Singular commutative rings in a braided category

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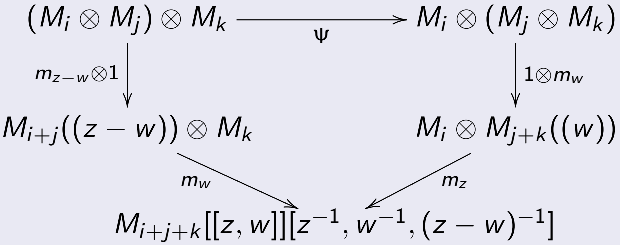
Outline

Classic question: Let N be a positive integer, V a vertex operator algebra, and $\{M_i\}_{i \in \mathbb{Z}/N\mathbb{Z}}$ a collection of modules, where $M_0 = V$. Assume the modules have integral conformal weight, and suppose we are given one dimensional spaces of intertwining maps $m_z : M_i \otimes M_j \to M_{i+j}((z))$ for all i and j. Can we choose nontrivial multiplication maps in these intertwiner spaces to endow $\bigoplus_i M_i$ with a vertex operator algebra structure?

Answer: Not necessarily. Locality requires that compositions yield commutative diagrams:

$$\begin{array}{ccc} M_i \otimes (M_j \otimes M_k) & \xrightarrow[\sigma_{12}]{} & M_j \otimes (M_i \otimes M_k) \\ & {\scriptstyle m_z \circ (1 \otimes m_w)} \searrow \quad \swarrow {\scriptstyle m_w \circ (1 \otimes m_z)} & \\ & M_{i+j+k}[[z,w]][z^{-1}, w^{-1}, (z-w)^{-1}] & \end{array}$$

We can write down an obstruction theory for the intertwiner maps. The Cousin property demands that we have a form of associativity:



If we assume there is a one dimensional space of maps

$$M_i \otimes M_j \otimes M_k \to M_{i+j+k}[[z,w]][z^{-1}, w^{-1}, (z-w)^{-1}]$$

compatible up to a constant with composition of intertwiners in our given spaces, then we have $\alpha_{i,j,k} \in \mathbb{C}^\times$ encoding the failure of Cousin.

In addition to $\alpha_{i,j,k}$, we also get nonzero constants $\beta_{i,j}$ by gluing two Cousin diagrams to the locality diagram. These encode a sort of commutativity failure.

These data (α and β) form a 4-cochain on (the Eilenberg-MacLane model of) $K(\mathbb{Z}/N\mathbb{Z}, 2)$, with coefficients in $\mathbb{C}^\times$.

If we assume the existence of a space of multiplications on $M_i \otimes M_j \otimes M_k \otimes M_l$ compatible with compositions of intertwiners, then this 4-cochain is a 4-cocycle.

There is a transitive action of the torus $(\mathbb{C}^\times)^{\mathbb{Z}/N\mathbb{Z}\times\mathbb{Z}/N\mathbb{Z}}$ on the intertwiner spaces, given by scalar multiplication. We identify this group with the 3-cochains on $K(\mathbb{Z}/N\mathbb{Z},2)$ with coefficients in $\mathbb{C}^\times$.

The induced action on $\alpha_{i,j,k}$ and $\beta_{i,j}$ give a surjective "coboundary" homomorphism to the 4-coboundary group, so there is a canonical class in $H^4(K(\mathbb{Z}/N\mathbb{Z},2),\mathbb{C}^\times)$ attached to the data of these modules with choices of intertwiner spaces.

Eilenberg and MacLane showed that $H^4(K(\mathbb{Z}/N\mathbb{Z},2),\mathbb{C}^\times)$ classifies $\mathbb{C}^\times$-valued quadratic forms on $\mathbb{Z}/N\mathbb{Z}$.

The quadratic form that determines the cohomology class attached to $\{M_i\}$ is determined by the self-braiding of each M_i, following the Cousin and locality diagrams.

This self-braiding of a module is in turn given by exponentiating ($2\pi i$ times) its conformal weight.

Since our conformal weights were integral, the cohomology class is trivial. In other words, there is a choice of intertwiners such that α and β are identically one. This yields a vertex operator algebra.

End of example.

We started by assuming the conformal weights of all M_i were integers. What happens when they are not?

Vertex operator algebras have integral conformal weights, so we can't assemble one from these modules.

However, we can try to make a more general version of vertex algebra, where locality is weakened, but we can still put modules together to form bigger algebras. There are many examples, coming from lattice and orbifold models.

Dong and Lepowsky made such a generalization, defining abelian intertwiner algebras, where $\mathbb{Z}/N\mathbb{Z}$ can be replaced by an arbitrary finite abelian group. Nontrivial 4-cocycles are encoded by correction terms in the Jacobi identity.

Bakalov and Kac have an alternative formulation using polylocal fields, where the group can be infinite, but α is trivial.

We'd like to consider a different point of view, where we make the algebra look as commutative as possible.

A hint comes from Joyal and Street. For an abelian group A, there is a bijection:

$$\left\{ \begin{array}{c} \text{equivalence classes} \\ \text{of braided tensor structures} \\ \text{on the abelian} \\ \text{category of } A\text{-graded} \\ \text{complex vector spaces} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \mathbb{C}^\times\text{-valued} \\ \text{quadratic} \\ \text{forms on } A \end{array} \right\}$$

In other words, braided tensor structures are described by elements of $H^4(K(A,2),\mathbb{C}^\times)$.

A braided tensor structure on a category $\mathcal{C}$ is the least structure you need to define a natural notion of commutative ring in $\mathcal{C}$:

- A tensor product $\otimes$ admits multiplication $R \otimes R \to R$.
- An associator $\Psi : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ lets us say that multiplication is associative.
- A commutor $\Phi : A \otimes B \to B \otimes A$ lets us say that multiplication is commutative.
- There is also a unit object, and some diagrams describing compatibility.

The notion of vertex algebra in a braided category of A-graded vector spaces is not particularly difficult to define using this structure. The Cousin diagram gets an associator $\Psi : (V_a \otimes V_b) \otimes V_c \to V_a \otimes (V_b \otimes V_c)$, and locality uses a commutor $\Phi : V_a \otimes V_b \to V_b \otimes V_a$. We need to add fractional powers of z, etc.

When trying to construct an A-graded vertex algebra from intertwiners, we find that the obstruction theory is essentially unchanged - the 4-cocycle is captured by the conformal weights.

In this case, we can choose the braided structure on the category to have the correct class in $H^4(K(A,2),\mathbb{C}^\times)$, so that α and β can be trivialized.

As before, the torus $(\mathbb{C}^\times)^{A\times A}$ acts by rescaling intertwiners. We can alternatively view this group as $\mathbb{C}^\times$-valued 2-cochains on BA, where the stabilizer of the associator α is the subgroup of cocycles.

Nontrivial cocycles yield analogues of twisted group algebras. This yields the action of $H^2(A,\mathbb{C}^\times)$ on isomorphism classes described by Bakalov and Kac.

The vanishing of obstructions lets us construct graded vertex algebras, but we need to find one-dimensional spaces of intertwiners with the correct compatibility properties.

Easiest example: Lattices - there is a direct construction, so we don't even need to calculate intertwiner compatibilities.

Second easiest example: Twisted modules of a holomorphic C_2-cofinite VOA V.

The intertwiner spaces come from a conformal block calculation.

Theorem: Let G be a finite group acting on V, and for each $g \in G$, let $V(g)$ be the unique irreducible g-twisted module. Let $g_1, \ldots, g_k \in G$ be a set of elements satisfying $g_1 \cdots g_k = 1$ and suppose the conformal weight of each $V(g_i)$ lies in $\frac{1}{|g_i|}\mathbb{Z}$. Then the space of conformal blocks associated to a G-cover of $\mathbb{P}^1$ with insertions $V(g_i)$ at ramification points q_i with monodromy g_i is one-dimensional.

The proof basically uses a twisted coordinate-free rewrite of Nagatomo and Tsuchiya's work on the untwisted case. The condition on conformal weight can be removed at the cost of some extra complexity.

This theorem lets us construct abelian intertwiner algebras associated to VOAs such as the moonshine module.

Once we have abelian intertwiner algebras, we can construct generalized Kac-Moody algebras with group actions using BRST cohomology. Combining this with a version of the no-ghost theorem yields a way to attack the generalized moonshine conjecture.

We can also use the theorem to precisely calculate characters of twisted modules. If g commutes with an element h in class 2B, then $V^\natural(g) \oplus V^\natural(gh)$ is a twisted module for Huang's algebra $V^\natural \oplus V^\natural(h)$. It is also a sum of two irreducible twisted modules for the Leech lattice algebra V_Λ, and those characters are known. This lets us determine a large number of previously ambiguous constants.

We now have a notion of vertex algebra in a braided category of A-graded vector spaces, together with a way to construct them.

This is a little unsatisfying for two reasons: these are not a genuinely new class of objects, and now that we've stepped into the braided category world, the restriction to A-graded vector spaces seems rather limiting.

The braiding in the A-graded categories is determined by L_0 on modules with conformal weight supported on a coset of $\mathbb{Z}$, and there are much richer braided categories, where commutors and associators are not described by constant multiplication.

There is another problem that arises when we try to interpret these algebras geometrically.

Beilinson and Drinfeld introduced chiral algebras on an algebraic curve X as an algebro-geometric manifestation of vertex algebras. They gave some equivalent definitions, e.g.:

- A D_X-module with a Lie bracket (in the $\otimes^{ch}$ pseudo-tensor structure).
- A collection of D_{X^n}-modules for $n \geq 0$ with gluing on diagonals and factorization on the open complement.
- A quasicoherent sheaf on a “space” of effective Cartier divisors.

If we try to generalize these definitions of chiral algebra to the A-graded setting, it is natural to replace D_X-modules with twisted D_X-modules, or equivalently, monodromic sheaves. Twisting accounts for non-integral conformal weight.

Unfortunately, we can't get a well-behaved pseudo-tensor category of twisted D_X-modules, because the definition of pseudo-tensor structure is manifestly symmetric. There are similar technical problems with the other definitions.

The easiest way to fix this is to impose ordering on inputs. Tensor products are over a linearly ordered set of D_X-modules, and factorization maps are over surjections of finite linearly ordered sets. We then add braid-equivariance afterward by hand.

This can be done similarly for Borcherds's treatment of vertex algebras as singular commutative rings.

Borcherds starts with functors from the category of finite sets to a symmetric category, and we replace it with functors from the category of finite linearly ordered sets (with arbitrary maps) to a braided category, with a braid group action by permutations.

If we want an interesting formal group action, we need to choose a symmetric tensor subcategory. There doesn't seem to be enough structure to get a good Hopf algebra otherwise.

The ring S describing singularities is still rather mysterious. I have yet to see really exotic cases.

Examples of interest:

1. $\mathcal{C} = D^\omega(G) - mod$ for G not necessarily abelian, where $\omega \in H^3(G, \mathbb{C}^\times)$. These come up in orbifold models. If V is a holomorphic C_2 VOA with G-action, we'd like to put a singular commutative algebra structure on the sum of all irreducible twisted modules. Work on transmutation by Majid makes this a reasonable idea. For example, $\mathbb{C}[G]$ can be written as a commutative ring in $D(G) - mod$, since the braiding takes $g \otimes h$ to $ghg^{-1} \otimes g$. The obstruction theory is more subtle - the analogue of $K(A, 2)$ is the double delooping of the ω-twisted inertia stack for G.

2. Vertex representations of quantum affine algebras - major headache. It would be nice if this could be made to work.

Ordered sets give a very *ad hoc* construction. It would be great to have a more geometric version of these operations, which can generalize to other contexts. Instead of ordered sets, one should really parametrize multiplications by spaces in the $\mathcal{E}_2$ operad. They should glue together to form some kind of sheaf with operations on moduli spaces of stable pointed genus zero curves.

This was the point behind Soibelman's pseudo-braided categories. Instead of having vector spaces $Mult(\{M_i\}, M)$ of multilinear operations (pretending to be maps $\bigotimes M_i \to M$), he added a configurational dependence.

Dream: Have a workable notion of singular $\mathcal{O}$-algebra in an $\mathcal{O}$-multicategory for any operad $\mathcal{O}$ (and interesting examples).