Arithmetic Meyer sets and finite automata

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Abstract

Non-standard number representation has proved to be useful in the speed-up of some algorithms, and in the modelization of solids called quasicrystals. Using tools from automata theory we study the set \( \mathbb{Z}_\beta \) of \( \beta \)-integers, that is, the set of real numbers which have a zero fractional part when expanded in a real base \( \beta \), for a given \( \beta > 1 \). In particular, when \( \beta \) is a Pisot number — like the golden mean —, the set \( \mathbb{Z}_\beta \) is a Meyer set, which implies that there exists a finite set \( F \) (which depends only on \( \beta \)) such that \( \mathbb{Z}_\beta = \mathbb{Z}_\beta + F \). Such a finite set \( F \), even of minimal size, is not uniquely determined.

In this paper we give a method to construct the sets \( F \) and an algorithm, whose complexity is exponential in time and space, to minimize their size. We also give a finite transducer that performs the decomposition of the elements of \( \mathbb{Z}_\beta + \mathbb{Z}_\beta \) as a sum belonging to \( \mathbb{Z}_\beta + F \).

1 Introduction

It is well known that the choice of an adequate number representation can speed-up some algorithms. For instance, the signed-digit number representation consists of an integer base \( \beta > 1 \) and a set of signed digits \( \{-a, -a+1, \ldots, a\} \) with \( \beta/2 < a < \beta - 1 \); in such a system a number may have several representations. This property of redundancy allows fast addition and multiplication, and also to design on-line algorithms, see [3, 8, 10]. A complex base like \(-1 + i\) allows to expand any complex number as a sequence of digits
0 and 1 with no splitting of the real and the imaginary part, and is convenient for some algorithms, see [26].

Special attention has been raised to the case where the base $\beta$ is a non-integer real number. In this case the number system is naturally redundant, see [25]. The well-known fact that addition is computable by a finite transducer when the base is an integer can be extended to some special type of non-integer base. A Pisot number (or a Pisot-Vijayaraghavanan number) is an algebraic integer $> 1$ such that all its algebraic conjugates have modulus strictly less than one. The natural integers and the golden mean are Pisot numbers. It happens that, when the base is a Pisot number, addition is computable by a finite transducer as well [11]. So Pisot numbers can be considered as a nice generalization of the natural integers.

Another domain where these numbers play an important role is the modelization of the so-called “quasicrystals”. The classical crystallography prescribes entirely the possible orders of symmetry of crystals: it can be 2, 3, 4 or 6. When physicists observed in the eighties new alloys presenting a symmetry of order 5, and a long-range aperiodic order, the mathematical notion of quasicrystals had already been introduced by Meyer [20, 21, 22, 23, 24] in order to define a generalization of ideal crystalline structures. So the name of Meyer set was given to a mathematical idealization of these solids.

A set $X$ of $\mathbb{R}^d$ is a Meyer set if it is a Delaunay set — that is, a set which is uniformly discrete and relatively dense — and if there exists a finite set $F$ such that the set of differences $X - X$ is a subset of $X + F$. Meyer [20] has shown that if $X$ is a Meyer set and if $\beta > 1$ is a real number such that $\beta X \subset X$ then $\beta$ must be a Pisot or a Salem number $^1$. Conversely for each $d$ and for each Pisot or Salem number $\beta$, there exists a Meyer set $X \subset \mathbb{R}^d$ such that $\beta X \subset X$. Note that all the quasicrystals observed in the real world are linked to quadratic Pisot numbers, namely $\frac{1+\sqrt{5}}{2}$, $1 + \sqrt{2}$ and $2 + \sqrt{3}$, see [4].

In classical crystallography, crystals are sitting in a lattice, whose vertices are indexed by integers. In quasicrystallography, the points of a quasicrystal are labelled by the so-called $\beta$-integers, which are real numbers such that their fractional part is equal to 0 when they are expanded in base $\beta$ (see Section 2 for definitions). So numeration in real base $\beta$ is an adequate tool for the description of these solids. As a consequence, $\beta$-integers are handled as words, and the set of the expansions of $\beta$-integers is known to be recognizable by a finite state automaton when $\beta$ is a Pisot number (see [13]).

When $\beta$ is a Pisot number, the set $\mathbb{Z}_\beta$ of $\beta$-integers is a Meyer set, see [7]. In this paper, by means of automata theory tools, we give an algorithm that computes a minimal set $F$ such that $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$.

With a geometrical approach, Lagarias [18] has given a general construction of a set $F$ satisfying $X - X \subset X + F$ for any Meyer set $X$. But the sets obtained are huge and no method of minimization of these sets is known. Minimal sets $F$ are given in [7] for $\mathbb{Z}_\beta$ when $\beta$ is a quadratic Pisot unit. When $\beta$ is a quadratic Pisot number, a possible set $F$ for $\mathbb{Z}_\beta$ is exhibited in [14]. The method consists in giving a bound on the length of the

$^1$A Salem number is an algebraic integer such that every conjugate has modulus smaller than or equal to 1, and at least one of them has modulus 1.
fractional part of the \( \beta \)-expansion of the sum (resp. the difference) of two \( \beta \)-expansions.

In this work we use different methods, coming from automata theory. We first give the minimal finite automata describing the formal addition and subtraction, that is the digit-sum and digit-difference, of \( \beta \)-integers in the case where \( \beta \) is a Parry number (see definition in Section 2). Every Pisot number is a Parry number, but the converse does not hold.

We then give a construction of a finite set \( F \) of minimal size such that \( \mathbb{Z} \beta - \mathbb{Z} \beta \subset \mathbb{Z}_\beta + F \) making use of automata. This algorithm of minimization, which is the first known, is exponential in time an space. It also computes a finite transducer that performs the decomposition of the result of the formal subtraction \( \mathbb{Z} \beta - \mathbb{Z} \beta \) into a sum belonging to \( \mathbb{Z}_\beta + F \).

A preliminary version of this work has been presented in [2].

2 Preliminaries

Let \( A \) be a finite alphabet. A concatenation of letters of \( A \) is called a word. The set \( A^* \) of all finite words equipped with the operation of concatenation and the empty word \( \varepsilon \) is a free monoid. We denote by \( a^k \) the word obtained by concatenating \( k \) letters \( a \). The length of a word \( w = w_0w_1 \cdots w_{n-1} \) is denoted by \( |w| = n \). One considers also infinite words \( v = v_0v_1v_2 \cdots \). The set of infinite words on \( A \) is denoted by \( A^\mathbb{N} \). An infinite word \( v \) is said to be eventually periodic if it is of the form \( v = wz^\omega \), where \( w \) and \( z \) are in \( A^* \) and \( z^\omega = zzz \cdots \). A factor of a finite or infinite word \( w \) is a finite word \( v \) such that \( w = uvz \) if \( u = \varepsilon \), the word \( v \) is a prefix of \( w \).

The lexicographic order for infinite words over an ordered alphabet is defined by \( v <_\text{lex} w \) if there exist factorizations \( v = uav_0 \) and \( y = ubw_0 \), for some word \( u \in A^* \), \( a, b \in A \) such that \( a < b \), and \( v', w' \in A^\mathbb{N} \).

Beta-expansions

Definitions and results can be found in [19, Chapter 7]. Let \( \beta > 1 \) be a real number. Any non-negative real number \( x \) can be represented in base \( \beta \) by the following greedy algorithm [27].

Denote by \( \lfloor . \rfloor \) and by \( \{ . \} \) the integral part and the fractional part of a number. There exists \( k \in \mathbb{Z} \) such that \( \beta^k \leq x < \beta^{k+1} \). Let \( x_k = \lfloor x/\beta^k \rfloor \) and \( r_k = \{ x/\beta^k \} \). For \( i < k \), put \( x_i = \lfloor \beta r_{i+1} \rfloor \), and \( r_i = \{ \beta r_{i+1} \} \). Then \( x = x_k \beta^k + x_{k-1} \beta^{k-1} + \cdots \). If \( x < 1 \), we get \( k < 0 \) and we put \( x_0 = x_1 = \cdots = x_{k+1} = 0 \). The sequence \( (x_i)_{k \geq i \geq -\infty} \) is called the (greedy) \( \beta \)-expansion of \( x \), and is denoted by

\[
\langle x \rangle_\beta = x_k x_{k-1} \cdots x_1 x_0 \cdot x_{-1} x_{-2} \cdots
\]

most significant digit first. The part \( x_{-1} x_{-2} \cdots \) after the “decimal” point is called the \( \beta \)-fractional part of \( x \).

The digits \( x_i \) are elements of the canonical alphabet \( A_\beta = \{0, \ldots, \lfloor \beta \rfloor \} \) if \( \beta \notin \mathbb{N} \) and \( A_\beta = \{0, \ldots, \beta - 1\} \) otherwise. When a \( \beta \)-expansion ends in infinitely many zeroes, it is said to be finite, and the 0’s are omitted.
A finite or infinite word \( w \) on \( A_\beta \) which is the \( \beta \)-expansion of some non-negative number \( x \) is said to be admissible. Leading 0’s are allowed. The normalization on an alphabet of digits \( D \supseteq A_\beta \) is the function that maps a word \( w = w_k \cdots w_0 \) on \( D \) onto the \( \beta \)-expansion of its numerical value \( \sum_{i=0}^{k-1} d_i \beta^i \) in base \( \beta \). The same notion exists for infinite words. Addition is a particular case of normalization: first add digit-wise two \( \beta \)-expansions; this gives a word on the alphabet \( \{0, \ldots, 2\lceil \beta \rceil \} \); then normalize to obtain the result. It is known that for every alphabet \( D \) normalization is computable by a finite transducer [11].

Denote by \( D_\beta \) the set of \( \beta \)-expansions of numbers of \([0, 1)\) and by \( \sigma \) the shift defined by \( \sigma(xkx_{k-1} \cdots) = x_{k-1}x_{k-2} \cdots \). Then \( D_\beta \) is shift-invariant. Let \( S_\beta \) be its closure in \( A_\beta^\mathbb{N} \). The set \( S_\beta \) is a symbolic dynamical system, called the \( \beta \)-shift. There is a peculiar representation of the number 1 which can be used to characterize the elements of the \( \beta \)-shift. It is denoted by \( d_\beta(1) \), and computed by the following process [27]. Let the \( \beta \)-transform be defined on \([0, 1)\) by \( T_\beta(x) = x \mod 1 \). Then \( d_\beta(1) = (t_i)_{i \geq 1} \), where \( t_i = \lfloor \beta T_\beta^{i-1}(1) \rfloor \). Note that \( \lfloor \beta \rfloor = t_1 \).

Set \( d_\beta^+(1) = (t_1 \cdots t_{m-1}(t_m - 1))\omega \) if \( d_\beta(1) = t_1 \cdots t_{m} \) is finite, and \( d_\beta^+(1) = d_\beta(1) \) if \( d_\beta(1) \) is infinite. Then a sequence \( s \) of natural integers is an element of \( D_\beta \) if and only if for every \( p \geq 1 \), \( \sigma^p(s) \) is strictly less in the lexicographic order than \( d_\beta^+(1) \), see Parry [25].

The numbers \( \beta \) such that \( d_\beta(1) \) is eventually periodic are called Parry numbers, and simple Parry numbers in the case where \( d_\beta(1) \) is finite. When \( \beta \) is a Pisot number then \( d_\beta(1) \) is finite or infinite eventually periodic [5, 29].

**Example 1** If \( \beta = \frac{1+\sqrt{5}}{2} \), then \( d_\beta(1) = 11 \) and \( d_\beta^+(1) = (10)\omega \).

If \( \beta = \frac{3+\sqrt{5}}{2} \), then \( d_\beta(1) = 21^\omega = d_\beta^+(1) \).

The set \( Z_\beta \) of \( \beta \)-integers is the set of real numbers \( x \) such that the \( \beta \)-fractional part of \( |x| \) is equal to 0,

\[
Z_\beta = \{ x \in \mathbb{R} \mid \langle |x| \rangle _\beta = x_k \cdots x_0 \} = Z_\beta^+ \cup Z_\beta^-
\]

where \( Z_\beta^+ \) is the set of non-negative \( \beta \)-integers, and \( Z_\beta^- = -Z_\beta^+ \). Observe that

\[
-Z_\beta = Z_\beta \quad \text{and} \quad \beta(Z_\beta) \subset Z_\beta.
\]

Notice that, if \( \beta \) is an integer, the set of \( \beta \)-integers is just \( \mathbb{Z} \).

Denote \( L_\beta^+ \) the set of \( \beta \)-expansions of the elements of \( Z_\beta^+ \) with possible leading 0’s; then \( L_\beta^+ \) is equal to the set of finite factors of \( S_\beta \).

**Meyer sets**

We recall here several definitions and results from Meyer that can be found in [20, 21, 22, 23, 24]. A set \( X \subset \mathbb{R}^d \) is uniformly discrete if there exists a positive real \( r \) such that for any \( x \in \mathbb{R}^d \), the open ball of center \( x \) and radius \( r \) contains at most one point of \( X \). If \( Y \subset X \) and \( X \) is uniformly discrete, then \( Y \) is uniformly discrete. A set \( X \subset \mathbb{R}^d \) is relatively dense if there exists a positive real \( R \) such that for any \( x \in \mathbb{R}^d \), the open ball
of center $x$ and radius $R$ contains at least one point of $X$. If $X \subset Y$ and $X$ is relatively
dense, then $Y$ is relatively dense. A set $X$ is a Delaunay set if it is uniformly discrete and relatively dense.

The set $X - X$ is the set $\{x - y \mid x \in X, y \in X\}$. A set $X$ is a Meyer set if it is a Delaunay set and there exists a finite set $F$ such that $X - X \subset X + F$. Lagarias has proved [18] that a set $X$ is a Meyer set if and only if both $X$ and $X - X$ are Delaunay sets. Note that when $X$ is a Delaunay set, then $X - X$ is relatively dense, but not necessarily uniformly discrete. For example $X = \{n + \frac{1}{|n|+2} \mid n \in \mathbb{Z}\}$ is a Delaunay set and $X - X$ has 1 as point of accumulation.

**Lemma 1** For $\beta$ a real number > 1, the set $\mathbb{Z}_\beta$ of $\beta$-integers is relatively dense.

**Proof.** Indeed any non-negative real number $x$ can be expanded as

$$\langle x \rangle_\beta = x_k x_{k-1} \cdots x_1 x_0 \cdot x_{-1} x_{-2} \cdots$$

thus $x = z + r$ with $z = \sum_{i=0}^{k} x_i \beta^i \in \mathbb{Z}_\beta^+$, and $0 \leq r = \sum_{i<0} x_i \beta^i < 1$ is the $\beta$-fractional part of $x$. Thus the maximal distance between two consecutive elements of $\mathbb{Z}_\beta$ is equal to 1. \qed

The following result is already proved in [7], but we give here a different proof.

**Proposition 1** If $\beta$ is a Pisot number, then the set $\mathbb{Z}_\beta$ of $\beta$-integers is a Meyer set.

**Proof.** Let us prove that $\mathbb{Z}_\beta$ is uniformly discrete when $\beta$ is a Pisot number. Indeed the minimal distance between two consecutive points $a$ and $b$ of $\mathbb{Z}_\beta$ with $\langle a \rangle_\beta = a_N \cdots a_0$ and $\langle b \rangle_\beta = b_N \cdots b_0$ is equal to the minimum of $|\sum_{i=0}^{N} (a_i - b_i) \beta^i|$.

Since an integral linear combination of algebraic integers is still an algebraic integer, $\sum_{i=0}^{N} (a_i - b_i) \beta^i$ is an algebraic integer. Let $\beta^{(2)}, \ldots, \beta^{(d)}$ be the conjugates of $\beta = \beta^{(1)}$. As the product of all the conjugates of an algebraic integer is a positive integer, we get

$$\prod_{j=1}^{d} \left( \sum_{i=0}^{N} (a_i - b_i) \beta^{(j)} \right) \geq 1.$$ 

As all conjugates of $\beta$ have a modulus strictly less than 1 and $|a_i - b_i| \leq 2[\beta]$, 

$$\sum_{i=0}^{N} (a_i - b_i) \beta^i \geq \frac{1}{\prod_{j=2}^{d} \frac{2[\beta]}{1-|\beta^{(j)}|}}.$$ 

Since this bound is independent of $N$, $\mathbb{Z}_\beta$ is uniformly discrete. Using Lemma 1, $\mathbb{Z}_\beta$ is a Delaunay set.

The uniform discretness of $\mathbb{Z}_\beta - \mathbb{Z}_\beta$ can be proved as above with $|a_i - b_i| \leq 4[\beta]$. Moreover as $\mathbb{Z}_\beta$ is a Delaunay set, $\mathbb{Z}_\beta - \mathbb{Z}_\beta$ is relatively dense, thus it is a Meyer set. \qed
3 Automata for $\mathbb{Z}_\beta - \mathbb{Z}_\beta$

In this section we construct automata that symbolically describe the elements of $\mathbb{Z}_\beta - \mathbb{Z}_\beta$ when $\beta$ is a Parry number. This simple symbolical description of the elements of $\mathbb{Z}_\beta - \mathbb{Z}_\beta$ will be used, in the following sections, to determine minimal sets $F$ associated with the Meyer set $\mathbb{Z}_\beta$ when $\beta$ is a Pisot number.

3.1 Minimal automaton for $\mathbb{Z}_\beta$

When $\beta$ is a Parry number, the set $L_\beta^+$ is recognizable by a minimal finite automaton [13], of which we recall the construction. The reader is referred to [9] and [28] for definitions and results in automata theory. Let us recall the classical construction of the minimal automaton recognizing a language $L$. The right congruence modulo $L$ is defined as follows: two words $v$ and $w$ are congruent modulo $L$ if they have the same right contextes, more precisely $v \sim_L w$ if $vu \in L$ if and only if $wu \in L$. The minimal automaton of $L$ is then constructed as follows: the states are the right classes mod $L$, denoted by $[v]_L$. There is a transition from $[v]_L$ to $[v']_L$ labelled by $a$ if $[v']_L = [va]_L$. The initial state is $[\varepsilon]_L$. A state $[v]_L$ is terminal if $v$ belongs to $L$.

If $d_\beta(1) = t_1 \cdots t_m$ is finite, the automaton $\mathcal{A}_{\mathbb{Z}_\beta^+}$ recognizing $L_\beta^+$ has $m$ states, denoted $0, 1, \ldots, m - 1$. The name of state $i$ stands for $[t_1 \cdots t_i]_{L_\beta^+}$, and $0 = [\varepsilon]_{L_\beta^+}$. Denote by suff$_k$ the suffix of $d_\beta^*(1)$ starting at index $k \geq 1$. Note that, because of the admissibility condition, the right context of state $i$ is entirely determined by suff$_{i+1}$, which is the greatest word in the lexicographic order that can be read from $i$. For each $0 \leq i \leq m - 2$ there is an edge between states $i$ and $i + 1$ labelled by $t_{i+1}$. For each $0 \leq i \leq m - 1$ there are $t_{i+1}$ edges between states $i$ and $0$ labelled by $0, 1, \ldots, t_{i+1} - 1$. The initial state is $0$; every state is terminal. The automaton is shown on Fig. 1.

![Figure 1: Automaton $\mathcal{A}_{\mathbb{Z}_\beta^+}$ when $d_\beta(1) = t_1 \cdots t_m$.](image)

The case where $d_\beta(1) = t_1 \cdots t_m (t_{m+1} \cdots t_{m+p})^\omega$ is infinite eventually periodic is similar. The automaton $\mathcal{A}_{\mathbb{Z}_\beta^+}$ recognizing $L_\beta^+$ has $m + p$ states $0, \ldots, m + p - 1$. For
each $0 \leq i \leq m + p - 2$ there is an edge between $i$ and $i + 1$ labelled by $t_{i+1}$. For each $0 \leq i \leq m + p - 1$ there are $t_{i+1}$ edges between $i$ and $0$ labelled by $0, \ldots, t_{i+1} - 1$. There is an edge from $m + p - 1$ to $m$ labelled by $t_{m+p}$. The initial state is 0; every state is terminal. The automaton is shown on Fig. 2.

![Automaton Diagram](image)

**Figure 2: Automaton $A_{Z_{\beta}}$ when $d_{\beta}(1) = t_{1} \cdots t_{m}(t_{m+1} \cdots t_{m+p})^\omega$.**

We introduce some notations. Set $\bar{k} = -k$, where $k$ is an integer, and let $A_{\beta} = \{[\beta], \ldots, 1, 0\}$. We denote by $L_{\beta}^- \subset A_{\beta}^+$ the set $\overline{\{w = w_N \cdots w_0 \mid w = w_N \cdots w_0 = (-x)_{\beta}, x \in \mathbb{Z}_{\beta}\}}$.

Clearly the set $L_{\beta}^-$ is recognizable by the same automaton as $L_{\beta}^+$, but with negative labels on edges. Then the set $L_{\beta} = L_{\beta}^+ \cup L_{\beta}^-$ of $\beta$-expansions of the elements of $\mathbb{Z}_{\beta}$ is recognized by the finite automaton $\mathcal{A}_{Z_{\beta}} = \mathcal{A}_{Z_{\beta}^+} \cup \mathcal{A}_{Z_{\beta}^-}$. By abuse we say that $\mathbb{Z}_{\beta}$ is recognized by $\mathcal{A}_{Z_{\beta}}$.

**Example 2** Take $\beta = \frac{1 + \sqrt{5}}{2}$. Minimal automata $A_{Z_{\beta}^+}, A_{Z_{\beta}^-}$ and $A_{Z_{\beta}}$ are given in Fig. 3. Initial states are indicated by an incoming arrow, and all states are terminal.

Since

$$Z_{\beta} - Z_{\beta} = (Z_{\beta}^+ - Z_{\beta}^-) \cup (Z_{\beta}^+ + Z_{\beta}^-) \cup -(Z_{\beta}^+ + Z_{\beta}^+) \quad (1)$$

we introduce symbolic representations of $Z_{\beta}^+ + Z_{\beta}^-$ and $Z_{\beta}^+ - Z_{\beta}^-$. More precisely the **formal addition** of elements of $\mathbb{Z}_{\beta}^+$ consists in adding elements without carry. More precisely,

$$L_{\beta}^+ + L_{\beta}^+ = \{(a_N + b_N) \cdots (a_0 + b_0) \mid N \geq 0, a_N \cdots a_0, b_N \cdots b_0 \in L_{\beta}^+\} \subset \{0, \ldots, 2[\beta]\}^*$$

Similarly the **formal subtraction** of elements of $\mathbb{Z}_{\beta}^+$ is defined by

$$L_{\beta}^+ - L_{\beta}^+ = \{(a_N - b_N) \cdots (a_0 - b_0) \mid N \geq 0, a_N \cdots a_0, b_N \cdots b_0 \in L_{\beta}^+\} \subset \{-[\beta], \ldots, [\beta]\}^*.$$
3.2 Minimal automaton of $L^+_\beta + L^+_\beta$

We give a direct construction of the minimal automaton of $L^+_\beta + L^+_\beta$ when $\beta$ is a Parry number. Let $Q = \{0, 1, \ldots, h - 1\}$ be the set of states of the minimal automaton of $L^+_\beta$ ($h = m$ or $h = m + p$ according to the value of $d_\beta(1)$, see Section 3.1).

We construct an automaton $S$ as follows.

The set of states is the set $Q_S = \{(i, j) \in Q^2 | i \leq j\}$. The cardinality of this set is equal to $h(h + 1)/2$. The initial state is $(0, 0)$ and every state is terminal.

Let $c$ be in $\{0, \ldots, 2|\beta|\}^*$, and let $(i, j)$ be in $Q_S$. Let $C_c(i, j) = \{(i', j') \in Q^2 | \exists a, b \in A_\beta, c = a + b, i \xleftarrow{a} i' \text{ and } j \xrightarrow{b} j' \text{ in } A_{Z_\beta^+}\}$. If $C_c(i, j)$ is empty there is no transition outgoing from state $(i, j)$ with label $c$.

Suppose that $C_c(i, j)$ is not empty. Let $(i', j') \in C_c(i, j)$. We have seen in Section 3.1 that the right context modulo $L^+_\beta$ of state $i'$ is entirely determined by suffix$_{i'+1}$, and similarly for $j'$. Take $(r, s) \in C_c(i, j)$ such that suffix$_{r+1} + \text{ suffix}_{s+1} \geq \text{ suffix}_{r'+1} + \text{ suffix}_{j'+1}$ for all $(i', j') \in C_c(i, j)$. This choice ensures that the future readings will be the greatest possible in the lexicographic order. Then we define in $S$ a transition $(i, j) \xrightarrow{c} (r, s)$ if $r \leq s$, or a transition $(i, j) \xleftarrow{c} (s, r)$ otherwise.

Thus the following holds true.

**Proposition 2** The automaton $S$ is the minimal automaton of $L^+_\beta + L^+_\beta$.

3.3 Minimal automaton of $L^+_\beta - L^+_\beta$

We construct an automaton $D$ for $L^+_\beta - L^+_\beta$ as follows.

The set of states is the set $Q_D = \{(i, 0), (0, i) \in Q^2 | 0 \leq i \leq h - 1\}$. The cardinality of this set is equal to $2h - 1$. The initial state is $(0, 0)$ and every state is terminal.

Let $c$ be in $\{0, \ldots, |\beta|\}^*$ and let $(i, j)$ be in $Q_D$. If $c = t_{i+1}$ and if $i \xrightarrow{c} i + 1$ in $A_{Z_\beta^+}$ we define in $D$ a transition $(i, j) \xrightarrow{c} (i + 1, 0)$. If $c < t_{i+1}$ we define a transition $(i, j) \xrightarrow{c} (0, 0)$. Symmetrically if $\bar{c} = -t_{j+1}$ and if $j \xrightarrow{c} j + 1$ in $A_{Z_\beta^-}$ we define a
transition \((i, j) \xrightarrow{c} (0, j + 1)\). If \(c > -t_{j+1}\) there is a transition \((i, j) \xrightarrow{c} (0, 0)\). In each case the future readings will be the greatest possible in the lexicographic order. Thus the following holds true.

**Proposition 3** The automaton \(D\) is the minimal automaton of \(L_\beta^+ - L_\beta^+\).

### 3.4 Fibonacci example

**Example 3** In Fig. 4 are drawn the minimal automata \(A_{Z_\beta^+ + Z_\beta^+}\) and \(A_{Z_\beta^+ - Z_\beta^+}\) in the case where \(\beta = \frac{1+\sqrt{5}}{2}\). Every state is terminal.

![Figure 4: Automata \(A_{Z_\beta^+ + Z_\beta^+}\) and \(A_{Z_\beta^+ - Z_\beta^+}\).](image)

### 4 A family of finite sets containing a minimal set \(F\)

When \(\beta\) is a Pisot number, the set of beta-integers \(Z_\beta\) is a Meyer set so there exists a finite set \(F\) such that \(Z_\beta - Z_\beta \subset Z_\beta + F\). Our goal is to construct sets \(F\) as small as possible for \(Z_\beta\).

Note the following property of minimal sets \(F\).

**Lemma 2** If \(F\) is a set of minimal size such that \(Z_\beta - Z_\beta \subset Z_\beta + F\) then

\[F \subset (Z_\beta - Z_\beta) - Z_\beta.\]

**Proof.** Let \(F\) be a set of minimal size such that \(Z_\beta - Z_\beta \subset Z_\beta + F\), that is

\[\forall x \in Z_\beta - Z_\beta, \exists (y, f) \in Z_\beta \times F \text{ such that } x = y + f.\]

If there exists \(f \in F\) such that for all \(x \in Z_\beta - Z_\beta\) and for all \(y \in Z_\beta\), \(f \neq x - y\) then \(F' = F \setminus \{f\}\) satisfies \(Z_\beta - Z_\beta \subset Z_\beta + F'\) and \(F'\) is strictly smaller than \(F\), that is contradictory with \(F\) minimal.

Note that there may exist several sets \(F\) of minimal size.

**Example 4** For \(\beta = (1+\sqrt{5})/2\) the possible minimal sets \(F\) such that \(Z_\beta - Z_\beta \subset Z_\beta + F\) are the following

1. \(F = \{0, \beta - 1, -\beta + 1\} = \{0, \frac{1}{\beta}, -\frac{1}{\beta}\}\), see [7]
2. $F = \{0, \beta - 2, -\beta + 2\} = \{0, \frac{1}{\beta}, -\frac{1}{\beta}\} \subset \left[-\frac{1}{2}, \frac{1}{2}\right]$, see [12]

3. $F = \{0, \beta - 1, -\beta + 2\} = \{0, \frac{1}{\beta}, \frac{1}{\beta}\} \subset [0, 1]$.

Proof. To prove 3., suppose from 1. that for $x$ and $y$ in $\mathbb{Z}_\beta$ there exists $z$ in $\mathbb{Z}_\beta$ such that $x - y = z - \frac{1}{\beta}$. Suppose first $z$ in $\mathbb{Z}_\beta^+$. Denote $(z)_{\beta} = z_k \cdots z_0$ and let $z_i$ be the rightmost non-zero digit. If $i$ is even, then $x - y = z^{(1)} + \frac{1}{\beta^2}$ where $z^{(1)}$ has for $\beta$-expansion the word $z_k \cdots z_{i+1}(01)^{i/2}0$, and is thus in $\mathbb{Z}_\beta^+$. If $i$ is odd, then $x - y = z^{(2)}$ where $z^{(2)}$ has for $\beta$-expansion $z_k \cdots z_{i+1}(01)^{i/2}$. Now suppose that $z$ belongs to $\mathbb{Z}_\beta^-$. Let $(z)_{\beta} = u = u_k \cdots u_0$. First suppose that $u_0 = 0$, then write $u$ in the form $u'0(01)^{\ell}0$ (if necessary $u$ can be prefixed by two zeroes); then $-(x - y) = -z + \frac{1}{\beta}$ is equal to $v^{(1)} - \frac{1}{\beta^2}$ where $v^{(1)}$ has for $\beta$-expansion the word $u'010^{2\ell}$. If $u_0 = 1$, then $u$ can be written as $u'0(01)^{\ell}$; then $-(x - y)$ has for $\beta$-expansion the word $u'010^{2\ell-1}$. \hfill \Box

Using properties of the algebraic conjugates of the elements of minimal sets $F$, we first define finite sets from which can be extracted the finite sets $F$.

**Lemma 3** Let $\beta$ be a Pisot number of degree $d$, let $I \subset \mathbb{R}$ be an interval of finite length greater than or equal to 1 and let $W$ be the following set

$$W = \left\{ x \in \mathbb{Z}[\beta] \mid x \in I \text{ and for } 2 \leq j \leq d, \ |x^{(j)}| < \frac{3|\beta|}{1 - |\beta|} \right\},$$

where $x^{(2)}, \ldots, x^{(d)}$ are the algebraic conjugates of $x$. Then $W$ is finite, and $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + W$.

**Proof.** From Lemma 1 the maximal distance between two consecutive points of $\mathbb{Z}_\beta$ is equal to 1, thus one can find a finite set $F$ such that $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$ in any interval $I$ of length greater than or equal to 1. Fix an interval $I$ of length $\geq 1$ and let $F$ be a finite subset of $I$ of minimal size such that $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$. Let $x \in F$, then from Lemma 2, $x \in (\mathbb{Z}_\beta - \mathbb{Z}_\beta) - \mathbb{Z}_\beta$ and can be written as

$$x = \sum_{i=0}^{N} (a_i - b_i)\beta^i \sum_{i=0}^{N} c_i\beta^i \quad \text{with } |a_i|, |b_i|, |c_i| \leq |\beta|.$$

So

for $2 \leq j \leq d$ \quad $x^{(j)} = \sum_{i=0}^{N} (a_i - b_i - c_i)(\beta^{(j)})^i \quad \text{with } |a_i - b_i - c_i| \leq 3|\beta|.$

As $\beta$ is a Pisot number, for all $j \geq 2$, $|\beta^{(j)}| < 1$ and $|\sum_{i=0}^{N}(\beta^{(j)})^i| < (1 - |\beta^{(j)}|)^{-1}$. We obtain in this way the announced bound on the moduli of the conjugates of $x$ and $x \in W$. So $F$ is a subset of $W$.

Since $\beta$ is a Pisot number the set $W$ contains only points of $\mathbb{Z}[\beta]$ with bounded modulus and whose all conjugates have bounded modulus, thus $W$ is finite. \hfill \Box

The choice of any interval $I \subset [-1, 1]$ of length 1 allows us to reduce the size of the set containing a minimal set $F$. 10
Lemma 4 \textit{Let }\beta\textit{ be a Pisot number of degree }d\textit{, let }I \subset ]-1,1[\textit{ be an interval of length 1 and let }U\textit{ be the following set}

\[ U = \left\{ x \in \mathbb{Z}[\beta] \mid x \in I \text{ and for } 2 \leq j \leq d, |x^{(j)}| < \frac{2|\beta|}{1-|\beta^{(d)}|} \right\}. \]

\textit{Then }U\textit{ is finite and }\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + U.\]

\textbf{Proof}. \textit{We choose here }I \subset ]-1,1[\textit{ of length 1 and improve the bound on the moduli of the conjugates of }x\textit{ given in Lemma 3 by considering the decomposition}

\[ \mathbb{Z}_\beta - \mathbb{Z}_\beta = (\mathbb{Z}_\beta^+ - \mathbb{Z}_\beta^+) \cup (\mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+) \cup -(\mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+). \]

\textit{More precisely let }F\textit{ be a finite subset of }I\textit{ of minimal size such that }\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F\textit{ and let }x \in F, \textit{ then }x \in (\mathbb{Z}_\beta - \mathbb{Z}_\beta) - \mathbb{Z}_\beta\textit{ and can be written as}

\[ x = \sum_{i=0}^{N} (a_i - b_i)\beta^i - \sum_{i=0}^{N} c_i\beta^i. \]

\textit{We study }|a_i - b_i - c_i|\textit{ according to the signs of }a_i, b_i\text{ and }c_i. \textit{Recall that }|a_i|, |b_i|\text{ and }|c_i|\textit{ are smaller than }|\beta|. \textit{In }\mathbb{Z}_\beta^+ - \mathbb{Z}_\beta^+\text{ and }\mathbb{Z}_\beta^+ - \mathbb{Z}_\beta^-\textit{, the products }a_ib_i\textit{ are non-negative and the coefficients satisfy }|a_i - b_i| \leq |\beta|. \textit{When }F \subset ]-1,1[, \mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^- \subset \mathbb{Z}_\beta^+ + F\textit{ and }-\left(\mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^-\right) \subset \mathbb{Z}_\beta^- + F, \textit{ so when }a_ib_i \leq 0, \textit{ then }a_ic_i \geq 0\textit{ and we have }|a_i - c_i| \leq |\beta|. \textit{Thus when }F \subset ]-1,1[, \textit{ we get in all cases }|a_i - b_i - c_i| \leq 2|\beta|. \textit{Thus}

\[ \text{for } 2 \leq j \leq d \quad x^{(j)} = \sum_{i=0}^{N} (a_i - b_i - c_i)(\beta^{(j)})^i \quad \text{with } |a_i - b_i - c_i| \leq 2|\beta|, \]

\textit{and the announced bound on the moduli of the conjugates of }x\textit{ holds true. The proof that }U\textit{ is finite is the same as for }W.\]

\textbf{Remark 1} \textit{In what follows we restrict our study to the sets }U\textit{ defined in Lemma 4 as finite subsets of intervals }I \subset ]-1,1[\textit{ of length 1, but all constructions remain valid with small changes for the finite sets }W\textit{ introduced in Lemma 3 as finite subsets of arbitrary intervals of length greater or equal to 1.}

\textbf{Quadratic Pisot numbers}

\textit{We now establish a bound on the size of the sets }U\textit{ of Lemma 4 for any quadratic Pisot number }\beta. \textit{Recall [13] that a quadratic Pisot number }\beta\textit{ has a minimal polynomial of the form }M_\beta = X^2 - aX - b, \textit{ with either }a \geq b \geq 1, \text{ or } a \geq 3 \text{ and } 0 > b \geq -a + 2. \textit{In the first case }d_\beta(1) = ab, \text{ and in the second one }d_\beta(1) = (a-1)(a+b-1)^2.
Proposition 4  Let $\beta$ be a quadratic Pisot number with minimal polynomial $M_\beta = X^2 - aX - b$. Then for any interval $I \subset [-1,1]$ of length 1, $\text{Card}(U) \leq 2|B - 1| + 1$, with

$$B = \begin{cases} \\frac{a}{a^2 + 1} + \frac{a(a+2)}{(a+1)(a^2 + 1)} + \frac{1}{a} & \text{when } a \geq b > \frac{a}{2}, \\ \frac{a}{a^2 + 1} + \frac{1}{a} & \text{when } 0 < b \leq \frac{a}{2}, \\ \frac{a}{a^2 + 1} + \frac{1}{a} & \text{when } -\frac{a}{2} < b < 0, \\ \frac{a}{a^2 + 1} + \frac{1}{a} & \text{when } -a + 2 \leq b \leq -\frac{a}{2}. \end{cases}$$

Proof. Denote by $\beta'$ the algebraic conjugate of $\beta$. Any point $x$ of $\mathbb{Z}[\beta]$ and its algebraic conjugate $x'$ can be written as $x = x_1 + x_2\beta$ and $x' = x_1 + x_2\beta'$ where $x_1, x_2 \in \mathbb{Z}$. Then

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\beta - \beta'} \begin{pmatrix} -\beta' & \beta \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}.$$ 

Note that for each value of $x_2$ there is only one possible value for $x_1$ such that $x \in U$ since $x_1$ is an integer and the interval $I$ is of length 1. So if, for all $x \in U$, $|x_2| < B$ then $|x_2| \leq \lfloor B - 1 \rfloor$ and $\text{Card}(U) \leq 2|B - 1| + 1$.

We establish the bound on the modulus of $x_2$ using the inequalities $|x| < 1$ and $|x'| \leq 2|\beta|/(1 - |\beta'|)$ with $|\beta| = a$ when $b > 0$ and $|\beta| = a - 1$ when $b < 0$. Setting $\Delta = a^2 + 4b$, we get

when $b > 0$,

$$|x_2| < \frac{1}{\sqrt{\Delta}} \left( 1 + \frac{4a(a + 2 + \sqrt{\Delta})}{(a + 2)^2 + \Delta} \right) \leq \frac{a}{a - b + 1} + \frac{a(a + 2)}{\sqrt{\Delta}(a - b + 1)} + \frac{1}{\sqrt{\Delta}}$$

and when $b < 0$,

$$|x_2| < \frac{1}{\sqrt{\Delta}} \left( 1 + \frac{4(a - 1)(a + 2 + \sqrt{\Delta})}{\Delta - (a - 2)^2} \right) \leq \frac{a - 1}{a + b - 1} + \frac{(a - 1)(a - 2)}{\sqrt{\Delta}(a + b - 1)} + \frac{1}{\sqrt{\Delta}}.$$ 

The announced bounds follow from the study of $\Delta$ according to the value of $b$. \qed

Remark 2  Specifying the values for $a$ and $b$ given above for $B$, we obtain the following bounds.

- If $a \geq b > \frac{a}{2}$, then $B \leq 2a + 1$ and $\text{Card}(U) \leq 4a + 1$.
- If $0 < b \leq \frac{a}{2}$, then $B < 4$ and $\text{Card}(U) \leq 7$.
- If $-\frac{a}{2} < b < 0$, $B < 7$ and $\text{Card}(U) \leq 13$.
- If $-a + 2 \leq b \leq -\frac{a}{2}$ then $B \leq 2a - 1$ and $\text{Card}(U) \leq 4a - 3$.

Corollary 1  Let $\beta$ be a quadratic Pisot unit, i.e., $|b| = 1$, and $I \subset [-1,1]$ be an interval of length 1, then the set $U$ contains at most 5 points.
Proof. From Proposition 4 when \( b = 1 \) or \( b = -1 \), \( B \leq 3 \), in all but two cases.

If \( M_\beta = X^2 - 3X + 1 \), then \( B \leq 4 \) and \( |x_2| \leq 3 \) but there is no corresponding value for \( x_1 \) when \( |x_2| = 3 \), thus \( |x_2| \leq 2 \) and \( \text{Card}(U) \leq 5 \).

If \( M_\beta = X^2 - 2X - 1 \), we obtain \( B \leq 3 \) if we do not approximate \( \Delta \) in the computation of the proof of Proposition 4.

\[ \]  

**Example 5** Let \( \beta = (1 + \sqrt{5})/2 \) then \( \beta' = (1 - \sqrt{5})/2 \). Then

\[ U = \{ x \in \mathbb{Z}[\beta] \mid x \in I \text{ and } |x'| < 2\beta + 2 \}. \]

- For \( I = [-1/2, 1/2] \), \( U = \{0, \beta - 2, 2\beta - 3, 2 - \beta, 3 - 2\beta\} \).
- For \( I = [0, 1] \), \( U = \{0, -1 + \beta, -3 + 2\beta, 2 - \beta\} \), since the conjugate \( 4 - 2\beta' \) of \( 4 - 2\beta \) has a modulus greater than \( 2\beta + 2 \).

Example 4 shows that the size of minimal sets \( F \) in this case is equal to 3.

## 5 A reduction of the sets containing minimal sets \( F \)

We present our constructions in the case where \( I \) is an interval of length 1 in \([-1, 1]\) and consider the finite subset \( U \) of \( I \) defined in Lemma 4. By construction a minimal set \( F \) is contained in \( U \) and from Lemma 2 \( F \) is a subset of \((\mathbb{Z}_\beta - \mathbb{Z}_\beta) - \mathbb{Z}_\beta \). Thus a minimal set \( F \) is included in \( U \cap ((\mathbb{Z}_\beta - \mathbb{Z}_\beta) - \mathbb{Z}_\beta) \).

In the following we give an algorithm that computes this intersection. Roughly speaking we construct an automaton that recognizes the Cartesian product \((L_\beta - L_\beta) \times L_\beta\) and whose each state \( q \) corresponds to the value of the subtraction of the elements of \( \mathbb{Z}_\beta - \mathbb{Z}_\beta \) and \( \mathbb{Z}_\beta \) whose representations label the paths from the initial state to \( q \).

The first step of the construction consists in associating to each element of a minimal set \( F \) at least a path labelled on \( \{-2[\beta], \cdots, 2[\beta]\}^* \times \{0, \cdots, [\beta]\}^* \) in a directed graph \( G \) whose set of vertices contains \( U \).

Following [15], we define the directed graph \( G \) as follows.

- The set of vertices is
  \[ V = \left\{ x \in \mathbb{Z}[\beta] \mid |x| < \frac{2[\beta]}{\beta - 1}, \text{ and for } 2 \leq j \leq d, |x^{(j)}| < \frac{2[\beta]}{1 - |\beta^{(j)}|} \right\}. \]

- The labels \((b, a)\) of the transitions belong to \( \{-2[\beta], \cdots, 2[\beta]\} \times \{0, \cdots, [\beta]\} \).

- There is a transition from \( x \in V \) to \( y \in V \) labelled by \((b, a)\), denoted \( x \xrightarrow{(b,a)} y \), if and only if \( y = \beta x + (b - a) \).

Note that \( 0 \in V \) and \( U \subset V \). The set \( V \) is finite.

**Remark 3** Transitions in \( G \) are defined in such a way that words will be processed most significant digit first (i.e., from left to right) as in the automata for \( \mathbb{Z}_\beta \) and \( \mathbb{Z}_\beta - \mathbb{Z}_\beta \).
Proposition 5 Let $F \subseteq U$ be a minimal set satisfying $Z_\beta - Z_\beta \subseteq Z_\beta + F$. Then for any $f \in F$ there is a path from 0 to $f$ whose label belongs to $(L_\beta - L_\beta) \times L_\beta$.

Proof. From Lemma 2, $F \subseteq (Z_\beta - Z_\beta) - Z_\beta$, so any element $f$ of $F$ can be written as $f = \sum_{i=0}^{N} (b_i - a_i)\beta^i$ where $x = \sum_{i=0}^{N} a_i\beta^i \in Z_\beta$ with $a_N \cdots a_0 \in L_\beta$ and $y = \sum_{i=0}^{N} b_i\beta^i \in Z_\beta - Z_\beta$ with $b_N \cdots b_0 \in L_\beta - L_\beta$.

With such an $f$ is associated a finite sequence

$$f_0 = 0, \quad \text{for } 0 \leq i \leq N \quad f_{i+1} = \beta f_i + (b_{N-i} - a_{N-i}).$$

Note that $f_{N+1} = f$.

Let us show that for any $f \in F$, the elements $f_1, \ldots, f_{N+1}$ of the sequence associated with $f$ belong to $V$. Note that the smallest $K$ such that $|x| < K$ implies $|(x-(b-a))/\beta| < K$ is $K = 2|\beta|/(\beta - 1)$. Since $f$ is in $U$, $|f| < K$, and so for all $0 \leq i \leq N$, $|f_i| < K$. Moreover from Lemma 4, when $F \subseteq U$, for all $i$, $|b_i - a_i| \leq 2|\beta|$, thus for $1 \leq i \leq N+1$ and $2 \leq j \leq d$, the conjugates $f_i^{(j)}$ of $f_i$ satisfy $|f_i^{(j)}| \leq 2|\beta|/(1 - |\beta^{(j)}|)$ and for $1 \leq i \leq N+1, f_i$ belongs to $V$.

Finally if $f \in F$ then there is in $G$ a path

$$0 = f_0 \xrightarrow{(b_N,a_N)} f_1 \xrightarrow{(b_{N-1},a_{N-1})} \cdots \xrightarrow{(b_0,a_0)} f_{N+1} = f$$

where the words $a_N \cdots a_0$ and $b_N \cdots b_0$ respectively belong to $L_\beta$ and $L_\beta - L_\beta$, concluding the proof. \hfill \Box

From Proposition 5 we can take into account in $G$ only the paths whose labels belong to $(L_\beta - L_\beta) \times L_\beta$. In order to compute such paths, we use the Cartesian product of the automata $A_{Z_\beta - Z_\beta}$ and $A_{Z_\beta}$. Recall the definition of the Cartesian product $\mathcal{P} = A \times B$ of two automata $A$ and $B$:

- the set of states of $\mathcal{P}$ is $Q_\mathcal{P} = Q_A \times Q_B$,
- there is an edge in $\mathcal{P}$ from $(p, q)$ to $(p', q')$ labelled by $(a, b)$ if and only if there is an edge from $p$ to $p'$ labelled by $a$ in $A$ and an edge from $q$ to $q'$ labelled by $b$ in $B$,
- the set of initial (resp. terminal) states of $\mathcal{P}$ is the Cartesian product of the sets of initial (resp. terminal) states of $A$ and $B$.

Note that in $A_{Z_\beta - Z_\beta} \times A_{Z_\beta}$ every state is terminal.

From all vertices $f$ of $G$ which are in $U$ we look for a path from 0 to $f$ in the directed graph $G$ which is successful in $A_{Z_\beta - Z_\beta} \times A_{Z_\beta}$. We find these paths making use of the intersection $I = A \cap B$ of two finite automata $A$ and $B$ defined as follows:

- all sets of states of $I$ are defined as the ones of the Cartesian product,
- there is an edge in $I$ from $(p, q)$ to $(p', q')$ labelled by $a$ if and only if there is an edge from $p$ to $p'$ in $A$ and an edge from $q$ to $q'$ in $B$ both labelled by $a$.
**Algorithm of reduction of the size of the sets containing a minimal set $F$**

**Input:** The set $U$ containing a minimal set $F$.

**Output:** A subset $U'$ of $U$ containing a minimal set $F$.

1. Build the automaton $G_U$ having as underlying transition graph $G$ with 0 as initial state and $U$ as set of terminal states.

2. Compute the intersection $I_U = (A_{Z_\beta-Z_\beta} \times A_{Z_\beta}) \cap G_U$. Note that the set of terminal states of $I_U$ is $Q_{Z_\beta-Z_\beta} \times Q_{Z_{\beta}} \times U$.

3. Prune $I_U$ into $I_{U'}$ (that is, keep only the states which belong to a path from the initial state to a terminal state).

4. Return the set $U'$ of the third components of terminal states of $I_{U'}$.

**Corollary 2** A minimal set $F$ is contained in $U' \subset U$.

**Remark 4** The number of states of the automaton $I_{U'}$ is $O(Q^3 \times |V|)$, where $Q$ is the number of states of $A_{Z_{\beta}+}$ and $|V|$ is the number of vertices of $G$.

Because of the large number of states of the automaton obtained in this way, we shall not illustrate the construction with a figure. Nevertheless we give an example of reductions that can be obtained.

**Example 6** When $\beta = (1 + \sqrt{5})/2$, we obtain

- For $I = [-1/2, 1/2[$ and $U = \{0, \beta - 2, 2\beta - 3, 2 - \beta, 3 - 2\beta\}$,
  $$U \cap (Z_\beta - Z_\beta) - Z_\beta = \{0, \beta - 2, 2 - \beta\}.$$

- For $I = [0, 1[$ and $U = \{0, -1 + \beta, -3 + 2\beta, 2 - \beta\}$,
  $$U \cap (Z_\beta - Z_\beta) - Z_\beta = \{0, \beta - 1, 2 - \beta\}.$$

A geometrical argument could also be used to prove that $2\beta - 3 = \frac{1}{\beta}$ and $-2\beta + 3 = -\frac{1}{\beta}$ are not in $(Z_\beta - Z_\beta) - Z_\beta$. Indeed the distance between two consecutive points of $Z_\beta$ is equal to $\frac{1}{\beta}$ or 1 = $\frac{1}{\beta} + \frac{1}{\beta^2}$, so $Z_\beta + \{\frac{1}{\beta^2}, -\frac{1}{\beta^2}\} \cap Z_\beta + \{0, \frac{1}{\beta}, -\frac{1}{\beta}\} = \emptyset$. Moreover $Z_\beta - Z_\beta \subset Z_\beta + \{0, \frac{1}{\beta}, -\frac{1}{\beta}\}$ (see Example 4), thus $Z_\beta - Z_\beta \cap Z_\beta + \{\frac{1}{\beta^2}, -\frac{1}{\beta^2}\} = \emptyset$ and $\pm \frac{1}{\beta^2} \notin (Z_\beta - Z_\beta) - Z_\beta$. 


6 Algorithm computing a minimal set $F$

The finite sets $U'$ obtained by the previous construction are not minimal. An element $y \in \mathbb{Z}_\beta - \mathbb{Z}_\beta$ can be close to two distinct points of $x$ and $x'$ of $\mathbb{Z}_\beta$, for example such that $x < y < x'$, and $y = x + f = x' + f'$ with $f, f' \in U'$.

**Theorem 1** A minimal set $F \subseteq U'$ can be computed by an algorithm which is exponential in time and space. It consists in building a transducer which rewrites a representation of an element of $\mathbb{Z}_\beta - \mathbb{Z}_\beta$ into its representation $\mathbb{Z}_\beta + F$.

**Proof.** To find a minimal set $F \subseteq U'$ we proceed in two steps.

First we define from the automaton $T_{U'}$, a deterministic automaton $R_{U'}$ that recognizes the set $L_\beta - L_\beta$. Note that the words of $L_\beta - L_\beta$ appear as the first component of the labels of the successful paths in $T_{U'}$. The automaton $R_{U'}$ is obtained by erasing the second component of the labels (that belongs to $L_\beta$) of the transitions of $T_{U'}$ and determining the automaton defined in this way. The determinization of automata is based on the so-called subset construction (see [9]), which is exponential in space, and the automaton $R_{U'}$ has $O(2^{Q_{U'}})$ states.

Next we look amongst all subsets of $U'$ for the smallest set $F$ such that the automaton $R_F$, obtained from $R_{U'}$ by keeping only as terminal states the terminal states of $R_{U'}$ in which occur an element of $F$, recognizes $L_\beta - L_\beta$. To test the inclusion, we compute the complement $C_F$ of $R_F$ by completing the automaton $R_F$ (when a transition is missing we add a transition ending in a new state called the sink) and replacing the set of terminal states $F$ by its complement (including the sink). Then the automaton $R_F$ recognizes $L_\beta - L_\beta$ if and only if the intersection of $C_F$ and $A_{\mathbb{Z}_\beta - \mathbb{Z}_\beta}$ is empty. Note that the complexity of the search amongst all subsets of $U'$ is exponential in time.

From the set $F$ obtained above, we define a transducer that provides, for any $b = b_N \ldots b_0 \in L_\beta - L_\beta$ and $y = \sum_{i=0}^{N} b_i \beta^i \in \mathbb{Z}_\beta - \mathbb{Z}_\beta$, a decomposition $(a_N \ldots a_0, f)$ where $a = a_N \ldots a_0 \in L_\beta$, $f \in F$ and $y = \sum_{i=0}^{N} a_i \beta^i + f$.

Consider $\mathcal{I}_F = (A_{\mathbb{Z}_\beta - \mathbb{Z}_\beta} \times A_{\mathbb{Z}_\beta}) \cap \mathcal{G}_F$ ($F$ is the set of terminal states of $\mathcal{G}_F$). For any element $b = b_N \ldots b_0 \in L_\beta - L_\beta$ there exists $f \in F$ such that $b$ is the first component of the label of a successful path $w$ ending in $(s, f)$ where $s$ is any state of $(A_{\mathbb{Z}_\beta - \mathbb{Z}_\beta}) \times A_{\mathbb{Z}_\beta}$ (by construction all states are terminal). Consequently we get $\sum_{i=0}^{N} b_i \beta^i = \sum_{i=0}^{N} a_i + f$ where $a_N \ldots a_0$ is the second component of the label of the same path $w$ and so belongs to $L_\beta$.

More generally the first component of the labels of the edges in $\mathcal{I}_F$ can be interpreted as the inputs in $\mathbb{Z}_\beta - \mathbb{Z}_\beta$ given by their representation in $L_\beta - L_\beta$ of the transducer, the second component as the corresponding outputs in $\mathbb{Z}_\beta$ given by their representation in $L_\beta$. The associated element of $F$ is given by the second component of the label of the state where the path ends.

To conclude, the method used here for determining minimal sets $F$ probably could be generalized to the following sets. Let $G$ be a strongly connected graph labelled by numbers taken from a finite alphabet, and let $\beta$ be the spectral radius of its adjacency matrix. Let us consider the set $X_G = \{ \sum_{i=0}^{k} x_i \beta^i \mid k \geq 0, \ x_k \ldots x_0 \text{ is the label of a path} \}$
in $G$. Under certain conditions on $G$ and $\beta$, $X_G$ is a Meyer set, and so the question of minimal $F$ makes sense. The characterization of these Meyer sets and the construction of associated minimal sets $F$ remain open problems.

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**References**


