FINITENESS AND PERIODICITY OF BETA EXPANSIONS - NUMBER THEORETICAL AND DYNAMICAL OPEN PROBLEMS[1]

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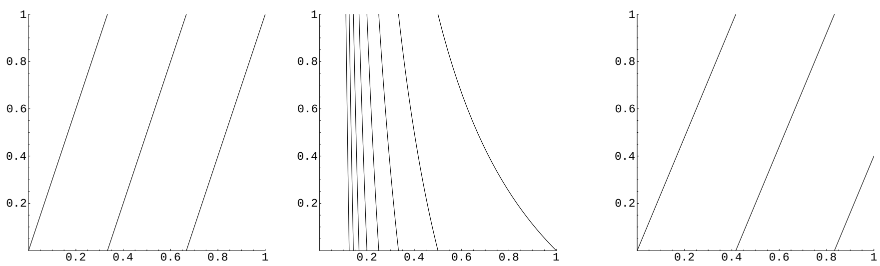
Let $y = f(x)$ be a positive real function. Consider a digital expansion of a real number x in a form:

$$x = \varepsilon_0 + f(\varepsilon_1 + f(\varepsilon_2 + f(\varepsilon_3 + \dots.$$

given by an algorithm with $\varepsilon_0 = \lfloor x \rfloor, r_0 = x - \lfloor x \rfloor$ and

$$\varepsilon_{n+1} = \lfloor f^{-1}(r_n) \rfloor, r_{n+1} = f^{-1}(r_n) - \lfloor f^{-1}(r_n) \rfloor.$$

This is **Rényi's f-expansion** ([24]). It is the usual b-adic expansion when $f(x) = x/b$ and the regular continued fraction when $f(x) = 1/x$. For $f(x) = x/\beta$ with a non-integer $\beta > 1$, it is called the **β-expansion**.



(a) 3-adic expansion (b) Continued fraction (c) Beta expansion

FIGURE 1. The graph of f^{-1} mod 1

Rényi studied ergodic properties of f-expansion for monotone function f with mild growth condition. The images of f^{-1} consists of 'full' copies of intervals $[0, 1]$ in left two graphs, which implies that the corresponding digits are independent, while the last case has dependent digits. We start with our fundamental question:

Problem 1. Find a good f-expansion.

Historically, several different 'goodness' of f-expansion are studied.

- Does it have a nice invariant measure ? We expect a measure equivalent to the Lebesgue measure with an explicit Radon-Nikodym derivative.

[1]Based on the talk on 24 March 2009 at 'Numeration: Mathematics and Computer Science' (CIRM)

- Does it have nice ergodic properties (Mixing, unique measure of maximum entropy) ?
- Does it require little memory to judge admissibility of words (i.e., certain Markovian property) ?
- Can we characterize periodic orbits and/or finite orbits ? (The expansion is finite, if the digits end up in zeros 0^∞.)
- Is there a useful small natural extension ?
- Is there an 'additive' dynamical system whose successive induced systems is described by f-expansion ? (Kamae's number system [15])
- Can we apply this expansion to Diophantine approximation ?

In fact, these questions are intimately related. Hereafter we discuss β-expansion. This is too simple and we can not expect last two number theoretical 'goodness' but others could be achieved for some β's.

Fix a real number $\beta > 1$ and consider a map $T_\beta(x) = \beta x - \lfloor \beta x \rfloor$ from $[0, 1)$ to itself. The trajectory of x is written as

$$x \xrightarrow{a_1} T_\beta(x) \xrightarrow{a_2} T_\beta^2(x) \xrightarrow{a_3} T_\beta^3(x) \xrightarrow{a_4} \dots$$

with

$$a_i = \left\lfloor \beta T_\beta^{i-1}(x) \right\rfloor \in \mathcal{A} := \mathbb{Z} \cap [0, \beta).$$

Then $x \in [0, 1)$ is uniquely expanded as:

$$x = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \frac{a_3}{\beta^3} + \dots$$

through greedy algorithm. We summarize its ergodic properties.

- Rényi (Rényi [24]) showed that it is ergodic with an absolute continuous invariant measure. From this proof, it is weak mixing. It has unique measure of maximum entropy ([28], [12]).
- Its Radon-Nikodym derivative is made explicit (Parry [23]).
- It is 'exact' in the sense of Rohlin [25]. This implies mixing of all degree.
- Its natural extension is a Bernoulli shift. (Smorodinski [27] , Fischer [10], Ito-Takahashi [14])
- However T_β itself is **not** Bernoulli except when β is an integer (Kubo-Murata-Totoki [17]).

The clue to analyze Markovian property of a piecewise linear transformation, is the orbit of discontinuity. In the case of beta expansion, the study of the orbits is reduced to so called 'expansion of 1'. Beta expansion defines a map $d_\beta : [0, 1) \to \mathcal{A}^{\mathbb{N}}$. This d_β is not surjective when $\beta \notin \mathbb{Z}$. The **expansion of** 1 is defined by $d_\beta(1-) = \lim_{\varepsilon \downarrow 0} d_\beta(1 - \varepsilon)$. Let $c_1 = \lfloor \beta \rfloor$. Then $d_\beta(1-)$ is the concatenation of c_1 and the beta expansion of $\beta - \lfloor \beta \rfloor$ when this expansion is infinite. If it is finite, i.e., $d_\beta(\beta - \lfloor \beta \rfloor) = c_2 c_3 \dots c_\ell$, then

$$d_\beta(1-) = (c_1 c_2 \dots c_{\ell-1}(c_\ell - 1))^\infty.$$

For $x = a_1 a_2 a_3 \dots \in \mathcal{A}^{\mathbb{N}}$, we have

$$x \in d_\beta([0, 1)) \iff \sigma^n(x) <_{\text{lex}} d_\beta(1-) \quad (n = 0, 1, 2, \dots)$$

where $\sigma(a_1 a_2 \dots) = a_2 a_3 \dots$ (shift operator) and $<_{\text{lex}}$ is the lexicographical order (Parry [23], Ito-Takahashi [14]). X_β is the closure of $d_\beta([0, 1))$ in $\mathcal{A}^{\mathbb{N}}$. This set is closed and $\sigma(X_\beta) = X_\beta$. This defines the **beta shift** (X_β, σ). The element in X_β is characterized by $\sigma^n(x) \leq_{\text{lex}} d_\beta(1-)$ $(\forall n)$.

Example 1. $d_\beta(1-) = (10)^\infty$ holds for $\beta = (1+\sqrt{5})/2$. $d_\beta([0,1))$ is the set of infinite words on $\{0,1\}$ that 11 and the tail $(10)^\infty$ are forbidden. X_β is exactly the set of infinite words on $\{0,1\}$ without 11.

Example 2. If $\beta > 1$ is the root of $x^3 - x^2 - x - 1$, then $d_\beta(1-) = (110)^\infty$ holds. $d_\beta([0,1))$ is the set of infinite words on $\{0,1\}$ that 111 and the tail $(110)^\infty$ are forbidden. X_β is exactly the set of infinite words on $\{0,1\}$ without 111.

From this $d_\beta(1-)$ we can introduce symbolic dynamical classification of β.

- β is a **simple Parry number** if $d_\beta(1-)$ is purely periodic. It is equivalent to the fact that X_β is a subshift of finite type (SFT).
- β is a **Parry number** if $d_\beta(1-)$ is eventually periodic. This is exactly the case when X_β is sofic, i.e., it is a factor of SFT.
- β is a **Delone number** if $d_\beta(1-)$ has bounded run of 0's. X_β has 'specified' property.

Many questions on this classification remain open (Blanchard [7]). We pick up a few of them:

Problem 2. (Salem Periodicity Problem 1) Is there a non-Parry Salem number ? How about $x^6 - 3x^5 - x^4 - 7x^3 - x^2 - 3x + 1$ (Boyd [8]) ?

Problem 3. Is there an algebraic Delone number except Parry numbers ? What about $3/2$ or $\sqrt{2}$?

If β is a Pisot number, then $d_\beta(x)$ is eventually periodic for $x \in \mathbb{Q}(\beta) \cap [0,1)$. Conversely if $d_\beta(x)$ is eventually periodic for $x \in \mathbb{Q} \cap [0,1)$ then β must be a Pisot or Salem number (Schmidt [26]). Therefore 'Pisot' implies 'Parry'.

Problem 4. (Salem Periodicity Problem 2) For Salem β, is $d_\beta(x)$ eventually periodic for all $x \in \mathbb{Q}(\beta) \cap [0,1)$?

The answer is expected negative and there are possible counter-examples (Boyd [8, 9]) in degree ≥ 6.

Problem 5. For Salem β, is there a way to prove that $d_\beta(x)$ is non-periodic for a fixed x ?

For Salem number of degree 4, the above periodicity conjecture seems valid both by numerical experiments and by heuristic consideration.

We say that the expansion of x is **finite**, if $d_\beta(x)$ ends up in 0^∞. Clearly in this case $x \in \mathbb{Z}[1/\beta]$. Frougny-Solomyak [11] studied the property

(F) $d_\beta(x)$ is finite for all $x \in \mathbb{Z}[1/\beta] \cap [0,1)$.

This implies that β is a Pisot number. Converse is not true. For e.g, if β has a positive conjugate other than itself, then it can not satisfy (F). The problem to characterize β's with (F) is open for degree ≥ 3, and it is transformed into the following **shift radix system** problem. The vector $(r_0, r_1, \dots, r_{d-1}) \in \mathbb{R}^d$ gives a shift radix system, **SRS**, if the integer sequence defined by

$$0 \leq r_0 a_n + r_1 a_{n+1} + \cdots + r_{d-1} a_{n+d-1} + a_{n+d} < 1$$

always falls into a trivial cycle $0^d \to 0^d$.

Example 3. $(1/2, 1)$ gives a shift radix system. For e.g.,

$$(-5,3) \to (3,0) \to (0,-1) \to (-1,1) \to (1,0) \to (0,0) \to (0,0)$$

by the recurrence:

$$0 \le a_n/2 + a_{n+1} + a_{n+2} < 1.$$

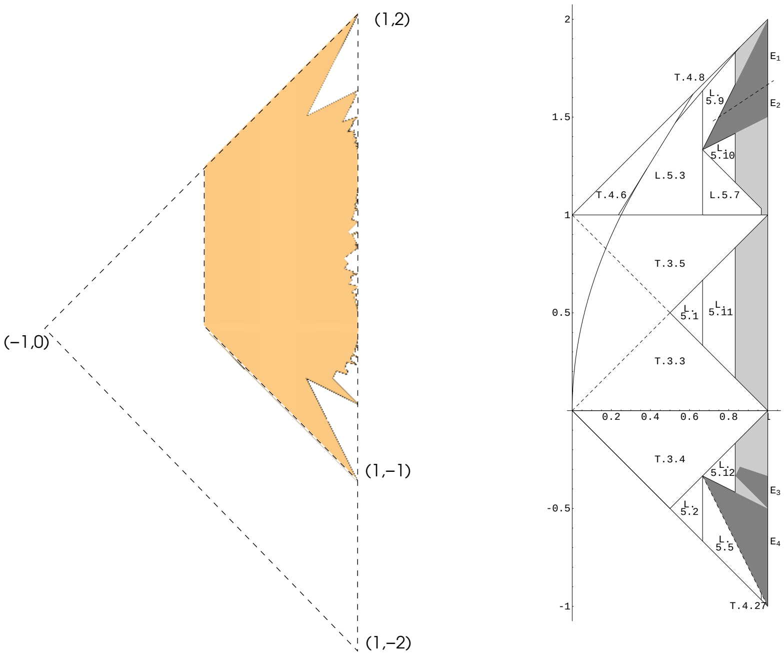
Indeed any orbits ends up in the trivial cycle $(0,0) \to (0,0)$.

A necessary condition for SRS $(r_0, r_1, \dots, r_{d-1})$ is that $x^d + r_{d-1}x^{d-1} + \dots + r_0$ is **semi-contractive**, i.e., all roots of the polynomial has modulus not greater than one. If it is **contractive**, i.e., all roots have modulus less than one, then the orbits must be eventually periodic. This is a simple fact on a contractive transformation acting on the lattice.

Take a Pisot number β and its minimal polynomial $p(x)$ which is factorized into

$$p(x) = (x - \beta)(x^{d-1} + r_{d-1}x^{d-1} + \dots + r_1 x + r_0)$$

in $\mathbb{C}$. Then one can prove that β has property (F) if and only if $(r_0, r_1, \dots, r_{d-1})$ gives a SRS. Semi-contractive cases correspond to the above-stated Salem periodicity problems.



(a) $2d$ SRS Approximation (b) Known regions

In Figure (a), the isosceles triangle surrounded by broken edges is the semi-contractive region while the shaded part gives an approximation of SRS parameters. White (mostly polygonal) regions in Figure (b) are shown (in pretty different ways!) to be in SRS and dark grey parts are not in SRS. Light grey parts are not settled.

Problem 6. Must SRS polynomials be contractive ?

This is true for $d \le 2$. We know that $r_0 \ne 1$ for SRS in general.

Problem 7. Is SRS region connected ?

This seems likely by observing the approximations. However a little skeptic evidence exists as well. P.Surer found a cut point $(\frac{40}{41}, \frac{30}{41})$ of SRS region in $d = 2$!

Problem 8. Prove (or disprove) that $\{(x, y) \mid 0 < 2x < y < x+1\}$ is a SRS region.

This seems rather tough and a brand new idea is expected for this possible SRS region in $d = 2$. Figure (b) reads one can substitute 0 by 0.8. The essential difficulty of this problem exists around $(1, 2)$.

Problem 9. Let $|\gamma| < 2$. Prove that each integer sequence $\{a_n\}_n$ satisfying $0 \leq a_n + \gamma a_{n+1} + a_{n+2} < 1$ is eventually periodic.

This easy-looking problem seems quite hard except for trivial cases $\gamma = 0, \pm 1$. This 'Salem type periodicity' problem is related to the orbit of piecewise isometry and affirmative answers are known for certain quadratic γ's (Kouptsov-Lowenstein-Vivaldi [16], Akiyama-Brunotte-Pethő-Steiner [4]).

Let us switch our topic and briefly discuss dual tilings due to Thurston. It is intimately related to the explicit construction of an algebraic natural extension of β-expansion (see [3] for detail.) To this matter, it is important to extend β-expansion to the other direction. We introduce a set of β-integers:

$$\mathbb{Z}_\beta = \left\{ \sum_{i=0}^{n} a_i \beta^i \;\middle|\; n \in \mathbb{N}, \quad a_n a_{n-1} \dots a_0 \in d_\beta([0,1)) \right\}$$

This set is a Delone set (relatively dense and uniformly discrete) in $\mathbb{R} \cap [0, \infty)$ if and only if β is a Delone number. Let β be a Pisot number of degree d with r_1 real conjugates and $2r_2$ complex conjugates. We assume that β is a **unit**. Consider an embedding

$$\Phi : \mathbb{Q}(\beta) \to \mathbb{R}^{r_1 - 1} \times \mathbb{C}^{r_2} \simeq \mathbb{R}^{d-1}$$

defined by $x \mapsto (x^{(2)}, \dots, x^{(r_1)}, x^{(r_1+1)}, \dots, x^{(r_1+r_2)})$ where $x^{(i)}$ are the non trivial Galois conjugates of x. Since β is Pisot, the set $\overline{\Phi(\mathbb{Z}_\beta)}$ becomes compact, which is called the **central tile**. This central tile has a natural self-similar structure arose from multiplication/division by β. In fact, we can construct a covering of $\mathbb{R}^{d-1}$ as in Figure 2 but we are not sure that it always gives a tiling, a covering of degree 1.

Since β is Pisot, $d_\beta(1-)$ is eventually periodic and consequently the corresponding symbolic dynamics is sofic. Under the **weak finite** condition:

(W) For any $z \in \mathbb{Z}[\beta] \cap [0, 1)$, there are x, y with finite expansion such that $z = x - y$,

this construction surely gives a tiling of $\mathbb{R}^{d-1}$ by finitely many tiles up to translation, which is a geometric realization of a sofic shift X_β ([2]). The condition (F) is clearly stronger than (W). Under (F), the origin is an inner point of the central tile. This condition (W) seems to hold for all Pisot numbers.

This (W) is understood as a special form of **Pisot conjecture** in substitutive dynamical system. An **arithmetic natural extension** of $([0, 1), T_\beta)$ in $\mathbb{R}^d$ is constructed by dual tiles as in Figure 3.

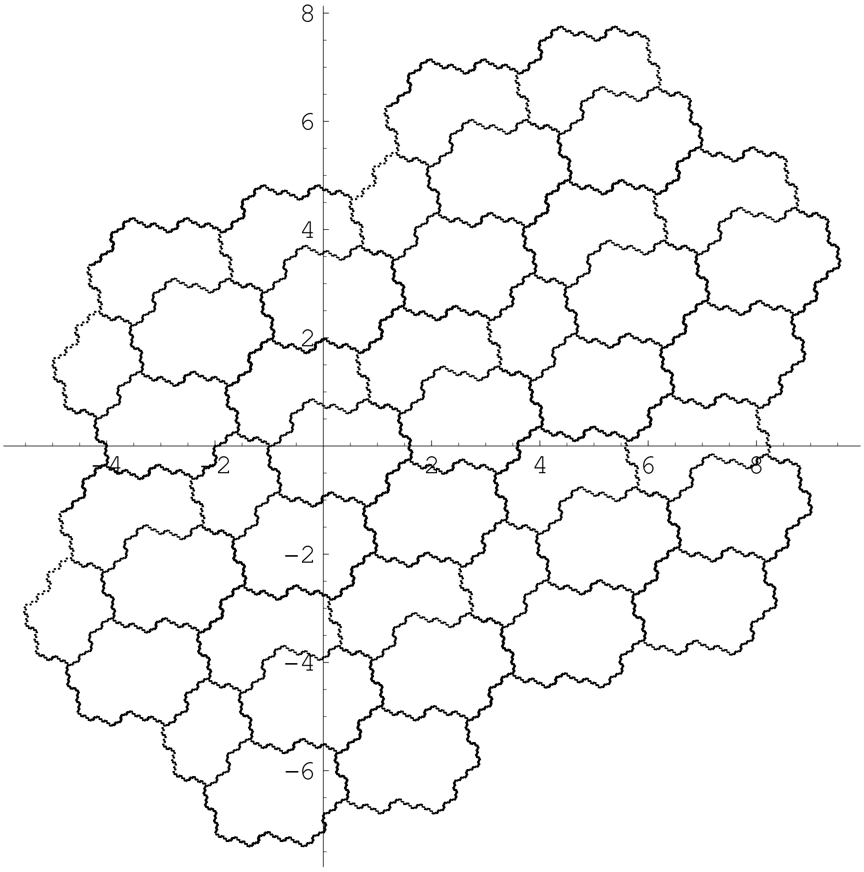


FIGURE 2. $[110]^\infty$: Tribonacci Case

As this Φ is basically composed by Galois conjugates, number theoretical information is preserved. Thanks to this advantage, one can characterize the set of periodic expansions as $\mathbb{Q}(\beta)$-points in the fractal shape (Ito-Rao [13]). The same idea works for non-unit β's by using p-adic embedding (see Berthé-Siegel [6]).

Let τ be an irreducible Pisot substitution on $\{0, 1, \ldots, d-1\}$. **Pisot conjecture** states that the dynamical system generated by shift closure of a fix point of τ has pure discrete spectrum. Many equivalent **coincidence conditions** are known. Here is one of them (a joint work with J.Y. Lee):

Problem 10. In the fixed point of τ, is there a non empty word w and a **relatively dense** occurrences of patterns (of infinite size) of the form:

$$wx_1wx_2wx_3\ldots$$

where x_i are arbitrary words satisfying $|wx_1wx_2\ldots wx_m| = |\tau^{m-1}(wy)|$ for some fixed y ?

One can show this criterion from (W) for β-substitution.

To attack (W) or coincidence, it is worthy to consider a little weaker problem of algebraic nature, which is called 'Height reducing problem'.

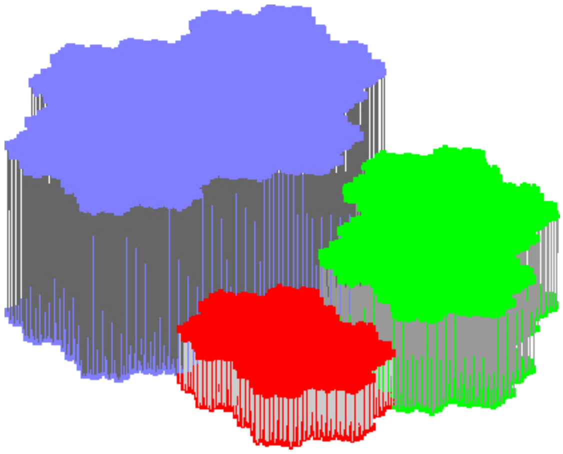


FIGURE 3. Natural extension

Problem 11. Let β be a Pisot number and put $\mathcal{B} = \{0, \pm 1, \pm 2, \dots, \pm\lfloor\beta\rfloor\}$. Prove $\mathbb{Z}[1/\beta] \cap [0,1) = \mathcal{B}[1/\beta] \cap [0,1)$.

Since we do not care that coefficients in the right side can be represented as a difference of admissible words, this is a weaker statement than (W). One can compare it with the same type of question in expanding case:

Problem 12. Let α be an algebraic integer whose all conjugates are greater than one in modulus and $N(\alpha)$ be the absolute norm of α over $\mathbb{Q}$. Set $\mathcal{B} = \{0, \pm 1, \pm 2, \dots, \pm(N(\alpha) - 1)\}$. Is there an easy proof of $\mathbb{Z}[\alpha] = \mathcal{B}[\alpha]$?

To prove tiling property in a different setting, Lagarias-Wang [21, 20, 22] gave an indirect proof of this fact using Wavelet analysis. Class number problems are related ([18, 19]).

To finish this note, I wish to mention my favorite conjecture posed in [1] in Japanese. Let β be a Parry number, i.e.,

$$d_\beta(1-) = a_1 a_2 \dots a_m [a_{m+1} a_{m+2} \dots a_{m+\ell}]^\infty \quad \text{with } a_m \neq a_{m+\ell}.$$

Problem 13. (Dynamical Norm Conjecture) Prove that $N(\beta) = |a_m - a_{m+\ell}|$.

This is supported by extensive numerical computation. When β is a simple-Parry Pisot unit, this conjecture implies that the associated tiles are connected. Here the simpleness is necessary since tiles could be disconnected in degree 4 (Akiyama-Gjini [5]) for non-simple Parry cases.

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