monstrous Moonshine: Controlled by the vertex operator algebra $V^\natural$. Hauptmodul property comes from string quantization and possibly 3d quantum gravity.
- Mathieu and umbral moonshine: possibly controlled by $V^{s\natural}$ and K3 surfaces.
- Thompson and skew-holomorphic moonshine: unknown.

Thank you.