Monstrous Moonshine over the Integers

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What is moonshine?

Strange connections between finite groups and modular forms

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The connections should be “very special”

Infinitely many cases $\Rightarrow$ not moonshine!

Monstrous Moonshine (1978-1992)

Classification of finite simple groups (1982 or 2004)

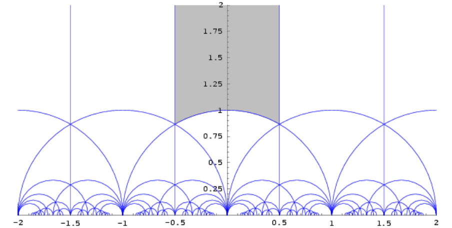
Any finite simple group is one of the following

- A cyclic group of prime order
- An alternating group A_n $(n \geq 5)$
- A group of Lie type (16 infinite families)
- One of 26 sporadic simple groups

Largest sporadic: Monster $\mathbb{M}$, about $8 \cdot 10^{53}$ elements (Griess 1982).
194 irred. repres. of dim $1, 196883, 21296876, \ldots$

$SL_2(\mathbb{Z})$ action on complex upper half-plane $\mathfrak{H}$

Generators: $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) : z \mapsto z + 1$, $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) : z \mapsto -1/z$



(Wikipedia)

J-function as Hauptmodul

The quotient space $SL_2(\mathbb{Z})\backslash\mathfrak{H}$ has genus zero. J generates the function field. Fourier expansion: $q^{-1} + 196884q + 21493760q^2 + \cdots (q = e^{2\pi i z})$

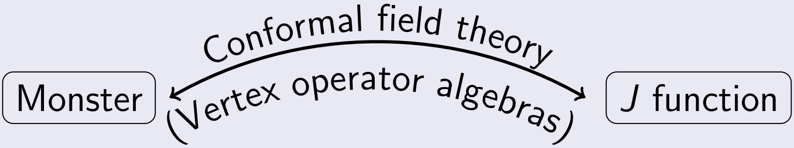
Coefficients of J and Irreducible Monster reps

$$\begin{aligned} 196884 &= 1 + 196883 \text{ (McKay, 1978)} \\ 21493760 &= 1+196883+21296876 \text{ (Thompson, 1979)} \\ 864299970 &= 2\times 1+2\times 196883+21296876+842609326 \\ \vdots & \quad \vdots \end{aligned}$$

How to continue this sequence?

McKay-Thompson conjecture: Natural graded rep $\bigoplus_{n=0}^{\infty} V_n$ of $\mathbb{M}$ such that $\sum \dim V_n q^{n-1} = J$.

Idea: Physics forms a bridge



Solution: Frenkel, Lepowsky, Meurman 1988

Constructed a vertex operator algebra $V^\natural = \bigoplus_{n \geq 0} V^\natural_n$ (the Moonshine Module), such that $\sum_{n \geq 0} (\dim V^\natural_n) q^{n-1} = J$ and $\mathrm{Aut}\, V^\natural = \mathbb{M}$.

Refined correspondence

Thompson's suggestion: replace graded dimension with graded trace of non-identity elements.

Monstrous Moonshine Conjecture (Conway, Norton 1979)

There is a faithful graded representation $V = \bigoplus_{n \geq 0} V_n$ of the monster $\mathbb{M}$ such that for all $g \in \mathbb{M}$, the series $T_g(\tau) = \sum_{n \geq 0} \mathrm{Tr}(g|V_n) q^{n-1}$ is the q-expansion of a congruence Hauptmodul (= "generates function field of genus 0 $\mathfrak{H}$-quotient").

First proof (Atkin, Fong, Smith 1980)

Theorem: A virtual representation of $\mathbb{M}$ exists yielding the trace functions T_g.
No construction.

Second proof (Borcherds 1992)

Theorem: The Conway-Norton conjecture holds for $V^\natural$.

More monstrous moonshines

Moonshine for other groups?

Conway-Norton 1979, computations by Queen 1980.
Example: Baby monster $\mathbb{B}$ has irreps of dim $1, 4371, 96255, \dots$, and the Hauptmodul for $\Gamma_0(2)^+$ is $q^{-1} + 4372q + 96256q^2 + \cdots$.

Main observation

If g has prime order p and g is in conjugacy class $p\mathrm{A} \subset \mathbb{M}$, then T_g has positive integer coefficients that "look like" representations of $C_{\mathbb{M}}(g)$.
For $p = 2$, $C_{\mathbb{M}}(g) \cong 2.\mathbb{B}$, a central extension of Baby monster.

Conj 1: Generalized Moonshine (Norton 1987)

For each $g \in \mathbb{M}$ exists $V(g)$ graded proj. rep. of $C_{\mathbb{M}}(g)$. Trace functions $Z(g, h; \tau)$ for commuting pairs satisfy modularity properties.

Conj 2: Modular Moonshine (Ryba 1994)

For each g in class pA, there is a vertex algebra V_g over $\mathbb{F}_p$ with $C_{\mathbb{M}}(g)$ action. For each p-regular $h \in C_{\mathbb{M}}(g)$, the graded Brauer character of h on V_g is equal to T_{gh}.

Interpretation of Generalized Moonshine

$V(g)$ - twisted sectors of a monster CFT.
$Z(g, h; \tau)$ - genus 1 partition functions (with twisted boundary conditions).
(Dixon, Ginsparg, Harvey 1988)

Interpretation of Modular Moonshine

$V_g = \hat{H}^0(g, V^\natural_{\mathbb{Z}})$ - Tate cohomology of $V^\natural_{\mathbb{Z}}$, a self-dual integral form of $V^\natural$ with $\mathbb{M}$ symmetry.
(Borcherds, Ryba 1996)

First advances (1990s)

Generalized moonshine: Existence and uniqueness (up to isom.) of $V^\natural(g)$ (Dong, Li, Mason 1997).
Modular moonshine: Good properties, assuming existence of $V^\natural_{\mathbb{Z}}$ (Borcherds, Ryba 1996, 1998).

Later advances (2010s)

Generalized: Good properties of $Z(g,h;\tau)$.
Modular: Existence of $V^\natural_{\mathbb{Z}}$.

Main breakthrough for both moonshines

- If V is strongly regular and holomorphic, and $g \in \mathrm{Aut}(V)$ is finite order, then there exists an abelian intertwining algebra structure on the direct sum of irreducible twisted modules

$$^{g}V := \bigoplus_{i=0}^{|g|-1} V(g^i)$$

(van Ekeren, Möller, Scheithauer 2015)

Corollary (Cyclic orbifold construction)

Let V be a strongly regular and holomorphic vertex operator algebra, and $g \in \mathrm{Aut}(V)$ finite order. Assume g is "anomaly-free" (i.e., eigenvalues of $L(0)$ on $V(g)$ are in $\frac{1}{|g|}\mathbb{Z}$).

Decompose ${}^gV := \bigoplus_{i=0}^{|g|-1} V(g^i)$ under canonical g action to get $\bigoplus V^{i,j}$, where $V = \bigoplus V^{0,j}$. Then $V/g := \bigoplus V^{i,0}$ is a strongly regular holomorphic vertex operator algebra, and there is a canonical automorphism g^* such that $V^{i,0}$ is the $e^{2\pi\sqrt{-1}i/|g|}$ eigenspace.

Cyclic orbifold constructions of $V^\natural$ from Leech lattice vertex operator algebra V_Λ

1. Order 2 orbifold (Frenkel, Lepowsky, Meurman 1988)
2. Order 3 orbifold (Chen, Lam, Shimakura 2016)
3. Orders 5, 7, 13 (Abe, Lam, Yamada 2017)
4. 46 classes of composite order (C 2017)

- confirms Tuite's orbifold correspondence (1992): Massless classes in $Co_0 \leftrightarrow$ non-Fricke classes in $\mathbb{M}$. For $p \in \{2,3,5,7,13\}$, $(V_\Lambda, pa) \leftrightarrow (V^\natural, pB)$.

Cyclic orbifolds over small rings

Vertex algebras

A vertex algebra over a commutative ring R is an R-module V, with an element $\mathbf{1} \in V$ and a multiplication map $V \otimes_R V \to V((z))$, written $u \otimes v \mapsto Y(u,z)v = \sum u_n v z^{-n-1}$, satisfying:

1. $Y(\mathbf{1}, z) = id_V z^0$ and $Y(a,z)\mathbf{1} \in a + zV[[z]]$.
2. For any $r, s, t \in \mathbb{Z}$, and any $u, v, w \in V$, $\sum_{i \geq 0} \binom{r}{i} (u_{t+i} v)_{r+s-i} w = \sum_{i \geq 0} (-1)^i \binom{t}{i} (u_{r+t-i}(v_{s+i} w) - (-1)^t v_{s+t-i}(u_{r+i} w))$

Example: Lattice vertex algebras over $\mathbb{Z}$

For any positive definite even unimodular lattice L there is a self-dual vertex algebra $(V_L)_\mathbb{Z}$ over $\mathbb{Z}$ (Borcherds 1986). It is a $\mathbb{Z}$-form of $\mathrm{Sym}(t^{-1}(\mathbb{C} \otimes L)[t^{-1}]) \otimes \mathbb{C}[L]$ spanned by $s_{\alpha_1,n_1} \cdots s_{\alpha_k,n_k} e^\alpha$, where e^α is a basis element of $\mathbb{C}[L]$, α_i are chosen from a basis of L, and the operator $s_{\alpha,k}$ is the z^k-coefficient of $\exp(\sum_{n>0} \frac{\alpha(-n)}{n} z^n)$. Here, $\mathrm{Sym}(t^{-1}(\mathbb{C} \otimes L)[t^{-1}])$ is a representation of the Heisenberg algebra, with generators $\alpha(n) = \alpha t^{-n} \in L[t, t^{-1}] \oplus \mathbb{C}K$.

Vertex operator algebras

A vertex operator algebra over R with half central charge c is a vertex algebra V over R equipped with a "conformal element" ω and a $\mathbb{Z}$-grading $V = \bigoplus V_n$, such that

1. If $u \in V_m$, $v \in V_n$, then $u_k v \in V_{m+n-k-1}$.
2. The coefficients of $Y(\omega, z) = \sum L_n z^{-n-2}$ satisfy Virasoro relations: $[L_m, L_n] = (m-n)L_{m+n} + c\binom{m+1}{3}\delta_{m+n,0}\,\mathrm{id}$.
3. Each V_n is a finite rank projective R-module, and L_0 acts on V_n by $n \cdot \mathrm{id}$.

Abelian intertwining algebras over subrings of $\mathbb{C}$

An abelian intertwining algebra is a "braided commutative" generalization of vertex operator algebra, graded by an abelian group A with Eilenberg-MacLane abelian 3-cocycle (F, Ω). These can be defined over any subring R of $\mathbb{C}$ that contains not only all values of $F : A^{\times 3} \to \mathbb{C}^\times$ and $\Omega : A^{\times 2} \to \mathbb{C}^\times$, but also $1/N$ and $e^{\pi\sqrt{-1}/N}$, where $\Omega(a, a)^N = 1$ for all $a \in A$.

Key lemma

Let $V = \bigoplus_{i,j \in \mathbb{Z}/N\mathbb{Z}} V^{i,j}$ be a self-dual abelian intertwining algebra over $\mathbb{C}$, where each $V^{i,j}$ is an irreducible $V^{0,0}$-module, and let $U = \bigoplus V^{0,j}$ and $W = \bigoplus V^{i,0}$. If R is a suitable subring of $\mathbb{C}$, and we are given self-dual R-forms U_R and W_R such that $U_R \cap V^{0,0} = W_R \cap V^{0,0}$, then they generate a self-dual R-form of V.

Intermediate orbifolds (after Abe, Lam, Yamada)

Let $P_0 = \{2, 3, 5, 7, 13\}$. If p, q are distinct in P_0, and $pq \notin \{65, 91\}$, then there is an automorphism $\bar{g}$ of the Leech lattice of order pq, such that no non-identity power of $\bar{g}$ has fixed points, and an order pq lift $g \in \mathrm{Aut}(V_\Lambda)$. Then:

1. $V_\Lambda/g^p \cong V_\Lambda/g^q \cong V^\natural$
2. $V_\Lambda/g \cong V_\Lambda$.

In particular, there are 2 copies of $V^\natural$ inside the abelian intertwining algebra $\bigoplus_i V_\Lambda(g^i)$, which is generated by 2 copies of V_Λ.

Corollary

Let p, q be distinct elements of $P_0 = \{2, 3, 5, 7, 13\}$, such that $pq \notin \{65, 91\}$, and let $R_{pq} = \mathbb{Z}[1/pq, e^{\pi\sqrt{-1}/pq}]$. Then, there is a self-dual R_{pq}-form of the abelian intertwining algebra $\bigoplus_i V_\Lambda(g^i)$, and it contains 2 isomorphic self-dual R_{pq}-forms of $V^\natural$.

Monster symmetry

Symmetries of V_Λ

The Leech lattice Λ has $Co_0 = 2.Co_1$ symmetry. $\mathrm{Aut}\, V_\Lambda \cong (\mathbb{C}^\times)^{24}.Co_0$ (non-split extension). Let $p \in P_0$, $\bar{g} \in Co_0$ fixed-point free, order p. Then any order p lift $g \in \mathrm{Aut}\, V_\Lambda$ has centralizer $(\mathbb{Z}/p\mathbb{Z})^{24/(p-1)}.C_{Co_0}(\bar{g})$. The same is true for suitably chosen automorphisms of the R-form, as long as R contains $1/p$ and $e^{\pi\sqrt{-1}/p}$.

Symmetries of $V^\natural_R$

The self-dual R_{pq}-forms of $V^\natural$ naturally inherit an action of $G_p = p^{1+24/(p-1)}.(C_{Co_0}(\bar{g}^q)/\bar{g}^q)$ from an abelian intertwining algebra containing $V^\natural_{R_{pq}}$ and $(V_\Lambda)_{R_{pq}}$ (and similarly for G_q).

Maximal subgroups (from Wilson 2015)

When $p \in P_0$, G_p contains the Sylow p-subgroup of $\mathbb{M}$, and when $p \in \{2,3,5\}$, G_p is contained in a unique maximal subgroup of $\mathbb{M}$.

Monster symmetry

If p, q are distinct elements of P_0 such that $pq \notin \{65, 91\}$, then G_p and G_q generate $\mathbb{M}$. In particular, the self-dual R_{pq}-forms of $V^\natural$ have $\mathbb{M}$ symmetry,

Gluing forms over small rings

Gluing data

Given a diagram $R_1 \to R_3 \leftarrow R_2$ of commutative rings, a gluing datum for vertex operator algebras is a triple (V^1, V^2, f), where

1. V^1 is a vertex operator algebra over R_1,
2. V^2 is a vertex operator algebra over R_2, and
3. $f : V^1 \otimes_{R_1} R_3 \to V^2 \otimes_{R_2} R_3$ is an isomorphism of vertex operator algebras over R_3.

These form a category, where morphisms are pairs of maps satisfying a commutative square condition.

Effective gluing lemma

Let $i_1 : R \to R_1$ and $i_2 : R \to R_2$ be maps of commutative rings, such that either

1. i_1 and i_2 form a Zariski open cover, or
2. i_1 and i_2 are faithfully flat.

Then, the category of gluing data for $R_1 \to R_1 \otimes_R R_2 \leftarrow R_2$ is equivalent to the category of vertex operator algebras over R.

Comparison of fixed points

Let $R_n = \mathbb{Z}[1/n, e^{\pi\sqrt{-1}/n}]$ and let $g \in p$B. Recall $(V^\natural, p\mathrm{B})$ is orbifold dual to (V_Λ, pa), and $V^g_{pq} \cong (V_\Lambda)^\sigma_{R_{pq}}$. Then $V^g_{pq} \otimes_{R_{pq}} R_{pqr} \cong (V_\Lambda)^\sigma_{R_{pqr}} \cong V^g_{pr} \otimes_{R_{pr}} R_{pqr}$.

pB-pure elementary subgroups (Wilson 1988)

For each $p \in P_0$, there is an elementary subgroup $H_p \subset \mathbb{M}$ of order p^2, whose non-identity elements lie in conjugacy class pB.

Construction of gluing datum

V_{pq} and V_{pr} are generated by g-fixed point subalgebras for g ranging over H_p, so $V_{pq} \otimes_{R_{pq}} R_{pqr} \cong V_{pr} \otimes_{R_{pr}} R_{pqr}$ by uniqueness of generated self-dual forms.

Sufficiency of gluing data

From our isomorphisms $V_{pq} \otimes_{R_{pq}} R_{pqr} \cong V_{pr} \otimes_{R_{pr}} R_{pqr}$, we may produce a self-dual $\mathbb{Z}$-form with $\mathbb{M}$-symmetry by repeated gluing. Uniqueness comes from the fact that $\mathbb{M} \backslash \mathbb{M} / \mathbb{M}$ is a singleton.

Main result

There is a unique self-dual $\mathbb{Z}$-form $V^\natural_\mathbb{Z}$ of $V^\natural$ such that $V^\natural_\mathbb{Z} \otimes R_{pq} \cong V_{pq}$. This form has $\mathbb{M}$-symmetry, and the natural inner product is positive definite.

Corollary

Modular moonshine conjecture.

Corollary

There exists a positive definite unimodular lattice of rank 196884 with a faithful monster action.

Further questions

Borcherds-Ryba proof of modular moonshine

1. Easy part: $\sum_n \tilde{\mathrm{Tr}}(h|\hat{H}^*(g, V^\natural_{\mathbb{Z},n}))q^{n-1} = \sum_n \mathrm{Tr}(gh|V^\natural_n)q^{n-1}$, where $\hat{H}^*$ is the virtual module $\hat{H}^0 \ominus \hat{H}^1$
2. Hard part: $\hat{H}^1(g, V^\natural_{\mathbb{Z}}) = 0$ for $g \in pA$.

Can we extend this to composite order g?

Conjecture (Borcherds-Ryba 1996)

$\hat{H}^1(g, V^\natural_{\mathbb{Z}}) = 0$ for all Fricke classes g.

Unifying Conjecture (Borcherds 1998)

For each $g \in \mathbb{M}$, let $R_g = \mathbb{Z}[e^{2\pi i/|g|}]$. Then there is a free $\frac{1}{|g|}\mathbb{Z}$-graded R_g-supermodule $\hat{V}_g$ with an action of $\mathbb{Z}/|g|\mathbb{Z}.C_{\mathbb{M}}(g)$, such that:

1. $\hat{V}_1 = V^\natural_{\mathbb{Z}}$.
2. If $h \in C_{\mathbb{M}}(g)$ satisfies $(|g|, |h|) = 1$, then $\hat{V}_{gh} \otimes_{R_{gh}} \mathbb{Z}/|h|\mathbb{Z} \cong \hat{H}^*(\tilde{h}, \hat{V}_g)$ for a lift $\tilde{h}$ of h.
3. If g is Fricke, then $\hat{V}_g \otimes_{R_g} \mathbb{C} \cong V^\natural(g)$.
4. If g is non-Fricke, then $\hat{V}_g$ is a self-dual conformal vertex superalgebra.

Connections to other moonshines

Numerical evidence relating
non-Fricke classes in $\mathbb{M} \leftrightarrow$ Umbral moonshine
Fricke classes $\leftrightarrow$ Skew-holomorphic moonshine
No concrete conjectures yet (as far as I know).

Thank you