ON THE TOPOLOGICAL STRUCTURE OF FRACTAL TILINGS GENERATED BY QUADRATIC NUMBER SYSTEMS

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Abstract. Let α be a root of an irreducible quadratic polynomial x^2+Ax+B with integer coefficients A, B and assume that α forms a canonical number system, i.e. each $x \in \mathbb{Z}[\alpha]$ admits a representation of the shape

$$x = a_0 + a_1\alpha + \cdots + a_h\alpha^h$$

with $a_i \in \{0, 1, \ldots, |B| - 1\}$. It is possible to attach a tiling to such a number system in a natural way. If $2A < B+3$ then we show that the fractal boundary of the tiles of this tiling is a simple closed curve and its interior is connected. Furthermore, the exact set equation for the boundary of a tile is given. If $2A \geq B + 3$ then the topological structure of the tiles is quite involved. In this case we prove that the interior of a tile is disconnected. Furthermore, we are able to construct finite labelled directed graphs which allow to determine the set of "neighbours" of a given tile $\mathcal{T}$, i. e. the set of all tiles which have nonempty intersection with $\mathcal{T}$. In a next step we give the structure of the set of points, in which $\mathcal{T}$ coincides with L other tiles. In this paper we use two different approaches: Geometry of numbers and finite automata theory. Each of these approaches has its advantages and allows to emphasize on different properties of the tiling. Especially the conjecture in Akiyama-Thuswaldner [1] that for $A \neq 0$ and $2A < B + 3$ there exist exactly 6 points where $\mathcal{T}$ coincides with 2 other tiles is solved in these two ways in Theorem 6.6 and Theorem 10.1.

Contents

1. Introduction

Let α be an algebraic integer and $\mathcal{N} = \{0, 1, \ldots, |N(\alpha)| - 1\}$ where $N(x)$ denotes the norm of x over $\mathbb{Q}(\alpha)/\mathbb{Q}$. Let $\mathbb{Z}[\alpha]$ be the smallest subring of $\mathbb{C}$ containing $\mathbb{Z}$ and

Date: October 12, 2004.

2000 *Mathematics Subject Classification.* 11A63, 37B50, 54D05, 68Q45, 28A80.

Key words and phrases. Radix representation, Tiling, Connectedness, Automata, Fractal.

The first author was supported by the Japanese Ministry of Education, Culture, Sports, Science and Technology, Grant in Aid for fundamental research, 14540015.

The second author was supported by the Austrian science foundation project S8310.

α. If for each element x of $\mathbb{Z}[\alpha]$ there exists a non-negative integer $\ell(x)$ such that

$$x = \sum_{i=0}^{\ell(x)} a_i \alpha^i \qquad (a_i \in \mathcal{N}) \tag{1.1}$$

then we call the pair $(\alpha, \mathcal{N})$ a *canonical number system*, or a CNS for short. To (1.1) we will refer as the *α-adic representation of x*. α is called the base of the CNS $(\alpha, \mathcal{N})$. Since $\mathcal{N}$ is completely determined by α, we sometimes simply say that α forms a CNS if $(\alpha, \mathcal{N})$ is a CNS. Knuth [26] pointed out that for each integer n greater than one $-n$ forms a CNS which has some applications in computer science. Even for quadratic number fields it is a nontrivial problem to characterize the bases of CNS. This characterization was given by Kátai-Kovács [22, 23] and Gilbert [11], generalizing Kátai-Szabó [20], where the characterization of all CNS in $\mathbb{Q}(\sqrt{-1})$ was established. For quadratic number fields this classification reads as follows. If α is a quadratic irrational whose minimal polynomial is $x^2 + Ax + B$ then α forms a CNS if and only if [1]

$$-1 \le A \le B \quad \text{and} \quad B \ge 2.$$

A proof of this fact using finite automata can be found in Thuswaldner [42].

In the present paper we are interested in the topological structure of a certain tiling, which can be attached to a given CNS in a rather natural way. Before we give an exact definition of this tiling we want to recall the general definition of a tiling (cf. for instance Grünbaum-Shephard [15] or Vince [44]). Two sets are called *non-overlapping* if their interiors are disjoint. A *tiling* is a decomposition of $\mathbb{R}^n$ into non-overlapping compact sets, each the closure of its interior and each with boundary having Lebesgue measure zero. Each set of a tiling is called *tile*. There exists a vast literature on tilings. Fundamental properties of a large class of tilings are shown for instance in Bandt [4], Lagarias-Wang [30] or Wang [45]. A large list of references can also be found for instance in Vince [44].

Now we are in a position to define the tiling associated to a CNS. Let the algebraic integer α of degree n be the base of a CNS and let Φ be the canonical embedding of $\mathbb{Q}(\alpha)$ into $\mathbb{R}^n$. Denote the conjugates of α by $\alpha^{(i)}$ $(i = 1, \ldots, n)$ ordered in a way such that $\alpha^{(i)}$ $(i = 1, \ldots, r_1)$ is real and $\alpha^{(i)}$ $(i = r_1 + 1, \ldots, r_1 + 2r_2 = n)$ is not real. Denote by $x^{(i)}$ the i-th conjugate of $x \in \mathbb{Q}(\alpha)$. Then Φ has the form

$$\Phi(x) = (x^{(1)}, \ldots, x^{(r_1)}, \Re x^{(r_1+1)}, \Im x^{(r_1+1)}, \ldots, \Re x^{(r_1+r_2)}, \Im x^{(r_1+r_2)}).$$

Since we are mainly concerned with quadratic CNS we note that for $n = 2$ we may write

$$\Phi(x) = \begin{cases} (\Re(x), \Im(x)) & \text{for } \alpha \in \mathbb{C} \setminus \mathbb{R}, \\ (x, x') & \text{for } \alpha \in \mathbb{R}, \end{cases} \tag{1.2}$$

where x' denotes the conjugate of x different from itself. If $\alpha \in \mathbb{C} \setminus \mathbb{R}$, then for $n = 2$ the image $\Phi(x)$ can be identified with the complex number $x \in \mathbb{C}$.

With this definition set

$$\mathcal{T} = \left\{ \sum_{i=1}^{\infty} a_{-i} \Phi(\alpha^{-i}) \;\middle|\; a_{-i} \in \mathcal{N} \right\}.$$

We say that $\mathcal{T}$ is the *central tile*. It was shown in Kátai-Kőrnyei [21] that the family of sets

$$\mathcal{T} + \Phi(\omega) \quad (\omega \in \mathbb{Z}[\alpha])$$

forms a tiling of $\mathbb{R}^n$ in the sense of the above definition. In Section 2 of the present paper we will give a slightly different proof of this result putting our stress on the existence of so-called exclusive inner points.

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[1] Note that the definition of CNS in [22] and [23] is slightly different from ours, as there is required that $\mathbb{Z}[\alpha]$ coincides with the whole integer ring of $\mathbb{Q}(\alpha)$.

Knuth [26] constructed the tiling corresponding to the CNS $-1+\sqrt{-1}$. He obtained the beautiful and well-known "twin dragon" tiling. Fundamental facts on tilings attached to CNS, generalizing the twin dragon, are studied for instance in Duvall et al. [7], Gilbert [12, 13], Gröchenig-Haas [14], Indlekofer et al. [17] and Kátai-Környei [21].

Before we give a detailed account on the results of the present paper we want to give some fundamental definitions. We are interested in the "neighbours" of the tile $\mathcal{T}$, i.e. in the translates $\mathcal{T}+\Phi(s)$, which have non-empty intersection with $\mathcal{T}$. Thus the set

$$S := \{s \in \mathbb{Z}[\alpha]\setminus\{0\} \mid \mathcal{T}\cap(\mathcal{T}+\Phi(s)) \neq \emptyset\}$$

of translates of all neighbours of $\mathcal{T}$ will play a prominent rôle in this paper. Furthermore, we call a point $v \in \mathcal{T}$ an *L-vertex*, if v is contained in at least L pairwise disjoint tiles different from $\mathcal{T}$. More precisely, for pairwise disjoint $s_1,\dots,s_L \in S$ we set

$$V_L(s_1,\dots,s_L) := \left\{ x \in \mathbb{R}^2 \;\middle|\; x \in \mathcal{T} \cap \bigcap_{j=1}^{L} (\mathcal{T}+\Phi(s_j)) \right\}.$$

The set of L-vertices of $\mathcal{T}$ is then defined by

$$V_L = \bigcup_{\{s_1,\dots,s_L\}\subset S} V_L(s_1,\dots,s_L)$$

where the union is extended over all subsets of S containing L elements. A 2-vertex is sometimes simply called vertex.

In Akiyama-Thuswaldner [1], generalizing a result of Gilbert [13], we proved certain results on the vertices of $\mathcal{T}$. In particular we obtained that each tile corresponding to a quadratic CNS has at least 6 vertices when $A \neq 0$. However, we also found curious phenomena. Namely, the number of vertices of $\mathcal{T}$ is infinite if $2A \geq B+3$ and it is even uncountable for $2A > B+3$.

To illustrate these results, it is worth demonstrating some computer graphics of tiles.

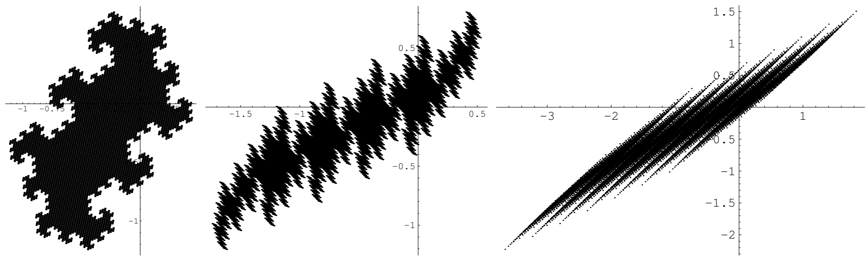

Figure 1. Tiles for the bases $-1+\sqrt{-2}$, $-2+\sqrt{-1}$ and $-3+\sqrt{2}$

The tile on the left hand side corresponds to the CNS arising from the minimal polynomial x^2+Ax+B with $A=2, B=3$ which fulfills $2A < B+3$, an "*ordinary*" case. It seems that this tile is surrounded by a simple closed curve and that its interior is connected. The tile in the middle corresponds to the choice $A=4, B=5$ and, hence, $2A = B+3$, a "*strange*" case. It seems to have a curious shape and its interior seems to be not connected. The last one is for $A=6, B=7$ and $2A > B+3$, a "*pathological*" case [2]. In this case one can not even imagine what happens on the boundary. The sign of $2A-B-3$ seems determinative on the difference. This speculation can be confirmed by a lot of numerical experiments.

[2]Note that we used a different embedding $((x+x')/2,(x-x')/2)$ to express the third one to compare with others. But it is just an image of $\mathcal{T}$ by an invertible affine transform which does not change its topological properties.

Based upon these observations, it is natural to conjecture that if $2A < B+3$ and $A \neq 0$ then the number of vertices is exactly 6. In the present paper, this conjecture will be settled at last. But our aim is not restricted to the solution of this single problem. In fact, we will solve a wealth of problems on the topological structure of $\mathcal{T}$. We will now give a short overview over the results of the present paper.

- Section 2 is devoted to the review of some fundamental results. Among other things we discuss the set equation of $\mathcal{T}$ and give a brief proof of the fact that $\mathcal{T}$ and its translates form a tiling putting our stress on exclusive inner points. Note that the results in this section are far from new since Theorem 2.1 is shown first by Kátai-Környei [21] and pretty general criteria of Bandt [4], Kenyon [24] and Lagarias-Wang [30] imply this theorem as well.
- In Section 3 the so-called fundamental inequality is proved (Theorem 3.3). It states that certain tiles of CNS fulfilling the inequality $2A < B + 3$ have empty intersection with the symmetry abscissa of the central tile. This result forms the basis for the proofs of many subsequent results.
- Using the fundamental inequality we show in Section 4 that the interior of $\mathcal{T}$ is a simply connected set for all CNS with $2A < B + 3$ (Theorem 4.1). To this matter a so-called skeleton of $\mathcal{T}$ is constructed. This skeleton is a connected union of line segments which is contained in the interior of $\mathcal{T}$.
- Section 5 contains the first main result of this paper. Namely, it is shown that for $2A < B + 3$ the boundary of $\mathcal{T}$ is a simple closed curve (Theorem 5.5) despite of its fractal nature. Of course, the connectedness of the interior of $\mathcal{T}$ proved in Section 4 is essential for showing this.
- In Section 6 we deal with the explicit set-equation of $\partial(\mathcal{T})$ for CNS with $2A < B + 3$ (Theorem 6.6). The method to give it as a graph directed set in the sense of Falconer [8] is already known in a more general context (c.f. for instance in Scheicher-Thuswaldner [38] and implicitly in Gröchenig-Haas [14]). However, here we employ a geometric proof using the preceding considerations which seems to be new and gives interesting insights in the structure of the tiling. For instance, we are able to give detailed information on the orientation and the symmetry of the pieces which comprise the boundary.
- In Section 7 we will prove that there exist infinitely many connected components of $\mathrm{Inn}(\mathcal{T})$ if $2A \geq B + 3$ (Theorem 7.1). In the proof we use the automata theoretic results shown in the later sections. The disconnectedness result of this section gives a good contrast to the results of the former sections treating the case $2A < B + 3$.
- In Section 8 we give some preparations and definitions in order to set up the environment for the automata theoretic approach. We show that L-vertices can be characterized algorithmically with help of graphs and dwell upon the connections between so-called counting automata and graphs describing $\partial(\mathcal{T})$ and V_L.
- In Section 9 we determine the set S of neighbours of $\mathcal{T}$ (Theorem 9.1). It is surprising, that S can be arbitrarily large depending on the quantity

$$J = \max\left\{1, \left[\frac{B-1}{B-A+1}\right]\right\} \tag{1.3}$$

  (here $[x]$ denotes the greatest integer not exceeding x). In particular, it turned out that the family of tilings generated by quadratic canonical number systems has an infinite hierarchy of complexity which is directed by J and has the case $J = 1$ (i.e. $2A < B + 3$) in its lowest layer.
- In Section 10 we deal with the set V_L. To each CNS we determine the values of L, for which V_L contains uncountably many, countably many,

finitely many or no elements (Theorem 10.1). Unexpectedly we will see that for any given integer L there exists a quadratic tiling which contains L-vertices, which may be difficult to imagine as we are concerned with plane tilings.

- In Section 11 we look more closely to the case $2A = B + 3$. We explicitly construct the automaton which accepts the elements of V_2 and list the elements of V_2. In principle, this procedure can be performed for any class with increasing effort depending on the value of J.

There exists an obvious distinction between real and complex bases α. However, the above results strongly suggest that a classification by the sign of $2A - B - 3$ is more essential in order to understand the topological structure of the tilings. An interesting question may be to find another algebraic and/or geometric explanation of the quantity $2A - B - 3$.

In some way the present paper may be a starting point for further research in several directions.

One of these directions may consist in extending the results of the present paper to more general classes of tilings. We mainly concentrate on the case of tilings induced by quadratic CNS. However, some of our results seem to be tractable by our methods also in the case of higher dimensional CNS. We need to study the non-planar topological structure of $\mathcal{T}$ and have to manage families of huge automata in a systematical way. We even expect that some results can be generalized to large classes of periodic self affine tilings. There exists a vast literature on self affine tilings. We refer for instance to Bandt [4], Lagarias-Wang [30] and Wang [45]. Among other things these papers provide certain characterization results for such tilings depending on the the digit set and the expanding matrix defining them. This is closely related to the construction of wavelet bases (cf. for example Gröchenig-Haas [14] and Wang [45]). Of course it would be an interesting task to classify these tilings by their topological structure. Recently we became aware of the paper Bandt-Wang [5] which forms a first step towards such a classification in providing certain criteria for the disclikeness of tilings of the plane in terms of their vertices. Putting together our results on the vertices of $\mathcal{T}$ with the results of [5] leads to a different proof of our Theorem 5.5.

Another direction may consist in exploring the influence of the topological structure of tilings to dynamical systems and diophantine approximations. For a survey on the connections between properties of tilings and the above mentioned fields we refer to the excellent book Fogg [9]. One class of tilings playing an important rôle in this context are aperiodic tilings related to so-called Pisot number systems generated by β-expansions of real numbers. They are studied from several points of view for instance in Akiyama [2], Praggastis [35], Rauzy [36] and Thurston [41]. Generalizing these, Kenyon-Vershik [25] studied a nice sofic partition attached to general hyperbolic polynomials, but there exists no topological study of this partition at present. Connections between tilings and dynamical systems are surveyed in Solomyak [40] which contains a lot of references. Furthermore, one can find number theoretical applications of Pisot number systems in consecutive works of Sh. Ito and his collaborators (e.g. Furukado-Ito [10] and Ito-Sano [18]). We expect that the knowledge of fine topological properties of the underlying tilings would cause a deeper understanding of some problems discussed in these works.

However for this purpose, CNS are somewhat easier objects than Pisot number systems. They correspond to full shifts in which symbolic dynamical researchers may find tiny interest. Tilings generated by CNS are also easier in the sense that they are periodic, i.e. they consist of a single tile up to translation. Nevertheless, computer experiments show that they have strong similarities with the aperiodic tilings which stem from Pisot number systems. So we hope that our methods will also lead to a description of their topological properties.

Last but not least, we expect that this study will lead to an idea how to construct a tractable Markov partition of a hyperbolic toral automorphism (cf. Arnoux-Ito [3]) or to understand the precise nature of periodic orbits of general arithmetic algorithms.

2. The tiling generated by a canonical number system

In this section we shall review some basic results on tilings generated by CNS of arbitrary dimension. For $\rho \in \mathbb{C}$ the symbol $|\rho|$ is used hereafter to indicate the standard absolute value of ρ, i.e. $|\rho| = \sqrt{\Re\rho^2 + \Im\rho^2}$. Let α be the base of a CNS. Since each integer has a representation of the form (1.1), it is easily seen that each conjugate of α has modulus greater than 1 (cf. Kovács-Pethő [28]). From this we can show that $\mathcal{T}$ is closed, bounded, and thus a compact set in $\mathbb{R}^n$ by standard arguments (cf. Müller et al. [31]).

As α forms a CNS, we see that $\mathbb{Z}[1/\alpha] = \left\{x = \sum_{i=n}^{m} a_i\alpha^i \mid n, m \in \mathbb{Z}, a_i \in \mathcal{N}\right\}$. Define a function $\deg : \mathbb{Z}[1/\alpha] \to \mathbb{Z}$ by assigning to each element of $\mathbb{Z}[1/\alpha]$ the greatest index i such that $a_i \neq 0$. For $x = \sum_{i=n}^{m} a_i\alpha^i \in \mathbb{Z}[1/\alpha]$, the *integer part* of x is defined to be $\sum_{i=0}^{m} a_i\alpha^i \in \mathbb{Z}[\alpha]$ and the *fractional part* of x is to be $\sum_{i=n}^{-1} a_i\alpha^i$. Let $\mathcal{S} = \{y \in \mathbb{Z}[1/\alpha] \mid \deg(y) < 0\}$, i.e., the subset of elements of $\mathbb{Z}[1/\alpha]$ with integer part 0. Denote by $\overline{A}$ the closure of the subset $A \in \mathbb{R}^n$ by the usual topology. Since Φ acts on finite power series of α with coefficients on $\mathbb{Z}$, we have $\mathcal{T} = \overline{\Phi(\mathcal{S})}$, which gives an alternative way to construct the central tile $\mathcal{T}$.

Note that Φ is a group homomorphism of the additive group. We also need to interpret the multiplication by a power of α in $\mathbb{Q}(\alpha)$ into $\mathbb{R}^n$. Let K be an integer. Then one can find a map ξ_K from $\mathbb{R}^n$ to itself which commutes the following diagram:

$$\begin{array}{ccc} \mathbb{Q}(\alpha) & \xrightarrow{\times\alpha^K} & \mathbb{Q}(\alpha) \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Phi} \\ \mathbb{R}^n & \xrightarrow[\xi_K]{} & \mathbb{R}^n \end{array} \tag{2.1}$$

For two square matrices M_1 and M_2, put

$$M_1 \otimes M_2 = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}$$

and let $\mathrm{diag}(d_1, \dots, d_s)$ be the $s\times s$ diagonal matrix with diagonal elements $d_1, \dots, d_s$. The concrete form of ξ_K can be calculated from

$$\xi_K(x_1, x_2, \dots, x_n) = (x_1, x_2, \dots, x_n)A_K,$$

where A_K is the $n \times n$ matrix

$$A_K = \mathrm{diag}((\alpha^{(1)})^K, \dots, (\alpha^{(r_1)})^K) \otimes B_1 \otimes \dots \otimes B_{r_2}$$

with

$$B_j = \begin{pmatrix} \Re((\alpha^{(r_1+j)})^K) & \Im((\alpha^{(r_1+j)})^K) \\ -\Im((\alpha^{(r_1+j)})^K) & \Re((\alpha^{(r_1+j)})^K) \end{pmatrix},$$

for $j = 1, \dots, r_2$. The eigenvalues of A_K are the conjugates $(\alpha^{(i)})^K$ $(i = 1, \dots, n)$. Let us fix an η such that

$$1 < \eta < \min_{i=1,\dots,n} |\alpha^{(i)}| \tag{2.2}$$

throughout this paper. Since $|a^{(i)}| > 1$ for all $i \in \{1,\dots,n\}$ we see that ξ_K is an expanding map for any positive integer K. In fact, for any $x, y \in \mathbb{R}^n$, we have

$$||\xi_K(x) - \xi_K(y)|| \geq \eta^K ||x-y||$$

for a suitably chosen norm $||\cdot||$ on $\mathbb{R}^n$ (c.f. [30]). Conversely, we have

$$||\xi_{-K}(x) - \xi_{-K}(y)|| \leq \eta^{-K} ||x-y||,$$

for any $x, y \in \mathbb{R}^n$ showing that ξ_{-K} is contractive. Obviously ξ_K is a homeomorphism and $\xi_{K_1+K_2} = \xi_{K_1} \circ \xi_{K_2}$ holds for any $K_1, K_2 \in \mathbb{Z}$. By the definition of CNS, we can derive the important *inflation subdivision principle*

$$\xi_1(\mathcal{T}) = \bigcup_{i=0}^{|N(\alpha)|-1} (\mathcal{T} + \Phi(i)) \,. \tag{2.3}$$

This is a set equation for $\mathcal{T}$. Occasionally we will identify an integer x with $\Phi(x)$ and write

$$\xi_1(\mathcal{T}) = \bigcup_{i=0}^{|N(\alpha)|-1} (\mathcal{T} + i) \tag{2.4}$$

if this will cause no confusion.

Let $\{f_i\}_{i=1}^m$ be a finite set of contractions of $\mathbb{R}^n$. By Hutchinson's theorem, there exists a unique non-empty compact set $K = K(f_1, f_2, \dots, f_m)$ satisfying

$$K = \bigcup_{i=1}^{m} f_i(K) \tag{2.5}$$

(see Falconer [8, Theorem 9.1]). Thus applying ξ_{-1} to (2.3), we see there exists a unique non-empty compact set $\mathcal{T}$ which satisfies (2.3). In many cases $\mathcal{T}$ as well as its boundary has fractal structure.

We will now state a theorem which contains the fundamental facts of the central tile $\mathcal{T}$ and its translates. To this matter let μ be the n-dimensional Lebesgue measure. Furthermore, an inner point of $\mathcal{T}$ is called *exclusive inner point* if it is not contained in a set of the shape $\mathcal{T} + \Phi(\omega)$ with $\omega \in \mathbb{Z}[\alpha] \setminus \{0\}$ (cf. Akiyama [2]). We will denote the set of exclusive inner points of $\mathcal{T}$ by $\mathrm{Inn}^*(\mathcal{T})$.

Theorem 2.1. *Let $(\alpha, \mathcal{N})$ be a CNS and let $\mathcal{T}$ be its central tile. Then the following assertions hold.*

(i) *The $\mathbb{Z}[\alpha]$-translates of $\mathcal{T}$ cover the whole space, i.e.*

$$\mathbb{R}^n = \bigcup_{\omega \in \mathbb{Z}[\alpha]} (\mathcal{T} + \Phi(\omega)).$$

(ii) *Two distinct $\mathbb{Z}[\alpha]$-translates of $\mathcal{T}$ have intersection of measure zero, i.e.*

$$\mu\left((\mathcal{T} + \Phi(\omega_1)) \cap (\mathcal{T} + \Phi(\omega_2))\right) = 0$$

if $\omega_1, \omega_2 \in \mathbb{Z}[\alpha]$ are distinct.

(iii) *For any $p \in \mathcal{S}$, the image $\Phi(p)$ is an inner point of $\mathcal{T}$. In particular, the origin $0 = \Phi(0)$ is an inner point of $\mathcal{T}$.*

(iv) *$\mathcal{T}$ is the closure of its interior, i.e.*

$$\mathcal{T} = \overline{\mathrm{Inn}(\mathcal{T})} = \overline{\mathrm{Inn}^*(\mathcal{T})}.$$

(v) *Each inner point of $\mathcal{T}$ is an exclusive inner point, i.e.*

$$\mathrm{Inn}(\mathcal{T} + \Phi(\omega_1)) \cap (\mathcal{T} + \Phi(\omega_2)) = \emptyset$$

if $\omega_1, \omega_2 \in \mathbb{Z}[\alpha]$ are distinct.

(vi) *The Lebesgue measure of the boundary of $\mathcal{T}$ is zero, i.e.*

$$\mu\left(\partial(\mathcal{T})\right) = 0.$$

Note that (i), (ii) and (iv) imply that $\mathcal{T}$ and its translates form a tiling of $\mathbb{R}^n$. As mentioned in the introduction, this was first shown by Kátai-Környei [21]. Since the assertions of Theorem 2.1 are very fundamental, for the sake of completeness we wish to give a new proof of them putting our stress on the existence of exclusive inner points.

First of all, we need two lemmas

Lemma 2.2. $\Phi(\mathbb{Z}[1/\alpha])$ *is dense in* $\mathbb{R}^n$.

Proof. As $\Phi(\mathbb{Z}[\alpha])$ forms a full rank lattice in $\mathbb{R}^n$, there exists a positive constant c such that for each $x \in \mathbb{R}^n$ there exists $y \in \mathbb{Z}[\alpha]$ such that $||x - \Phi(y)|| < c$. Let m be a positive integer and select $y_m \in \mathbb{Z}[\alpha]$ such that $||\xi_m(x) - \Phi(y_m)|| < c$, hence,

$$||x - \Phi(\alpha^{-m} y_m)|| < \eta^{-m} c \quad \text{with } \eta \text{ as in (2.2).}$$

Taking m sufficiently large, we get the assertion. □

Lemma 2.3. *For any positive* B*, there exists an integer* M *such that each* $y \in \mathbb{Z}[1/\alpha]$ *satisfying* $\deg y \geq M$ *fulfills* $||\Phi(y)|| > B$.

Proof. It suffices to prove the assertion for the subset $\mathbb{Z}[\alpha] \subset \mathbb{Z}[1/\alpha]$, since the image by Φ of the fractional parts is bounded. Suppose the assertion does not hold. Then we may assume that there exists a positive constant B and a sequence $(y_i)_{i=1}^{\infty}$ with $\deg y_i \geq i$ in $\mathbb{Z}[\alpha]$ such that $||\Phi(y_i)|| \leq B$. The last inequality shows that $\{y_i \mid i = 1, 2, \dots\}$ forms a finite set. On the other hand, $\deg y_i \to \infty$ implies that the set $\{y_i \mid i = 1, 2, \dots\}$ contains infinitely many elements. This is a contradiction. □

After these preparations we prove the theorem.

Proof of (i): Classifying the set $\mathbb{Z}[1/\alpha]$ by the integer parts of its elements yields the partition

$$\mathbb{Z}[1/\alpha] = \bigcup_{\omega \in \mathbb{Z}[\alpha]} (\mathcal{S} + \omega).$$

Applying Φ, we get

$$\Phi(\mathbb{Z}[1/\alpha]) = \bigcup_{\omega \in \mathbb{Z}[\alpha]} (\Phi(\mathcal{S}) + \Phi(\omega)). \tag{2.6}$$

We say that a family $\{A_i\}_i$ of closed subsets of $\mathbb{R}^n$ is *locally finite* if for any bounded open set $U \in \mathbb{R}^n$, the set $\{i \mid A_i \cap U \neq \emptyset\}$ is finite. It is easy to see that $\overline{\bigcup_i A_i} = \bigcup_i \overline{A_i}$ if the family is locally finite. Our tiling is locally finite. Indeed, by the compactness of $\mathcal{T}$ and Lemma 2.3 we easily see that the set of neighbours S is finite. Thus taking closures of both sides of (2.6), we have

$$\mathbb{R}^n = \bigcup_{\omega \in \mathbb{Z}[\alpha]} (\mathcal{T} + \Phi(\omega)).$$

Proof of (iii): This assertion is proved in Akiyama-Thuswaldner [1] in a different way by using the results of Kátai-Kőrnyei [21]. Here we use another method. First we show that the origin is an inner point of $\mathcal{T}$. By Lemma 2.3, for any positive constant B there exists an integer M such that $\deg(y) \geq M$ implies $||\Phi(y)|| > B$. Replacing y with $\alpha^{M-1} y$ yields that $\deg(y) > 0$ implies $||\Phi(y)|| > B/\eta^{M-1}$ with η as in (2.2). This proves that if $\omega \neq 0$ then for any $x \in \mathcal{T} + \Phi(\omega)$ we have $||x|| \geq B/\eta^{M-1}$. Using assertion (i) we see that the open disc $\{x \in \mathbb{R}^n \mid ||x|| < B/\eta^{M-1}\}$ is contained in $\mathcal{T}$ and has empty intersection with the translates $\mathcal{T} + \Phi(\omega)$ for $\omega \neq 0$. This shows that the origin is an inner point of $\mathcal{T}$. Keeping in mind the set equation (2.3), this fact implies that $\Phi(i)$ $(0 \leq i \leq |N(\alpha)| - 1)$ are inner points of $\xi_1(\mathcal{T})$, so equivalently, $\Phi(i/\alpha)$ $(i = 0, 1, \dots, |N(\alpha)| - 1)$ are inner points of $\mathcal{T}$. Similar, repeated application of (2.3) yields the assertion.

Proof of (iv): The proof of assertion (iii) also implies that each $\Phi(p)$ with $p \in \mathcal{S}$ is an exclusive inner point of $\mathcal{T}$. Thus $\mathcal{T} = \overline{\mathrm{Inn}(\mathcal{T})} = \overline{\mathrm{Inn}^*(\mathcal{T})}$.

Proof of (v): (cf. also AKIYAMA [2]). Let $x \in \mathrm{Inn}(\mathcal{T} + \Phi(\omega_1)) \cap (\mathcal{T} + \Phi(\omega_2))$. Then there exists a sequence $(\theta_i)_{i=1}^{\infty}$ in $\mathrm{Inn}^*(\mathcal{T} + \Phi(\omega_2))$ which converges to x. Thus there exists an index i such that $\theta_i \in \mathrm{Inn}(\mathcal{T} + \Phi(\omega_1)) \cap \mathrm{Inn}^*(\mathcal{T} + \Phi(\omega_2))$. This contradicts to the definition of an exclusive inner point.

Proof of (vi): First we show that there exist finitely many nonzero elements $\nu_1, \nu_2, \dots, \nu_m \in \mathbb{Z}[\alpha]$ such that

$$\partial(\mathcal{T}) = \bigcup_{i=1}^{m} \left(\mathcal{T} \cap (\mathcal{T} + \Phi(\nu_i))\right). \tag{2.7}$$

The inclusion $\subset$ follows from the local finiteness together with the compactness of $\mathcal{T}$. The inclusion $\supset$ is a consequence of assertion (v) (cf. AKIYAMA [2, Corollary 2]). Assertion (vi) is now easily seen from (2.7) by comparing the measure of both sides of (2.3) and its iterations.

Proof of (ii): This is an immediate consequence of the proof of assertion (vi). $\square$

To make clear the geometric meaning of $\ell(x)$ we finally give a new proof of the result KÁTAI-KŐRNYEI [21, Lemma 1] on the length $\ell(x)$ of the α-adic representation.

Proposition 2.4. *Let α form a CNS and $x \in \mathbb{Z}[\alpha] \setminus \{0\}$. Then there exists a positive constant $C(\alpha)$ depending only on α such that the length $\ell(x)$ of the α-adic representation of x with respect to α satisfies the inequality*

$$\left| \ell(x) - \max_j \frac{\log |x^{(j)}|}{\log |\alpha^{(j)}|} \right| < C(\alpha).$$

Proof. Let $B(x, \rho)$ denote an open disc with radius ρ centered at x. In light of Theorem 2.1, one can take two positive radii R_1 and R_2 such that

$$B(0, R_1) \subset \mathrm{Inn}(\mathcal{T}) \subset \mathcal{T} \subset B(0, R_2).$$

Expanding $x \neq 0$ in the form

$$x = \sum_{i=0}^{\ell(x)} a_i \alpha^i, \qquad a_{\ell(x)} \neq 0$$

we have $\Phi(x\alpha^{-\ell(x)}) \in \mathcal{T}$ and, hence,

$$\left| \frac{x^{(j)}}{(\alpha^{(j)})^{\ell(x)}} \right| \leq R_2. \tag{2.8}$$

On the other hand since $\Phi(x\alpha^{-\ell(x)+1}) \notin \mathrm{Inn}(\mathcal{T})$, there exists some j such that

$$\frac{R_1}{r_1 + r_2} \leq \left| \frac{x^{(j)}}{(\alpha^{(j)})^{\ell(x)-1}} \right|. \tag{2.9}$$

The assertion now follows from (2.8) and (2.9). $\square$

3. A FUNDAMENTAL INEQUALITY

Hereafter assume that α is the base of a CNS whose minimal polynomial is $x^2 + Ax + B$. In this section we shall prove a crucial inequality, to which we will refer as the "fundamental inequality".

Definition 3.1 (Generalized Imaginary Part)**.** *Let I be a function from $\mathbb{Q}(\alpha)$ to $\mathbb{Q}$ defined by*

$$I(z) = \begin{cases} \Im(z), & \alpha \in \mathbb{C} \setminus \mathbb{R} \\ (z - z')/2, & \alpha \in \mathbb{R} \end{cases}. \tag{3.1}$$

Without loss of generality we may assume in the following that $I(\alpha) > 0$.

Lemma 3.2. *The tile $\mathcal{T}$ is symmetric with respect to the point $\Phi(\frac{B-1}{2(\alpha-1)})$.*

Proof. Let $x \in \mathcal{S}$ and write $x = \sum_{i=1}^{n} a_{-i}\alpha^{-i}$ with $a_{-i} \in \mathcal{N}$. For $m \geq n$, each

$$y_m = \sum_{i=1}^{n}(B-1-a_{-i})\alpha^{-i} + \sum_{i=n+1}^{m}(B-1)\alpha^{-i}$$

is clearly contained in $\mathcal{S}$. This shows that if $x \in \mathcal{S}$ then

$$\Phi\left(\frac{B-1}{\alpha-1} - x\right) = \sum_{i=1}^{\infty}(B-1)\Phi(\alpha^{-i}) - \sum_{i=1}^{n} a_{-i}\Phi(\alpha^{-i}) \in \mathcal{T}.$$

This shows the required symmetry since $\mathcal{T} = \overline{\Phi(S)}$. □

Theorem 3.3 (Fundamental Inequality)**.** *Let α be the base of a CNS with minimal polynomial $x^2 + Ax + B$ and assume that $2A < B + 3$. Then there exists a constant $\epsilon = \epsilon(\alpha) > 0$ such that for any $x \in \mathcal{S} + \alpha$ we have*

$$I(x) > I\left(\frac{B-1}{2(\alpha-1)}\right) + \epsilon.$$

Hereafter we use the symbol ϵ in a generic way just to indicate some positive constant depending only on α and do not distinguish them like $\epsilon_1, \epsilon_2, \ldots$. The proof of the fundamental inequality is divided into two parts. First we show the easier case $\alpha \in \mathbb{R}$. The case $\alpha \in \mathbb{C} \setminus \mathbb{R}$ is unexpectedly tedious and we need the major part of this section to settle it.

Proof for the case $\alpha \in \mathbb{R}$. Since we assume w.l.o.g. that $I(\alpha) > 0$ we have $\alpha - \alpha' > 0$. Moreover, because α forms a CNS, we have $\alpha' < \alpha < -1$. Indeed, if α forms a CNS then there exists no conjugate of α whose modulus is less or equal to 1 by [28] and there is no positive real conjugate. Thus

$$I(\alpha^{-i}) = \frac{\alpha^{-i} - (\alpha')^{-i}}{2} \quad \begin{cases} < 0 & \text{for odd } i \\ > 0 & \text{for even } i \end{cases}. \tag{3.2}$$

Since each $x \in \mathcal{S} + \alpha$ has the form $x = \alpha + \sum_{i=1}^{m} a_{-i}\alpha^{-i}$ with $a_{-i} \in \mathcal{N}$,

$$\begin{aligned} I(x) &\geq I(\alpha) + (B-1)\sum_{i:\text{odd}} I(\alpha^{-i}) \\ &= \frac{\alpha - \alpha'}{2} + \frac{B-1}{2}\left(\frac{\alpha^{-1}}{1-\alpha^{-2}} - \frac{(\alpha')^{-1}}{1-(\alpha')^{-2}}\right). \end{aligned}$$

Thus it suffices to show that

$$\frac{\alpha - \alpha'}{2} + \frac{B-1}{2}\left(\frac{\alpha^{-1}}{1-\alpha^{-2}} - \frac{(\alpha')^{-1}}{1-(\alpha')^{-2}}\right) > \frac{B-1}{4}\left(\frac{1}{\alpha-1} - \frac{1}{\alpha'-1}\right) + \epsilon. \tag{3.3}$$

Using $\alpha + \alpha' = -A$ and $\alpha\alpha' = B$ we can rewrite (3.3) as

$$\frac{\alpha - \alpha'}{4}\frac{B+3-2A}{B-A+1} > \epsilon$$

which is obviously valid. □

In the case $\alpha \in \mathbb{C} \setminus \mathbb{R}$, there is no easy way to control the signs of the generalized imaginary parts like (3.2) which causes the main difficulty. In fact, for instance if $A \geq 0$, a direct calculation shows

$$I(\alpha^{-1}) = -\frac{\Im(\alpha)}{B} < 0, \tag{3.4}$$

$$I(\alpha^{-2}) = \frac{\Im(\alpha)A}{B^2} \geq 0 \tag{3.5}$$

but

$$I(\alpha^{-3}) = -\frac{\Im(\alpha)(A^2 - B)}{B^3}$$

may be positive or negative. Our aim is to show the inequality

$$I(z) + I(\alpha) > I\left(\frac{B-1}{2(\alpha-1)}\right) + \epsilon \tag{3.6}$$

for $z \in \mathcal{S}$. Putting $z = \sum_{i=1}^{\infty} a_{-i}\alpha^{-i}$, (3.6) may be rewritten as

$$I(\alpha)\left\{\sum_i a_{-i}\frac{\alpha^{-i} - (\alpha')^{-i}}{\alpha - \alpha'} + 1 + \frac{B-1}{2(\alpha-1)(\alpha'-1)}\right\} > \epsilon$$

with $a_{-i} \in \mathcal{N}$. Dividing by $I(\alpha)(B-1)$, what we have to show becomes

$$\frac{1}{B-1} + \frac{1}{2(\alpha-1)(\alpha'-1)} + \sum{}^{*} \frac{\alpha^{-i} - (\alpha')^{-i}}{\alpha - \alpha'} > \epsilon, \tag{3.7}$$

where the last summation $\sum^*$ is taken over all $\frac{\alpha^{-i}-(\alpha')^{-i}}{\alpha-\alpha'}$ which are negative.

Lemma 3.4. *If $\alpha \in \mathbb{C} \setminus \mathbb{R}$ then $2A \le B+3$ and equality holds if and only if $(A, B) = (3, 3), (4, 5), (5, 7)$.*

Proof. Assume that $2A \ge B + 3$ and $A^2 - 4B < 0$. Then $((B + 3)/2)^2 \le A^2 < 4B$ implies $2 \le B \le 8$ and $3 \le A \le 5$. If $A = 3$ then $B \le 2A - 3 = 3$ and $B > A^2/4 = 9/4$ so $B = 3$. In a similar manner, if $A = 4$ then $B = 5$. If $A = 5$ then $B = 7$. Thus $2A = B+3$ holds for the constellations $(A, B) = (3, 3), (4, 5), (5, 7)$. □

In order to establish the remaining part of the proof of our theorem, assume for the moment that $A \ge 0$ and $B \ge 8$. The remaining cases will be treated in the final part of this proof. In this case, by using (3.4) and (3.5) we see that $\arg\alpha \in [\pi/2, \pi)$.

Lemma 3.5. *For $\arg\alpha \in [\pi/2, \pi)$, we have*

$$|I(\alpha^{-i}) + I(\alpha^{-i-1})| \le \frac{1}{\sqrt{B}^i} + \frac{1}{\sqrt{B}^{i+2}}.$$

Proof. Let $\theta = \arg\alpha^i$ and $\omega = \arg\alpha$. If $I(\alpha^{-i})I(\alpha^{-i-1}) < 0$ then the assertion is clear. Thus without loss of generality we may assume that $0 \le \theta \le \pi/2 \le \theta+\omega < \pi$. It suffices to show

$$\frac{\sin(\theta)}{\sqrt{B}^i} + \frac{\sin(\theta+\omega)}{\sqrt{B}^{i+1}} \le \frac{1}{\sqrt{B}^i} + \frac{1}{\sqrt{B}^{i+2}}$$

or equivalently

$$\sin(\theta) + \frac{\sin(\theta+\omega)}{\sqrt{B}} \le 1 + \frac{1}{B}. \tag{3.8}$$

The function $F(x) = \sin(\theta) + \sin(\theta+x)/\sqrt{B} - 1 - 1/B$ is decreasing for $x \in [\pi/2, \pi)$. Thus it is enough to check the validity of (3.8) for $x = \pi/2$ which is seen from

$$\begin{aligned} F(\pi/2) &= \sin(\theta) + \frac{\cos(\theta)}{\sqrt{B}} - 1 - \frac{1}{B} \\ &\le \sqrt{1 + \frac{1}{B}} - 1 - \frac{1}{B} < 0. \end{aligned}$$

□

Lemma 3.6. *Assume that $B \ge 7$. For $\arg\alpha \in [\pi/2, \pi)$, we have*

$$|I(\alpha^{-i}) + I(\alpha^{-i-1}) + I(\alpha^{-i-3})| \le \frac{1}{\sqrt{B}^i} + \frac{1}{\sqrt{B}^{i+2}}.$$

Proof. Let θ and ω be as above. If $I(\alpha^{-i})$, $I(\alpha^{-i-1})$ and $I(\alpha^{-i-3})$ do not have the same sign, then the required inequality is trivial in light of Lemma 3.5. Hence without loss of generality we may assume $0 \le \theta \le \pi/2 \le \theta+\omega < \pi$ and $2\pi \le \theta + 3\omega \le 3\pi$. By an argument similar to Lemma 3.5, it suffices to show that

$$\sin(\theta) + \frac{\sin(\theta+\omega)}{\sqrt{B}} + \frac{\sin(\theta+3\omega)}{B\sqrt{B}} \le 1 + \frac{1}{B}. \tag{3.9}$$

Let $F(x) = \sin(\theta) + \sin(\theta+x)/\sqrt{B} + \sin(\theta+3x)/\sqrt{B^3} - 1 - 1/B$ for a fixed θ. Then x is restricted to the interval

$$\max\{\pi/2, (2\pi-\theta)/3\} \le x \le \pi - \theta. \tag{3.10}$$

Since $F''(x) = -\sin(\theta+x)/\sqrt{B} - 9\sin(\theta+3x)/\sqrt{B^3} < 0$ holds together with $F'(\pi/2) = -(B-1)\sin(\theta)/\sqrt{B^3} < 0$ and

$$F'((2\pi-\theta)/3) = \cos(\theta + (2\pi-\theta)/3)/\sqrt{B} + 3/\sqrt{B^3} \le -1/(2\sqrt{B}) + 3/\sqrt{B^3} < 0$$

for $B \ge 7$, we see that $F(x)$ is a decreasing function in the interval (3.10). Thus the remaining task is to show that $F(\pi/2) \le 0$ and $F((2\pi-\theta)/3) \le 0$. The first one follows similarly to the last part of the proof of Lemma 3.5. The second one is a consequence of (3.8), since $\sin(\theta+3x) = 0$ for $x = (2\pi-\theta)/3$. $\square$

Lemma 3.7. *For $B \ge 8$, there exists a positive constant $\epsilon = \epsilon(\alpha)$ such that*

$$\frac{1}{B-1} + \frac{1}{2(\alpha-1)(\alpha'-1)} - \frac{1}{B} - \sum_{i:\text{odd},\ i\ge 3} \frac{1}{\sqrt{B}^i I(\alpha)} > \epsilon.$$

Proof. Using $\alpha\alpha' = B$ and $\alpha + \alpha' = -A$, it suffices to show

$$\frac{1}{B-1} + \frac{1}{2(B+A+1)} - \frac{1}{B} - \frac{2}{\sqrt{4B-A^2}}\frac{1/\sqrt{B}^3}{1-1/B} > 0$$

with $A^2 - 4B < 0$ and $B \ge 8$. Let $F(x) = 1/(B(B-1)) + 1/(2(B+x+1)) - 2/(\sqrt{4B-x^2}\sqrt{B}(B-1))$. Since $F(x)$ is a decreasing function in $-1 \le x \le \sqrt{4B-1}$, it is sufficient to show $F([\sqrt{4B-1}]) > 0$. First let $x = \sqrt{4B-1}$. Then it is easily shown that the function

$$\frac{1}{B(B-1)} + \frac{1}{2(B+\sqrt{4B-1}+1)} - \frac{2}{\sqrt{B}(B-1)}$$

is positive for $B \ge 29$, proving the lemma for $B \ge 29$. Secondly for $8 \le B \le 28$, putting $x = [\sqrt{4B-1}]$, direct calculations yield the desired inequality. $\square$

We are now in a position to treat the case $A \ge 0, B \ge 8$ and $I(\alpha^{-3}) < 0$. Since $\arg(\alpha) \in [\pi/2, \pi)$, there are no three consecutive equal signs in the sequence $\{I(\alpha^{-i})\}_{i=1}^\infty$. Hence combining (3.4), (3.5), Lemma 3.5 and 3.6, there exists an increasing sequence $\{c_i\}_{i=1}^\infty$ of positive integers with $c_1 = 3$ and $c_{i+1} - c_i \ge 2$ for all $i = 1, 2, \dots$ such that

$$\sum^{*} \frac{\alpha^{-i} - (\alpha')^{-i}}{\alpha - \alpha'} \ge -\frac{1}{B} - \frac{1}{\sqrt{B}^{c_1} I(\alpha)} - \frac{1}{\sqrt{B}^{c_2} I(\alpha)} - \frac{1}{\sqrt{B}^{c_3} I(\alpha)} - \cdots.$$

The right hand side is estimated from below by

$$-\frac{1}{B} - \sum_{i:\text{odd},\ i\ge 3} \frac{1}{\sqrt{B}^i I(\alpha)}$$

Combining this with (3.7), it suffices to show that there exist a positive constant ϵ such that

$$\frac{1}{B-1} + \frac{1}{2(\alpha-1)(\alpha'-1)} - \frac{1}{B} - \sum_{i:\text{odd},\ i\ge 3} \frac{1}{\sqrt{B}^i I(\alpha)} > \epsilon,$$

which is already proved in Lemma 3.7.

Now we deal with the case $A \ge 0, B \ge 8$ and $I(\alpha^{-3}) \ge 0$. In this case, we have

$$\sum^{*} \frac{\alpha^{-i} - (\alpha')^{-i}}{\alpha - \alpha'} \ge -\frac{1}{B} - \sum_{i \ge 4} \frac{1}{\sqrt{B}^{i} I(\alpha)}.$$

We claim that

$$\sum_{i \ge 4} \frac{1}{\sqrt{B}^{i} I(\alpha)} < \sum_{i:\text{odd},\ i \ge 3} \frac{1}{\sqrt{B}^{i} I(\alpha)},$$

which completes the proof in light of Lemma 3.7. Indeed, we see that

$$\sum_{i \ge 4} \frac{1}{\sqrt{B}^{i}} = \frac{1/B^2}{1 - 1/\sqrt{B}} < \frac{1/\sqrt{B}^{3}}{1 - 1/B} = \sum_{i:\text{odd},\ i \ge 3} \frac{1}{\sqrt{B}^{i}}.$$

Thus we completed the proof of Theorem 3.3 for $A \ge 0$ and $B \ge 8$. Next we assume $A = -1$ and $B \ge 8$. As $\alpha + \alpha' = 1, \alpha\alpha' = B$, we see that

$$\begin{aligned}
\frac{\alpha^{-1} - (\alpha')^{-1}}{\alpha - \alpha'} &= -\frac{1}{B}, \\
\frac{\alpha^{-2} - (\alpha')^{-2}}{\alpha - \alpha'} &= -\frac{1}{B^2}, \\
\frac{\alpha^{-3} - (\alpha')^{-3}}{\alpha - \alpha'} &= \frac{B-1}{B^3} > 0.
\end{aligned}$$

We have

$$\begin{aligned}
&\frac{1}{B-1} + \frac{1}{2(\alpha - 1)(\alpha' - 1)} + \sum^{*} \frac{\alpha^{-i} - (\alpha')^{-i}}{\alpha - \alpha'} \\
&> \frac{1}{B-1} + \frac{1}{2(B+1)} - \frac{1}{B} - \frac{1}{B^2} - \frac{2}{B^2(1 - B^{-1/2})\sqrt{4B-1}} \\
&> \frac{1}{2(B+1)} - \frac{2}{B^2(1 - B^{-1/2})\sqrt{4B-1}} > 0
\end{aligned}$$

which shows the desired inequality (3.7). In fact, the last inequality is already valid for $B \ge 4$.

For the remaining case $B \le 7$, we directly show the validity of the inequality (3.7). There are the 30 cases

- $B = 2, 3$ and $A = -1, 0, 1, 2$
- $B = 4, 5$ and $A = -1, 0, 1, 2, 3$
- $B = 6, 7$ and $A = -1, 0, 1, 2, 3, 4$

to examine. To describe this routine work, we only give an example. Let us consider the twin dragon case $A = B = 2$, i.e. $\alpha = -1 + \sqrt{-1}$. For $i \le 7$,

$$\frac{\alpha^{-i} - (\alpha')^{-i}}{\alpha - \alpha'} < 0$$

holds if and only if $i = 1, 3, 6$. Thus the left hand side of (3.7) is estimated from below by

$$\frac{1}{B-1} + \frac{1}{2(\alpha - 1)(\alpha' - 1)} + \sum_{i=1,3,6} \frac{\alpha^{-i} - (\alpha')^{-i}}{\alpha - \alpha'} - \frac{B^{-8/2}}{(1 - B^{-1/2}) I(\alpha)} > 0.116,$$

showing the inequality. The other cases are proved in an analogous way. Thus we have finished the whole proof of the fundamental inequality. □

It can be confirmed that the fundamental inequality in Theorem 3.3 or in its equivalent form (3.7) is not valid for $(A, B) \in \{(3, 3), (4, 5), (5, 7)\}$ by a calculation similar to the one in the above case $A = B = 2$. This is consistent with Lemma 3.4. In fact, if $A = B = 3$ and $\alpha = (-3 + \sqrt{-3})/2$ then we have

$$\frac{1}{B-1} + \frac{1}{2(\alpha-1)(\alpha'-1)} + \sum_{i=1,3,5} \frac{\alpha^{-i} - (\alpha')^{-i}}{\alpha - \alpha'} = -\frac{4}{189},$$

if $A = 4, B = 5$ and $\alpha = -2 + \sqrt{-1}$ then

$$\frac{1}{B-1} + \frac{1}{2(\alpha-1)(\alpha'-1)} + \sum_{i=1,3,5} \frac{\alpha^{-i} - (\alpha')^{-i}}{\alpha - \alpha'} = -\frac{7}{6250},$$

and if $A = 5, B = 7$ and $\alpha = (-5 + \sqrt{-3})/2$ then

$$\frac{1}{B-1} + \frac{1}{2(\alpha-1)(\alpha'-1)} + \sum_{i=1,3,5,7,9} \frac{\alpha^{-i} - (\alpha')^{-i}}{\alpha - \alpha'} = -\frac{26833}{1573790673}.$$

In the case $2A \geq B + 3$, the corresponding tile has a strange shape which will be seen in Section 7 and the sections following it.

4. On the connectedness of the interior of a tile $\mathcal{T}$ for $2A < B + 3$

Remember that α is the base of a CNS whose minimal polynomial is $x^2 + Ax + B$. We wish to discuss topological properties of the tile $\mathcal{T} \subset \mathbb{R}^2$. In particular, in the present section we shall prove the following result.

Theorem 4.1. *Let α be the base of a CNS with minimal polynomial $x^2 + Ax + B$ and let $\mathcal{T}$ be the corresponding central tile. If $2A < B + 3$ then $\mathrm{Inn}(\mathcal{T})$ is simply connected.*

It is well known that for an open set arcwise connectedness is equivalent to connectedness. Moreover, a domain $\mathcal{A}$ in $(\mathbb{R}^2)^*$ is simply connected if and only if $(\mathbb{R}^2)^* \setminus \mathcal{A}$ is connected. Here $(\mathbb{R}^2)^* = \mathbb{R}^2 \cup \{\infty\}$ is the Riemann sphere, i.e., the one point compactification of $\mathbb{R}^2$ (see Newman [32, Chapter 6, Claim 4.1]). Thus a bounded domain $\mathcal{A}$ in $\mathbb{R}^2$ is simply connected if and only if $\mathbb{R}^2 \setminus \mathcal{A}$ is connected.

Lemma 4.2. *If $\mathrm{Inn}(\mathcal{T})$ is connected then it is simply connected.*

Proof. What we have to prove is that $\mathbb{R}^2 \setminus \mathrm{Inn}(\mathcal{T})$ is arcwise connected if $\mathrm{Inn}(\mathcal{T})$ is connected. Using Theorem 2.1 (v) we see that

$$\mathbb{R}^2 \setminus \mathrm{Inn}(\mathcal{T}) = \bigcup_{0 \neq x \in \mathbb{Z}[\alpha]} (\mathcal{T} + \Phi(x)).$$

Furthermore, each $\mathcal{T} + \Phi(x)$ is arcwise connected by Akiyama-Thuswaldner [1, Theorem 3.1]. Thus the remaining task is to show that for any two tiles $\mathcal{T} + \Phi(a)$ and $\mathcal{T} + \Phi(b)$ with $a \neq 0, b \neq 0$, one can find a non zero sequence $(x_i)_{i=1}^m$ in $\mathbb{Z}[\alpha]$ that that $x_1 = a$, $x_m = b$ and $(\mathcal{T} + \Phi(x_i)) \cap (\mathcal{T} + \Phi(x_{i+1})) \neq \emptyset$ for $i = 1, \dots, m-1$. But this can be easily established by [1, Theorem 2.1] where we explicitly constructed the vertices of $\mathcal{T}$. □

So Theorem 4.1 follows if we can show that $\mathrm{Inn}(\mathcal{T})$ is connected. To this matter we construct a concrete arcwise connected subset of $\mathrm{Inn}(\mathcal{T})$ which is dense in $\mathrm{Inn}(\mathcal{T})$ showing that $\mathrm{Inn}(\mathcal{T})$ is arcwise connected. We prepare the proof with some lemmas.

Lemma 4.3. *Let $\gamma \in \mathbb{Z}[\alpha]$ and put $\gamma = u + v\alpha$ with u, v in $\mathbb{Z}$. Then there exist a constant $\epsilon = \epsilon(\alpha) > 0$ such that for any $x \in \mathcal{S}$,*

$$\begin{cases} I(x) + I(\gamma) > I(\frac{B-1}{2(\alpha-1)}) + \epsilon & \text{if } v \geq 1 \\ I(x) + I(\gamma) < I(\frac{B-1}{2(\alpha-1)}) - \epsilon & \text{if } v \leq -1. \end{cases}$$

Proof. Remember that $I(\alpha) > 0$ and $I(\gamma) = vI(\alpha)$. Thus Theorem 3.3 implies the assertion for $v \geq 1$. From now on assume that $v \leq -1$. Then of course $-\gamma = u' + v'\alpha$ with $v' \geq 1$. Put $g(z) = (B-1)/(\alpha-1) - z$ for $z \in \mathcal{S}$. Note that $g(z)$ is not always contained in $\mathcal{S}$ but in $\mathbb{Q}(\alpha)$. All the same, Lemma 3.2 implies $\overline{\Phi(g(\mathcal{S}))} = \overline{\Phi(\mathcal{S})} = \mathcal{T}$. This means that any $\Phi(x)$ for $x \in \mathcal{S}$ is an accumulation point of $\{\Phi(g(x)) \mid x \in \mathcal{S}\}$. Hence for any $x \in \mathcal{S}$, $I(x)$ is an accumulation point of $\{I(g(x)) \mid x \in \mathcal{S}\}$. Note that

$$g(\mathcal{S} + \gamma) = g(\mathcal{S}) - \gamma$$

and

$$I(g(z)) = I\left(\frac{B-1}{\alpha-1}\right) - I(z).$$

Thus Theorem 3.3 implies

$$\begin{aligned}
\sup_{x \in \mathcal{S}}\{I(x) + I(\gamma)\} &= \sup_{x \in \mathcal{S}}\left\{I\left(\frac{B-1}{\alpha-1}\right) - I\left(g(x+\gamma)\right)\right\} \\
&= I\left(\frac{B-1}{\alpha-1}\right) - \sup_{x \in \mathcal{S}}\{I(x-\gamma)\} \\
&< I\left(\frac{B-1}{\alpha-1}\right) - I\left(\frac{B-1}{2(\alpha-1)}\right) - \varepsilon \\
&= I\left(\frac{B-1}{2(\alpha-1)}\right) - \varepsilon.
\end{aligned}$$

This shows the lemma. $\square$

Let

$$\mathcal{L} = \left\{\Phi\left(\frac{B-1}{2(\alpha-1)}\right) + w\Phi(1) \,\middle|\, w \in [0, B-1]\right\}.$$

$\mathcal{L}$ is the closed line segment joining the center of symmetry of $\mathcal{T}$ and that of $\mathcal{T} + \Phi(B-1)$. The following Lemma is a remarkable consequence of Theorem 3.3.

Lemma 4.4. *If* $2A < B + 3$ *then* $\mathcal{L} \subset \mathrm{Inn}(\xi_1(\mathcal{T}))$.

Proof. Using Lemma 4.3 we see that for any $\gamma = u + v\alpha$ with $u, v \in \mathbb{Z}$ and $v \neq 0$

$$(\mathcal{T} + \Phi(\gamma)) \cap \mathcal{L} = \emptyset.$$

By (2.3), and Theorem 2.1 (i) and (v),

$$\begin{aligned}
\mathrm{Inn}(\xi_1(\mathcal{T})) &= \mathbb{R}^2 \setminus \bigcup_{0 \neq x \in \mathbb{Z}[\alpha]} (\xi_1(\mathcal{T}) + \Phi(\alpha x)) \\
&= \mathbb{R}^2 \setminus \left(\bigcup_{(u,v) \in \mathbb{Z}^2, v \neq 0} (\mathcal{T} + \Phi(u + v\alpha)) \cup \bigcup_{j \leq -1 \text{ or } j \geq B} (\mathcal{T} + \Phi(j))\right).
\end{aligned}$$

Thus it suffices to show that for any integer j with $j \leq -1$ or $j \geq B$.

$$(\mathcal{T} + \Phi(j)) \cap \mathcal{L} = \emptyset. \tag{4.1}$$

By Lemma 4.3 for $r \leq -1$, there exists a positive constant ϵ such that for any $x \in \alpha^{-1}\mathcal{S} + r$ we have

$$I(\alpha x) < I\left(\frac{B-1}{2(\alpha-1)}\right) - \epsilon. \tag{4.2}$$

Let us define P, P_+ and P_- by

$$P := \overline{\Phi\left(\left\{x \in \mathbb{Q}(\alpha) \mid I(\alpha x) = I\left(\frac{B-1}{2(\alpha-1)}\right)\right\}\right)},$$

$$P_+ := \overline{\Phi\left(\left\{x \in \mathbb{Q}(\alpha) \mid I(\alpha x) > I\left(\frac{B-1}{2(\alpha-1)}\right)\right\}\right)} \setminus P$$

and

$$P_- := \overline{\Phi\left(\left\{x \in \mathbb{Q}(\alpha) \mid I(\alpha x) < I\left(\frac{B-1}{2(\alpha-1)}\right)\right\}\right)} \setminus P.$$

Obviously P forms a line in $\mathbb{R}^2$ and (4.2) implies that $\overline{\Phi(\alpha^{-1}\mathcal{S}+r)} = \xi_{-1}(\mathcal{T})+\Phi(r)$ satisfies

$$\xi_{-1}(\mathcal{T}) + \Phi(r) \subset P_- \tag{4.3}$$

for $r \le -1$. Clearly from (4.3) and the definition of P_-,

$$\xi_{-1}(\mathcal{T}) + \Phi(\alpha^{-1}r' + r) \subset P_-$$

for any $r, r' \in \mathbb{Z}$ with $r \le -1$. Since the set equation (2.3) gives

$$\mathcal{T} = \bigcup_{i=0}^{B-1} \left(\xi_{-1}(\mathcal{T}) + \Phi(\alpha^{-1}i)\right),$$

we have

$$(\mathcal{T} + \Phi(r)) \subset P_-$$

for $r \le -1$. On the other hand, it is easily seen that $\mathcal{L}$ is a subset of $P \cup P_+$. Indeed, if $y \in [0, B-1] \cap \mathbb{Q}$ then

$$\alpha\left(\frac{B-1}{2(\alpha-1)} + y\right) = \alpha y + \frac{B-1}{2} + \frac{B-1}{2(\alpha-1)}$$

which shows

$$I\left(\alpha\left(\frac{B-1}{2(\alpha-1)} + y\right)\right) \ge I\left(\frac{B-1}{2(\alpha-1)}\right)$$

since $I(\alpha) > 0$. Thus

$$(\mathcal{T} + \Phi(r)) \cap \mathcal{L} = \emptyset$$

for $r \le -1$. The case $r \ge B$ is shown likewise. □

By Lemma 4.4 we know that the segment $\xi_{-1}(\mathcal{L})$, which we shall call the *backbone* of $\mathcal{T}$, is contained in the interior of $\mathcal{T}$. Let u, v $(u < v)$ be distinct positive integers and

$$\mathcal{L}_{u,v} = \left\{\Phi\left(\frac{B-1}{2(\alpha-1)}\right) + w\Phi(1) \,\middle|\, w \in [u, v]\right\}.$$

The next lemma is shown in a way similar to Lemma 4.4 and we omit its proof.

Lemma 4.5. *If* $2A < B + 3$ *then* $\mathcal{L}_{u,v} \subset \mathrm{Inn}\left(\bigcup_{k=u}^{v-1}(\mathcal{T} + \Phi(k))\right)$.

m-fold iteration of the set equation (2.3) gives us the subdivision

$$\mathcal{T} = \bigcup_{y=.a_{-1}a_{-2}\dots a_{-m}} \left(\xi_{-m}(\mathcal{T}) + \Phi(y)\right), \tag{4.4}$$

where $.a_{-1}a_{-2}\dots a_{-m} = \sum_{i=1}^m a_{-i}\alpha^{-i}$ with $a_{-i} \in \mathcal{N}$. Each tile of the subdivision $\xi_m(\mathcal{T}) + \Phi(.a_{-1}a_{-2}\dots a_{-m})$ has a backbone $\Phi(.a_{-1}a_{-2}\dots a_{-m}) + \xi_{-m-1}(\mathcal{L})$ (i.e. a small segment contained in its interior), since each ξ_i $(i \in \mathbb{Z})$ is a homeomorphism. Composing these small pieces of backbones, let us define the *n-skeleton* by

$$\mathcal{K}_n = \bigcup_{m=0}^{n} \left(\bigcup_{.a_{-1}a_{-2}\dots a_{-m}} \Phi(.a_{-1}a_{-2}\dots a_{-m}) + \xi_{-1-m}(\mathcal{L})\right).$$

Lemma 4.6. *$\mathcal{K}_n$ is arcwise connected and $\mathcal{K}_n \subset \mathrm{Inn}(\mathcal{T})$.*

Proof. $\mathcal{K}_n \subset \text{Inn}(\mathcal{T})$ is obvious since each backbone is contained in the interior of some tile of the subdivision (4.4) of $\mathcal{T}$ by Lemma 4.4 and Lemma 4.5. Thus we only need prove that $\mathcal{K}_n$ is arcwise connected. Clearly $\mathcal{K}_0$ is arcwise connected. Assume that $\mathcal{K}_{n-1}$ is arcwise connected. It suffices to show that each additional segment $\Phi(.a_{-1}a_{-2}\dots a_{-n}) + \xi_{-1-n}(\mathcal{L})$ appearing in $\mathcal{K}_n$ has non-empty intersection with $\mathcal{K}_{n-1}$. For this purpose, consider its middle point

$$M_{.a_{-1}\dots a_{-n}} = \Phi(.a_{-1}a_{-2}\dots a_{-n}) + \Phi\left(\alpha^{-1-n}\left(\frac{B-1}{2} + \frac{B-1}{2(\alpha-1)}\right)\right).$$

Since

$$\begin{aligned} M_{.a_{-1}\dots a_{-n}} &= \Phi(.a_{-1}\dots a_{-n+1}) + \Phi(\alpha^{-n}a_{-n}) + \Phi\left(\alpha^{-n}\left(\frac{B-1}{2(\alpha-1)}\right)\right) \\ &\in \Phi(.a_{-1}\dots a_{-n+1}) + \xi_{-n}(\mathcal{L}) \end{aligned}$$

the proof is completed. $\square$

Proof of Theorem 4.1. By definition, $\bigcup_{m=0}^{\infty}\{\Phi(y) \mid y = .a_{-1}\dots a_{-m}\}$ is dense in $\mathcal{T}$. Thus the set of middle points of all backbones

$$M = \bigcup_{m=0}^{\infty}\left\{\Phi\left(y + \alpha^{-m}\frac{B-1}{2(\alpha-1)}\right) \;\middle|\; y = .a_{-1}\dots a_{-m} \in \mathbb{Z}[1/\alpha]\right\}$$

is also dense in $\mathcal{T}$. Take $x, y \in \text{Inn}(\mathcal{T})$ and let $B(z,\delta)$ denote the open ball centered at z with radius δ. Then for any positive ε, we have $B(x,\varepsilon) \cap M \neq \emptyset$ and $B(y,\varepsilon) \cap M \neq \emptyset$. In light of Lemma 4.6, this means that $\text{Inn}(\mathcal{T})$ is arcwise connected and the result follows from Lemma 4.2. $\square$

5. The boundary of a tile is a simple closed curve

Heavily depending on the fundamental inequality, it was shown in the previous section that $\text{Inn}(\mathcal{T})$ is simply connected if the minimal polynomial $x^2 + Ax + B$ of the CNS base α fulfills $2A < B+3$. Now we are in a position to establish topological properties of the boundary of the tile $\mathcal{T}$. First we recall some basic definitions.

A *continuum* is a connected compact set with at least two points. Let C be a set in $\mathbb{R}^n$. Then C is *locally connected* at $x \in \mathbb{R}^n$ if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that any two points in $C \cap B(x,\delta)$ are contained in a connected subset of $C \cap B(x,\varepsilon)$. We say that C is locally connected if it is locally connected at all $x \in C$. A *cut point* x of a connected set C is a point for which $C \setminus \{x\}$ is not connected. Keeping in mind the definition of $K = K(f_1,\dots,f_m)$ in (2.5) we can state the following result.

Lemma 5.1 (Hata [16, Theorem 4.6])**.** *The set $K = K(f_1,\dots,f_m)$ is a locally connected continuum if and only if for any $1 \le i \le j \le m$ there exists sequence $\{r_1, r_2, \dots, r_n\} \subset \{1,2,\dots,m\}$ with $r_1 = i, r_n = j$ such that*

$$f_{r_\ell}(K) \cap f_{r_{\ell+1}}(K) \neq \emptyset$$

for any $1 \le \ell \le n-1$.

This lemma was used in the proof of [1, Theorem 3.1] to show that if $n = 2$ then the tile $\mathcal{T}$ corresponding to a quadratic number system is arcwise connected by checking the conditions of Lemma 5.1. So it is already shown that $\mathcal{T}$ is a locally connected continuum. Here we may recall the famous theorem of Hahn and Mazurkiewicz saying that a locally connected continuum is the continuous image of a closed unit interval $[0,1]$ and vice versa, which gives us a characterization of a "curve" (see Newman [32, Chapter 4, Section 9].) Now let us show the following result.

Lemma 5.2. *If $2A < B+3$, there are no cut points in $\mathcal{T}$.*

Proof. Suppose that $x \in \mathcal{T}$ is a cut point. Clearly an inner point of $\mathcal{T}$ can not be a cut point, thus $x \in \partial(\mathcal{T})$ and $\mathcal{T} \setminus \{x\}$ is a union of non-empty (relative) disjoint closed sets U and V. Thus we have

$$\mathrm{Inn}(\mathcal{T}) = (U \cap \mathrm{Inn}(\mathcal{T})) \cup (V \cap \mathrm{Inn}(\mathcal{T})).$$

Assume, for e.g., $U \cap \mathrm{Inn}(\mathcal{T}) = \emptyset$ then $V \supset \mathrm{Inn}(\mathcal{T})$. Recalling $\mathcal{T} = \overline{\mathrm{Inn}(\mathcal{T})}$, this implies $V = \mathcal{T} \setminus \{x\}$ and $U = \emptyset$, contradicting to the choice of U and V. Thus we have $U \cap \mathrm{Inn}(\mathcal{T}) \neq \emptyset$ and $V \cap \mathrm{Inn}(\mathcal{T}) \neq \emptyset$ which gives a contradiction with Theorem 4.1. This proves that there are no cut points in $\mathcal{T}$. □

We quote another auxiliary result.

Lemma 5.3 (Kuratowski [29, Chap. 10, §61 II, Theorem 4])**.** *If a set C in the* Riemann *sphere $(\mathbb{R}^2)^*$ is a locally connected continuum without cut points, the boundary $\partial(R)$ of every component R of $(\mathbb{R}^2)^* \setminus C$ is a simple closed curve.*

Lemma 5.4. *If $2A < B + 3$, then $(\mathbb{R}^2)^* \setminus \mathcal{T}$ is connected.*

Proof. Let C be a component of $(\mathbb{R}^2)^* \setminus \mathcal{T}$ and take a point $x \in C$. Then by Theorem 2.1 (i) there exists $u + v\alpha \in \mathbb{Z}[\alpha]$ such that $x \in \mathcal{T} + \Phi(u+v\alpha)$. It suffices to show that x and the point of infinity can be joined by a curve in $(\mathbb{R}^2)^* \setminus \mathcal{T}$. Consider the case $v \neq 0$. By Lemma 4.5 and Theorem 2.1 (iv) $(\mathcal{L}_{u,\infty} + \Phi(v\alpha)) \cap \mathcal{T} = \emptyset$. As $\mathbb{R}^2 \setminus \mathcal{T}$ is open, one can take a positive ε such that $B(x, \varepsilon) \subset \mathbb{R}^2 \setminus \mathcal{T}$. Since $\mathcal{T} = \overline{\mathrm{Inn}(\mathcal{T})}$ we can find $y \in B(x,\varepsilon) \cap (\mathrm{Inn}(\mathcal{T}) + \Phi(u + v\alpha))$. By using Theorem 4.1 and its proof, there exists a curve C in $\mathrm{Inn}(\mathcal{T}) + \Phi(u + v\alpha)$ joining y and the symmetric center $\Phi(u+v\alpha+(B-1)/(2(\alpha-1))) \in (\mathcal{L}_{u,\infty} + \Phi(v\alpha))$. This symmetry center can be joined with the point of infinity along $\mathcal{L}_{u,\infty} + \Phi(v\alpha)$. This shows that x and the point of infinity can be joined by a curve in $(\mathbb{R}^2)^* \setminus \mathcal{T}$, as desired. If $v = 0$ then $u \neq 0$. If $u < 0$ then for any integer $p < -1$, $\mathcal{L}_{p,-1} \cap \mathcal{T} = \emptyset$. If $u > 0$ then for any integer $p > 1$, $\mathcal{L}_{1,p} \cap \mathcal{T} = \emptyset$. The proof is completed similarly in both cases. □

Summing up, we have reached the main result of this section.

Theorem 5.5. *Let α be the base of a CNS with minimal polynomial x^2+Ax+B and let $\mathcal{T}$ be the associated central tile. If $2A < B+3$, then $\partial(\mathcal{T})$ is a simple closed curve separating two domains* $\mathrm{Inn}(\mathcal{T})$ *and $(\mathbb{R}^2)^* \setminus \mathcal{T}$. Furthermore, $\mathcal{T}$ is homeomorphic to a closed unit disc.*

Proof. By Lemma 5.3 and Lemma 5.4, $(\mathbb{R}^2)^* \setminus \mathcal{T}$ is a domain whose boundary C is a simple closed curve. Since $\mathcal{T}$ is closed, $C \subset \mathcal{T}$. Also C can not contain points of $\mathrm{Inn}(\mathcal{T})$ because each inner point of $\mathcal{T}$ is exclusive. Thus $C \subset \partial(T)$. The inverse inclusion is trivial. This shows that the simple closed curve C is the common boundary of the two components of $(\mathbb{R}^2)^* \setminus C$, i.e. $(\mathbb{R}^2)^* \setminus \mathcal{T}$ and $\mathrm{Inn}(\mathcal{T})$. The last statement is a consequence of Newman [32, Theorem 17.1]. □

A point x of the boundary of a domain $D \in \mathbb{R}^2$ is said to be *accessible* from D if there exist a simple arc in $\mathbb{R}^2$ whose one end point is x and all other points are in D.

Corollary 5.6. *Each point on $\partial(\mathcal{T})$ is accessible from* $\mathrm{Inn}(\mathcal{T})$.

Proof. As each point of a simple closed curve in $(\mathbb{R}^2)^*$ is accessible from both residual domains, it is a consequence of Theorem 5.5. □

6. The set equation of the boundary of a tile

We have shown that under the hypothesis $2A < B+3$ the boundary $\partial(\mathcal{T})$ is just a simple closed curve. The aim of this section is to calculate the concrete form of this curve. We have already mentioned in the introduction that the set equation of $\partial(\mathcal{T})$ has been given in a general setting. In this section we give a proof of this set equation in a concrete and precise form adding topological informations as orientation and symmetry. Moreover, due to the results of the previous section this proof gives new insights in the structure of $\mathcal{T}$. In fact, Lemma 6.4 and Lemma 6.5 seem not so easy to establish without the topological study of the previous sections.

In the remaining part of the paper we will write α-adic representations in the form

$$a_H a_{H-1} \dots a_0 . a_{-1} a_{-2} \dots := \sum_{j=-\infty}^{H} a_j \alpha^j .$$

For the r-fold repetition of a substring $a_k \dots a_1$ we will write $[a_k \dots a_1]^r$, $[a_k \dots a_1]^\infty$ means that the string is repeated infinitely often.

If $A = 0$ then the problem is quite easy. In fact, if $\alpha = \sqrt{-B}$ then for every element x of $\mathbb{Z}[\alpha]$ there exists a smallest non-negative integer n such that x can be written uniquely in the form

$$x = a_0 + a_2(-B) + \cdots + a_{2n}(-B)^{2n} + \left(a_1 + \cdots + a_{2n-1}(-B)^{2n-1}\right)\sqrt{-B}$$

with $a_i \in \mathcal{N}$. This means that this canonical number system is just a "combination" of two canonical number systems in base $-B$. Thus the tile is a rectangle in the complex plane given by

$$\begin{aligned} \mathcal{T} &= \left\{ x + y\sqrt{-B} \mid x = \sum_{i=1}^{\infty} a_{-i}(-B)^{-i} \text{ and } y = \sum_{i=1}^{\infty} b_{-i}(-B)^{-i} \right\} \\ &= \left\{ x + y\sqrt{-B} \;\middle|\; x, y \in \left[-\frac{B}{1+B}, \frac{1}{1+B} \right] \right\} \end{aligned}$$

where $a_{-i}, b_{-i} \in [0, B-1] \cap \mathbb{Z}$. So in this case the tiling is made of a rectangle and its translations. Obviously, 4 tiles meet at the corners of each tile. Hereafter we want to exclude this trivial case. We shall call a CNS *nontrivial* if $A \neq 0$.

Now we determine the "neighbours" of $\mathcal{T}$ in the sense of (2.7). To this matter we need the following auxiliary result.

Lemma 6.1. *If* $|v| \geq 2$ *then* $\mathcal{T} \cap (\mathcal{T} + \Phi(u + v\alpha)) = \emptyset$.

Proof. By Lemma 3.2, we only need to show the case $v \geq 2$. Consider the line L which contains the backbone of $\mathcal{T} + \Phi(\alpha)$. By the same method as in the proof of Lemma 4.3 and 4.4, it can be shown that $\mathcal{T}$ and $\mathcal{T} + \Phi(u + v\alpha)$ are located on the different sides of two half planes separated by L. □

The characterization of the "neighbours" of $\mathcal{T}$ reads as follows.

Lemma 6.2. *If* $2A < B + 3$ *and* $A \neq 0$, *then* $\#S = 6$. *In particular, if* $A > 0$ *then*

$$S = \{(A,1), (A-1,1), (-1,0), (1,0), (-A,-1), (-A+1,-1)\}.$$

If $A = -1$ *we have*

$$S = \{(-1,1), (0,1), (-1,0), (1,0), (1,-1), (0,-1)\}.$$

If two distinct tiles have a common point then there are at least two common points.

Proof. By Lemma 6.1, if $(\mathcal{T} + \Phi(u+v\alpha)) \cap \mathcal{T} \neq \emptyset$ then v must be ± 1 or 0. Applying ξ_1 we have

$$\xi_1(\mathcal{T}) = \bigcup_{i=0}^{B-1} (\mathcal{T} + \Phi(i))$$

and

$$\xi_1(\mathcal{T} + \Phi(u + v\alpha)) = \bigcup_{i=0}^{B-1} \left(\mathcal{T} + \Phi(i - Bv + (u - A)\alpha)\right).$$

Again by Lemma 6.1 we have $u - A = \pm 1$ or 0. Reviewing [1, Theorem 2.1 and Corollary 2.1] we find that the above mentioned 6 tiles have common points with $\mathcal{T}$. The remaining task is to show that if $A > 0$ then

$$(\mathcal{T} + \Phi(A + 1 + \alpha)) \cap \mathcal{T} = \emptyset \tag{6.1}$$

and

$$(\mathcal{T} + \Phi(-A - 1 - \alpha)) \cap \mathcal{T} = \emptyset,$$

and if $A = -1$ then

$$(\mathcal{T} + \Phi(-2 + \alpha)) \cap \mathcal{T} = \emptyset$$

and

$$(\mathcal{T} + \Phi(2 - \alpha)) \cap \mathcal{T} = \emptyset.$$

The proofs of these cases are similar to each other, thus we want to show (6.1) and omit the proofs of the other cases. Applying ξ_2 to the left hand side of (6.1), we have

$$\left(\bigcup_{i,j=0}^{B-1} \mathcal{T} + \Phi(-B + j + (i - A - B)\alpha) \right) \cap \left(\bigcup_{i,j=0}^{B-1} \mathcal{T} + \Phi(j + i\alpha) \right).$$

Since $i - A - B \leq -2$ and $i \geq 0$, this set is empty by Lemma 6.1. The last statement of the present lemma is an immediate consequence of the fact that these 6 tiles are the only ones which touch $\mathcal{T}$ combined with [1, Theorem 2.1 and Corollary 2.1]. □

Lemma 6.3. *If $2A < B + 3$ and $A \neq 0$ then $V_4 = \emptyset$.*

Proof. Since our tiling is periodic with two periods $\Phi(1)$ and $\Phi(\alpha)$, Lemma 6.2 describes all tiles which have common points with a fixed tile $\mathcal{T} + \Phi(u + v\alpha)$ with integers u, v. Assume that the four distinct tiles $\mathcal{T}_i = \mathcal{T} + \Phi(u_i + v_i\alpha)$ with $i = 1, 2, 3, 4$ have a common point. Consider the case $A > 0$. By the above mentioned translation, we may assume that $u_1 = v_1 = 0$ and $(u_2, v_2) \in \{(1, 0), (A, 1), (A - 1, 1)\}$. Let $\mathcal{T}_2 = \mathcal{T} + \Phi(1)$. Then using Lemma 6.2 to the tiles $\mathcal{T}$ and $\mathcal{T}_2$, we have

$$(u_i, v_i) \in \{(A, 1), (-A + 1, -1)\}$$

for $i = 3, 4$. Again by Lemma 6.1,

$$(\mathcal{T} + \Phi(A + \alpha)) \cap (\mathcal{T} + \Phi(-A + 1 - \alpha)) = \emptyset,$$

showing that $\cap_{i=1}^{4} \mathcal{T}_i = \emptyset$ which contradicts the assumption. The remaining part of the proof is left to the reader. □

Lemma 6.4. *The intersection of three distinct tiles is a single point or empty.*

Proof. Assume that the three distinct tiles $\mathcal{T}_i = \mathcal{T} + \Phi(u_i + v_i\alpha)$ with $i = 1, 2, 3$ have two common points x, y $(x \neq y)$. By Corollary 5.6 and Theorem 4.1, there exist three simple arcs C_i $(1 \leq i \leq 3)$ in $\mathcal{T}_i$ from x to y such that each point of C_i apart from its end points is an inner point of the corresponding tile. We denote by $-C$ the reversed arc of C. Then by [29, Chapter 10, Section 61 II, Theorem 2], one of the following three statements is valid:

- $C_1 \setminus \{x, y\}$ is contained in the bounded residual component of the simple closed curve $C_2 \cup -C_3$.
- $C_2 \setminus \{x, y\}$ is contained in the bounded residual component of the simple closed curve $C_3 \cup -C_1$.
- $C_3 \setminus \{x, y\}$ is contained in the bounded residual component of the simple closed curve $C_1 \cup -C_2$.

Let $a, b, c \in \{1, 2, 3\}$. Since each $\mathrm{Inn}(\mathcal{T})$ is connected and $\mathcal{T} = \overline{\mathrm{Inn}(\mathcal{T})}$, this means that there exists a simple closed curve C in $\mathcal{T}_a \cup \mathcal{T}_b$ with $a \neq b$ that the bounded residual component of C contains $\mathrm{Inn}(\mathcal{T}_c)$ with $c \neq a, b$. In short, we say that *two tiles T_a and T_b surround another tile T_c* to express this situation. We wish to give a sketch of the proof that this is impossible. By considerations similar to the proof of Lemma 6.3, we may take $u_1 = v_1 = 0$ and $u_2 = 1, v_2 = 0$. If $A > 0$ then we may also assume that

$$(u_3, v_3) \in \{(A, 1), (A - 1, 1), (-A, -1), (-A + 1, -1)\}.$$

Now we treat the case that $u_3 = A, v_3 = 1$. We will freely employ Lemmas 4.3, 4.4 and 4.5 hereafter. The line containing the backbone of $\mathcal{T}_3$ can not have any intersection with $\mathcal{T}_1 \cup \mathcal{T}_2$. The half line $\mathcal{L}_{1,\infty}$ containing the backbone of $\mathcal{T}_2$ can not have a common point with $\mathcal{T}_1$ and $\mathcal{T}_3$. The half line $\mathcal{L}_{-\infty,0}$ containing the backbone of $\mathcal{T}_1$ can not have a common point with $\mathcal{T}_2$ and $\mathcal{T}_3$. This shows that no two tiles of $\mathcal{T}_i$ $(i = 1, 2, 3)$ surround the remaining tile, proving the result in the case $A > 0, u_3 = A, v_3 = 1$. The other cases are proved likewise. □

Lemma 6.5. *The intersection of two tiles is a simple arc or empty.*

Proof. Let $\mathcal{T}_i$ $(i = 1, 2)$ be two tiles having non-empty intersection. There are at least two points in $\mathcal{T}_1 \cap \mathcal{T}_2$ by Lemma 6.2. We may assume that $\mathcal{T}_1 = \mathcal{T}$ by a translation. Using (2.7) and Theorem 5.5, $\mathcal{T} \cap \mathcal{T}_2$ is a subset of the simple closed curve $\partial(\mathcal{T})$. If $\mathcal{T} \cap \mathcal{T}_2$ is connected then it is arcwise connected and, hence, a simple arc. Thus it suffices to show that $\mathcal{T}_1 \cap \mathcal{T}_2$ is connected. Assume that $\mathcal{T}_1 \cap \mathcal{T}_2$ is not connected and x, y are two points in the distinct components of $\mathcal{T}_1 \cap \mathcal{T}_2$. Then by Corollary 5.6, there exist two simple arcs C_i $(i = 1, 2)$ joining x and y in $\mathcal{T}_i$ such that each point of C_i apart from its end points is an inner point of the tile $\mathcal{T}_i$. Let us consider the bounded residual component D of $C_1 \cup -C_2$. Since $(C_1 \cup -C_2) \cap \partial(\mathcal{T}) = \{x, y\}$, $\partial(T)$ is separated into two simple arcs by $C_1 \cup -C_2$. Denote by A the arc in D. We claim that $D \not\subset \mathcal{T}_1 \cup \mathcal{T}_2$. For if $D \subset \mathcal{T}_1 \cup \mathcal{T}_2$, then $A \subset D$ implies that each point on A must lie on $\mathcal{T}_1 \cap \mathcal{T}_2$ contradicting to the assumption that x, y can not be joined by a curve in $\mathcal{T}_1 \cap \mathcal{T}_2$. Let us take a point $z \in D \setminus (\mathcal{T}_1 \cup \mathcal{T}_2)$. Then z must lie in a tile $\mathcal{T}_3$ different from $\mathcal{T}_i$ $(i = 1, 2)$. Since z is an accumulation point of $\mathrm{Inn}(\mathcal{T}_3)$ we have $\mathrm{Inn}(\mathcal{T}_3) \subset D$ by Theorem 4.1. In other words, $\mathcal{T}_1$ and $\mathcal{T}_2$ surround $\mathcal{T}_3$. This gives the same contradiction as in Lemma 6.4. □

We say that a *boundary section* is a non-empty intersection of two tiles (note that Lemma 6.2 assures that any boundary section contains two points, as we assumed $A \neq 0$). Lemma 6.3, Lemma 6.4 and Lemma 6.5 imply that a vertex is an isolated point and a boundary section is a simple arc. Let us define 6 boundary sections for $A > 0$ by

$$\begin{aligned}
E_1 &= \mathcal{T} \cap (\mathcal{T} + \Phi(A + \alpha)), \\
E_2 &= \mathcal{T} \cap (\mathcal{T} + \Phi(A - 1 + \alpha)), \\
E_3 &= \mathcal{T} \cap (\mathcal{T} + \Phi(-1)), \\
E_4 &= \mathcal{T} \cap (\mathcal{T} + \Phi(-A - \alpha)), \\
E_5 &= \mathcal{T} \cap (\mathcal{T} + \Phi(-A + 1 - \alpha)), \\
E_6 &= \mathcal{T} \cap (\mathcal{T} + \Phi(1)),
\end{aligned}$$

and for $A=-1$ by

$$\begin{aligned}
E_1 &= \mathcal{T}\cap(\mathcal{T}+\Phi(\alpha))\,,\\
E_2 &= \mathcal{T}\cap(\mathcal{T}+\Phi(-1+\alpha))\,,\\
E_3 &= \mathcal{T}\cap(\mathcal{T}+\Phi(-1))\,,\\
E_4 &= \mathcal{T}\cap(\mathcal{T}+\Phi(-\alpha))\,,\\
E_5 &= \mathcal{T}\cap(\mathcal{T}+\Phi(1-\alpha))\,,\\
E_6 &= \mathcal{T}\cap(\mathcal{T}+\Phi(1))\,.
\end{aligned}$$

We will write $\overrightarrow{E_i}$ or $\overleftarrow{E_i}$ to clarify the orientation naturally inherited from the appropriate orientation of the simple closed curve $\partial(\mathcal{T})$. The symbol $\uplus$ will be used in the following way. $A_1\uplus A_2\uplus\cdots\uplus A_n$ is, as a set, $A_1\cup A_2\cup\ldots A_n$ with the condition that $A_i\cap A_{i+1}$ consists of one point for $i=1,\ldots,n-1$.

Theorem 6.6. *Let α be the base of a nontrivial CNS whose minimal polynomial x^2+Ax+B fulfills $2A<B+3$ and let $\mathcal{T}$ be its corresponding central tile. Then V_2 consists of exactly 6 points and the simple closed curve $\partial(\mathcal{T})$ is divided into 6 boundary sections E_i $(i=1,2,\ldots,6)$ whose end points are vertices. Each E_i is a simple arc which satisfies the following set equations: for $A>0$*

$$\xi_1(\overrightarrow{E_1}) = \overrightarrow{E_3}, \tag{6.2}$$

$$\begin{aligned}\xi_1(\overrightarrow{E_2}) = {}& \overrightarrow{E_4}\uplus\overrightarrow{E_5}\uplus(\overrightarrow{E_4}+1)\uplus(\overrightarrow{E_5}+1)\uplus\ldots\\ &\uplus(\overrightarrow{E_4}+A-2)\uplus(\overrightarrow{E_5}+A-2)\uplus(\overrightarrow{E_4}+A-1),\end{aligned} \tag{6.3}$$

$$\begin{aligned}\xi_1(\overrightarrow{E_3}) = {}& (\overrightarrow{E_5}+A-1)\uplus(\overrightarrow{E_4}+A)\uplus(\overrightarrow{E_5}+A)\uplus\ldots\\ &\uplus(\overrightarrow{E_4}+B-1)\uplus(\overrightarrow{E_5}+B-1),\end{aligned} \tag{6.4}$$

$$\overrightarrow{E_4} = \overleftarrow{E_1}-\Phi(A+\alpha), \tag{6.5}$$

$$\overrightarrow{E_5} = \overleftarrow{E_2}-\Phi(A-1+\alpha), \tag{6.6}$$

$$\overrightarrow{E_6} = \overleftarrow{E_3}+1, \tag{6.7}$$

and for $A=-1$,

$$\begin{aligned}
\xi_1(\overrightarrow{E_1}) &= \overrightarrow{E_2},\\
\xi_1(\overrightarrow{E_2}) &= \overrightarrow{E_3},\\
\xi_1(\overrightarrow{E_3}) &= \overrightarrow{E_4}\uplus\overrightarrow{E_5}\uplus(\overrightarrow{E_4}+1)\uplus(\overrightarrow{E_5}+1)\uplus\ldots\\
&\quad\uplus(\overrightarrow{E_4}+B-2)\uplus(\overrightarrow{E_5}+B-2)\uplus(\overrightarrow{E_4}+B-1),\\
\overrightarrow{E_4} &= \overleftarrow{E_1}-\Phi(\alpha),\\
\overrightarrow{E_5} &= \overleftarrow{E_2}-\Phi(-1+\alpha),\\
\overrightarrow{E_6} &= \overleftarrow{E_3}+1.
\end{aligned}$$

Each boundary section E_i $(i=1,2,\ldots,6)$ is symmetric with respect to the middle point of the line segment joining its end points. The boundary sections E_i $(i=1,2,\ldots,6)$ form a graph directed set in the sense of Falconer [8] *which is defined by the above set equation. Furthermore, the set of vertices V_2 can be calculated explicitly by the above set equations and clearly coincides with the ones obtained in*

AKIYAMA-THUSWALDNER [1]. *Namely, for $A > 0$,*

$$V_2 = \{\Phi(0.[0(A-1)(B-1)]^\infty), \Phi(0.[0(B-1)(B-A)]^\infty), \Phi(0.[(A-1)(B-1)0]^\infty), \Phi(0.[(B-1)(B-A)0]^\infty), \Phi(0.[(B-1)0(A-1)]^\infty), \Phi(0.[(B-A)0(B-1)]^\infty)\}, \tag{6.8}$$

and for $A = -1$

$$V_2 = \{\Phi(0.[000(B-1)(B-1)(B-1)]^\infty), \Phi(0.[00(B-1)(B-1)(B-1)0]^\infty), \Phi(0.[0(B-1)(B-1)(B-1)00]^\infty), \Phi(0.[(B-1)(B-1)(B-1)000]^\infty), \Phi(0.[(B-1)(B-1)00(B-1)]^\infty), \Phi(0.[(B-1)00(B-1)(B-1)]^\infty)\}.$$

Moreover, $V_L = \emptyset$ for $L \geq 3$.

Remark 6.7. *The order of the vertices in (6.8) is determined by the orientation of $\partial(\mathcal{T})$. Thus they coincide with $\{p_1, p_2, \dots, p_6\}$ which will appear in the later proof including its order.* [3]

Remark 6.8. *If we employ the above set equations to reconstruct E_i $(i = 1, \dots, 6)$, then the orientation is useless. But to recover the set of vertices and to construct a concrete homeomorphism from $\mathbb{R}/\mathbb{Z}$ to $\partial(\mathcal{T})$ it is important. In fact, starting from a directed hexagon whose vertices are the elements of V_2, successive use of the set equation of Theorem 6.6 will produce polygons approximating $\partial(\mathcal{T})$. Taking the limit we can construct a continuous map from $\mathbb{R}/\mathbb{Z}$ to $\partial(\mathcal{T})$, which is shown to be a bijection by Theorem 5.5. Note that the orientation is necessary in this procedure.*

Proof. We shall only prove the case $A > 0$. The proof for the case $A = -1$ runs along the same lines. By Lemma 6.2 and Lemma 6.5, E_i $(i = 1, 2, \dots, 6)$ are simple arcs. An end point of E_i must belong to a third tile and, hence, is a vertex. Consider two end points p_1, p_2 of E_1. By Lemma 6.2 and Lemma 6.4, we have

$$\{p_1\} = \mathcal{T} \cap (\mathcal{T} + \Phi(A + \alpha)) \cap (\mathcal{T} + 1)$$

and

$$\{p_2\} = \mathcal{T} \cap (\mathcal{T} + \Phi(A - 1 + \alpha)) \cap (\mathcal{T} + \Phi(A + \alpha)).$$

We fix the orientation of $\partial(\mathcal{T})$ by assigning to it the orientation of the boundary section E_1. So let p_1 be starting point of the boundary section E_1. Thus by $\overrightarrow{E_1}$ we denote the simple arc E_1 from p_1 to p_2 and by $\overleftarrow{E_1}$ the reversed simple arc. Then we can naturally define p_3, p_4, p_5, p_6 inductively by the vertices on $\partial(\mathcal{T})$ ordered according to this orientation. In other words, we have

$$\begin{aligned}
\{p_3\} &= \mathcal{T} \cap (\mathcal{T} - 1) \cap (\mathcal{T} + \Phi(A - 1 + \alpha)), \\
\{p_4\} &= \mathcal{T} \cap (\mathcal{T} + \Phi(-A - \alpha) \cap (\mathcal{T} - 1)), \\
\{p_5\} &= \mathcal{T} \cap (\mathcal{T} - \Phi(-A + 1 - \alpha)) \cap (\mathcal{T} + \Phi(-A - \alpha)), \\
\{p_6\} &= \mathcal{T} \cap (\mathcal{T} + 1)) \cap (\mathcal{T} + \Phi(-A + 1 - \alpha))
\end{aligned}$$

and letting $p_7 = p_1$, p_i, p_{i+1} are the end points of E_i for $i = 1, \dots 6$. The three relations (6.5), (6.6) and (6.7) are shown easily under this orientation. In fact, for instance, $E_4 = E_1 - \Phi(A + \alpha)$, $p_1 - \Phi(A + \alpha) = p_5$ and $p_2 - \Phi(A + \alpha) = p_4$ is clear by definition, proving (6.5). Since $\alpha(A + \alpha) = -B$, with the help of Lemma 6.2 we see that

$$\begin{aligned}
\xi_1(E_1) &= \xi_1(\mathcal{T}) \cap (\xi_1(\mathcal{T}) - B) \\
&= \left(\bigcup_{i=0}^{B-1} \mathcal{T} + i \right) \cap \left(\bigcup_{i=0}^{B-1} \mathcal{T} + i - B \right) \\
&= \mathcal{T} \cap (\mathcal{T} - 1) = E_3,
\end{aligned}$$

[3]A different order is used in [1]. And there is an error that the values of P_5 and P_6 is to be exchanged in the statement of Theorem 2.1 in [1], obviously seen from its proof.

which proves (6.2). A similar calculation shows

$$\xi_1(E_2) \quad = \quad \left(\bigcup_{i=0}^{B-1} \mathcal{T}+i\right) \cap \left(\bigcup_{i=0}^{B-1} \mathcal{T}+\Phi(i-B-\alpha)\right).$$

Considering the surrounding tiles of $\mathcal{T}+i$ $(i=0,\dots,B-1)$, by Lemma 6.2 we observe that

$$\begin{aligned}\xi_1(E_2) \quad &= \quad (\mathcal{T}\cap(\mathcal{T}+\Phi(-A-\alpha))) \uplus (\mathcal{T}\cap(\mathcal{T}+\Phi(-A+1-\alpha))) \uplus \dots\\ &\quad \uplus ((\mathcal{T}+1)\cap(\mathcal{T}+\Phi(-A+1-\alpha)))\\ &\quad \uplus ((\mathcal{T}+A-2)\cap(\mathcal{T}+\Phi(-1-\alpha)))\\ &\quad \uplus ((\mathcal{T}+A-1)\cap(\mathcal{T}+\Phi(-1-\alpha)))\end{aligned}$$

and, hence,

$$\begin{aligned}\xi_1(\overrightarrow{E_2}) \quad &= \quad \overrightarrow{E_4} \uplus \overrightarrow{E_5} \uplus(\overrightarrow{E_4}+1) \uplus (\overrightarrow{E_5}+1) \uplus \dots\\ &\quad \uplus(\overrightarrow{E_4}+A-2) \uplus (\overrightarrow{E_5}+A-2) \uplus (\overrightarrow{E_4}+A-1),\end{aligned}$$

which proves (6.3). Note that here we used $A>0$ to deduce possible intersections. Furthermore, note that ξ_1 preserves the orientation of $\partial(\mathcal{T})$ since the matrix representation of ξ_1 has positive determinant. In the same manner we derive

$$\begin{aligned}\xi_1(E_3) \quad &= \quad \left(\bigcup_{i=0}^{B-1} \mathcal{T}+i\right) \cap \left(\bigcup_{i=0}^{B-1} \mathcal{T}+\Phi(i-\alpha)\right)\\ &= \quad ((\mathcal{T}+A-1)\cap(\mathcal{T}+\Phi(-\alpha))) \uplus ((\mathcal{T}+A)\cap(\mathcal{T}+\Phi(-\alpha))) \uplus \dots\\ &\quad \uplus ((\mathcal{T}+B-2)\cap(\mathcal{T}+\Phi(B-A+1-\alpha)))\\ &\quad \uplus ((\mathcal{T}+B-1)\cap(\mathcal{T}+\Phi(B-A-1-\alpha)))\\ &\quad \uplus ((\mathcal{T}+B-1)\cap(\mathcal{T}+\Phi(B-A-\alpha))),\\ \xi_1(\overrightarrow{E_3}) \quad &= \quad (\overrightarrow{E_5}+A-1) \uplus (\overrightarrow{E_4}+A) \uplus (\overrightarrow{E_5}+A) \uplus \dots\\ &\quad \uplus(\overrightarrow{E_4}+B-1) \uplus (\overrightarrow{E_5}+B-1),\end{aligned}$$

which shows (6.4). By Lemma 3.2, the involution map $\iota : x \to \Phi(\frac{B-1}{\alpha-1}) - x$ has the property

$$\iota(\mathcal{T}+\Phi(x)) = \mathcal{T}-\Phi(x)$$

for any x. Thus we have $\iota(\overrightarrow{E_i}) = \overrightarrow{E_{i+3}}$ for $i=1,2,3$. Together with (6.5),(6.6) and (6.7) we see that each arc E_i $(i=1,\dots,6)$ is symmetric with respect to the middle point of the segment joining its end points. In fact, for instance,

$$\Phi\left(\frac{B-1}{\alpha-1}+A+\alpha\right) - \overrightarrow{E_1} = \overleftarrow{E_1}$$

shows the required symmetry. Putting (6.5),(6.6) and (6.7) into (6.2),(6.3) and (6.4), applying ξ_{-1}, we get a set equation of E_i $(i=1,2,3)$ by contracting maps. Roughly speaking, to define each E_i $(i=1,2,3)$ we need all E_j $(j=1,2,3)$. Thus these set equations give a graph directed set in the sense of Falconer [8, Chapter 3]. The exact values of p_i $(i=1,\dots,6)$ are determined by the set equations with the orientation (c.f. Remark 6.8). Indeed, as p_1 is the starting point of E_1, using (6.2) and (6.4) yields

$$\begin{aligned}\xi_1(p_1) \quad &= \quad p_3,\\ \xi_1(p_3) \quad &= \quad p_5+\Phi(A-1),\\ p_5+\Phi(A-1+\alpha) \quad &= \quad p_3.\end{aligned}$$

Our aim is to find an exact value $y_i \in \mathbb{Q}(\alpha)$ with $\Phi(y_i) = p_i$ for $i = 1, \dots, 6$. Thus we have

$$\begin{aligned} \alpha y_1 &= y_3, \\ \alpha y_3 &= y_5 + A - 1, \\ y_5 + A - 1 + \alpha &= y_3, \end{aligned}$$

which is uniquely solvable in y_1, y_3 and y_5:

$$y_1 = -\frac{1}{\alpha - 1}, \quad y_3 = -\frac{\alpha}{\alpha - 1}, \quad y_5 = \frac{\alpha + A + B - 1}{\alpha - 1}.$$

Similarly from

$$\begin{aligned} \alpha y_2 &= y_4, \\ \alpha y_4 &= y_6 + B - 1, \\ y_6 &= y_4 + 1, \end{aligned}$$

we have

$$y_2 = \frac{B}{\alpha^2 - \alpha}, \quad y_4 = \frac{B}{\alpha - 1}, \quad y_6 = \frac{\alpha + B - 1}{\alpha - 1}.$$

It is an easy task to see that these values coincide with (6.8) including its order (cf. Remark 6.7). □

7. On the disconnectedness of the interior of a tile $\mathcal{T}$ for $2A \geq B + 3$

So far we treated the case $2A < B+3$ and showed somewhat 'ordinary' properties of the tile $\mathcal{T}$. If $2A \geq B + 3$, we have a big difference. The aim of this section is to shed some light on this difference. In particular, we will show that $\mathrm{Inn}(\mathcal{T})$ is no longer connected if $2A \geq B + 3$. To this matter we will use some results from the automata theoretic approach which will be extensively developed in later sections. [4]

First of all we want to give some definitions which will be needed in this section. Let $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$ be *broken lines*, i.e. directed curves consisting of a finite number of consecutive finite line segments, in the Riemann sphere $(\mathbb{R}^2)^*$. If the end point of $\overrightarrow{F_1}$ and the starting point of $\overrightarrow{F_2}$ coincide, we can compose them into a broken line $\overrightarrow{F_1} + \overrightarrow{F_2}$. Take a point p near to the end point of $\overrightarrow{F_1}$ and a point q near to the starting point of $\overrightarrow{F_2}$. Then we can consider a deformation of the broken line $\overrightarrow{F_1} + \overrightarrow{F_2}$. This deformation starts at the starting point of $\overrightarrow{F_1}$ and goes through $\overrightarrow{F_1}$ till p, then goes along the segment $\overrightarrow{pq}$ and finally through $\overrightarrow{F_2}$ to the end point of $\overrightarrow{F_2}$. The operation of replacing $\overrightarrow{F_1} + \overrightarrow{F_2}$ by this deformation is called a *short cut deformation*. We will need a slight generalization of the notion of broken line. Let C be a curve each of whose sub curves which does not contain the endpoints of C is a broken line. Such a curve will be called *infinite broken line*. In other words, an infinite broken line is a simple arc consisting of successive infinitely many consecutive line segments whose only accumulation points of end points of them, are the end points of the curve.

Let $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$ be infinite broken lines without self intersections consisting of the line segments $L_1^{(1)}, L_1^{(2)}, \dots$ and $L_2^{(1)}, L_2^{(2)}, \dots$, respectively. Then an intersection point y of $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$ is called a *normal crossing* if there exist $j_1, j_2 \in \mathbb{N}$ such that y is an element of $L_1^{(j_1)}$ and $L_2^{(j_2)}$ but is not contained in any other line segment $L_i^{(j)}$.

[4] Despite we have to use a result from later sections we decided to treat the disconnectedness for the case $2A \geq B + 3$ already here. In our opinion it fits very well to the results of the previous sections.

Theorem 7.1. *Let α be the base of a CNS with minimal polynomial x^2+Ax+B and let $\mathcal{T}$ be the corresponding central tile. If $2A \geq B+3$ then the interior of $\mathcal{T}$ is disconnected. It even has infinitely many components.*

Proof. **Case (i): $\mathbb{R}^2 \setminus \mathcal{T}$ is connected.** Since $\mathbb{R}^2 \setminus \mathcal{T}$ is open, it is also arcwise connected. In [1, Theorem 2.2 and 2.3], it is shown that $\mathcal{T}$ contains infinitely many vertices. More concretely it will be shown later in Lemma 10.6 that, in the notation of this lemma, $\mathcal{T} \cap (\mathcal{T}+\Phi(-P_1)) \cap (\mathcal{T}+\Phi(Q_1))$ contains infinitely many elements where $X^{(0)} := -P_1 = -1$ and $X^{(1)} := Q_1 = -1+A+\alpha$. [5] As in the previous sections, we may assume that $I(\alpha)>0$. Using (2.3), let

$$V^{(0)} := \xi_2(\mathcal{T}) - (B-1)\Phi(1+\alpha) = \bigcup_{i,j\in\mathcal{N}} (\mathcal{T} - \Phi(i+j\alpha))$$

and

$$V^{(1)} := \xi_2(\mathcal{T}) = \bigcup_{i,j\in\mathcal{N}} (\mathcal{T} + \Phi(i+j\alpha)).$$

Moreover, define

$$\begin{aligned} A^{(0)} &:= \mathbb{R}^2 \setminus V^{(1)} \quad \text{and} \\ A^{(1)} &:= \mathbb{R}^2 \setminus V^{(0)}. \end{aligned}$$

By the assumption of this case (i), $A^{(j)}$ is connected ($j \in \{0,1\}$). We also have

$$A^{(j)} \supset \mathrm{Inn}(\mathcal{T}+\Phi(X^{(j)})) \quad \text{and} \quad A^{(j)} \cap \Big(\mathcal{T} \cup (\mathcal{T}+\Phi(X^{(1-j)}))\Big) = \emptyset \quad (j \in \{0,1\}).$$

Let N be an arbitrary positive integer. Now fix N vertices $x_1,\dots,x_N$ of $\mathcal{T}$ contained in $\mathcal{T} \cap (\mathcal{T}+\Phi(X^{(0)})) \cap (\mathcal{T}+\Phi(X^{(1)}))$. Let $P^{(j)} \in \mathrm{Inn}(\mathcal{T}+\Phi(X^{(j)}))$ ($j \in \{0,1\}$) and take a positive ε such that $B(P^{(j)},\varepsilon) \subset \mathrm{Inn}(\mathcal{T}+\Phi(X^{(j)}))$. For example we may choose $P^{(j)} = \Phi(X^{(j)})$ by Theorem 2.1 (iii).

For $i \in \{1,\dots,N\}$ and $j \in \{0,1\}$ we wish to construct simple arcs $C_i^{(j)}$ joining x_i and $P^{(j)}$ in such a way that

$$C_i^{(j)} \setminus \{P^{(j)}\} \subset A^{(j)} \setminus (C_1^{(j)} \cup \ldots \cup C_{i-1}^{(j)}).$$

The construction of $C_i^{(j)}$ is done in the following way. Note that x_i is accessible from $A^{(j)}$ in view of Kuratowski [29, Chapter 10, Section 61 II, Theorem 11] since $\xi_2(\mathcal{T})$ is closed and locally connected. First let us take a simple arc $K_1^{(j)}$ which joins x_1 and $P^{(j)}$. Take the first intersection point $P_1^{(j)}$ of $K_1^{(j)} \cap \partial(B(P^{(j)},\varepsilon))$ and denote the sub arc of $K_1^{(j)}$ joining x_1 and $P_1^{(j)}$ by the same symbol $K_1^{(j)}$. After that we proceed inductively. Assume that $K_i^{(j)}$ ($i=1,2,\dots,r$; $j=0,1$) are curves joining x_i and $P_i^{(j)} \in \partial(B(P^{(j)},\varepsilon))$ which satisfy

$$K_i^{(j)} \subset A^{(j)} \setminus (K_1^{(j)} \cup \ldots \cup K_{i-1}^{(j)}) \quad \text{and} \quad K_i^{(j)} \cap \partial(B(P^{(j)},\varepsilon)) = \{P_i^{(j)}\}.$$

As $V^{(0)} \cup (\bigcup_{i=1}^r K_i^{(1)})$ and $V^{(1)} \cup (\bigcup_{i=1}^r K_i^{(0)})$ are again locally connected and closed, and the residual domain of them is also connected, we can find simple arcs $K_{r+1}^{(j)}$ and $P_{r+1}^{(j)} \in \partial(B(P^{(j)},\varepsilon))$ ($j \in \{0,1\}$) in a similar manner. As each $P_i^{(0)} (i=1,\dots,N)$ is contained in $\partial(B(P^{(0)},\varepsilon))$ and $B(P^{(0)},\varepsilon)$ does not contain points of $C_i^{(1)} \subset A^{(1)}$ by this construction, $P_i^{(0)}$ and $P^{(0)}$ can be joined by line segments to get desired curves $C_i^{(0)} (i=1,\dots,N)$. Proceeding in the same way also yields the curves $C_i^{(1)} (i=1,\dots,N)$.

Now set

$$E_i := -C_i^{(1)} \cup C_i^{(0)} \quad (i \in \{1,\dots,N\}).$$

[5] Note that the notation $-P_1$, Q_1 used in Lemma 10.6 is not convenient in the present proof because it would cause more tedious formulas.

Since $A^{(0)} \cap A^{(1)} \neq \emptyset$, each of the E_i may have self intersections, i.e., may be a non simple curve. From these E_i by some deformations, we wish to construct simple arcs $C_i \subset A^{(0)} \cup A^{(1)}$ $(i = 1, \dots, N)$ passing through x_i and joining $P^{(0)}$ and $P^{(1)}$ which are pairwise disjoint apart from their end points. $C_i^{(j)}$ is a simple arc from a point in $A^{(j)}$ to x_i. Seeing the argument of NEWMAN [32, Chapter 6, Section 14, Theorem 14.4], we may assume that each $C_i^{(j)}$ is an infinite broken line whose closed sub arcs contained in $A^{(j)}$ are broken lines, since $A^{(j)} \cup \{x_i\}$ is locally connected by KURATOWSKI [29, Chapter 10, Section 61 II, Theorem 11]. Moreover by using appropriate short cut deformations we may assume that each self intersection of E_i is a normal crossing. Assume that E_i has $m \in \mathbb{Z} \cup \{\infty\}$ such self intersections. If $m = 0$ then let $C_i = E_i$. If $m \geq 1$, then let $p_1, p_2, \dots$ be the self intersection points. The curves C_i^- and C_i^+ are subdivided into broken lines at p_i $(i = 1, 2, \dots)$, say

$$\begin{aligned} C_i^{(0)} &= \overrightarrow{f_1} + \overrightarrow{f_2} + \overrightarrow{f_3} + \dots \\ C_i^{(1)} &= \overrightarrow{g_1} + \overrightarrow{g_2} + \overrightarrow{g_3} + \dots . \end{aligned}$$

Then we consider the 'switched' infinite broken lines

$$D_i^{(0)} := \overrightarrow{f_1} + \overrightarrow{g_2} + \overrightarrow{f_3} + \overrightarrow{g_4} + \dots$$

and

$$D_i^{(1)} := \overrightarrow{g_1} + \overrightarrow{f_2} + \overrightarrow{g_3} + \overrightarrow{f_4} + \dots$$

and perform short cut deformations around p_i $(i = 1, 2, \dots)$ on $D_i^{(j)}$ to get infinite broken lines $R_i^{(0)}$ and $R_i^{(1)}$. It is easy to see that $C_i = -R_i^- + R_i^+$ gives a simple closed curve having the desired property.

Let $\mathcal{C}$ be the collection of all C_i $(1 \leq i \leq N)$. Note that C_i does not contain points of $\mathrm{Inn}(\mathcal{T})$ by definition. It is clearly seen that $\mathcal{C}$ divides $(\mathbb{R}^2)^*$ into a union of N distinct domains homeomorphic to a unit disk, by successive use of KURATOWSKI [29, Chapter 10, Section 61 II, Theorem 2]. Renumbering indices and considering indices mod N, we may assume that each C_i $(i = 1, \dots, N)$ is contained in the domain surrounded by $-C_{i-1} \cup C_{i+1}$. As x_i is the only point of $\mathcal{T}$ on the curve C_i and $\mathcal{T} = \overline{\mathrm{Inn}(\mathcal{T})}$, at least $\lceil N/2 \rceil$ domains in $(\mathbb{R}^2)^*$ contain an inner point of $\mathcal{T}$. Here $\lceil u \rceil$ is the minimum integer not less than u. As $C_i \subset A^{(0)} \cup A^{(1)}$ $(i = 1, \dots, N)$ does not contain points of $\mathrm{Inn}(\mathcal{T})$, the interior of $\mathcal{T}$ has at least $\lceil N/2 \rceil$ components. Since N was arbitrary, the result follows.

Case (ii): $\mathbb{R}^2 \setminus \mathcal{T}$ is not connected. Since $\mathcal{T}$ is compact this implies that $\mathbb{R}^2 \setminus \mathcal{T}$ has at least one bounded component T_0. As $\mathbb{R}^2 \setminus \mathcal{T}$ is locally connected since it is open, every component of $\mathbb{R}^2 \setminus \mathcal{T}$ must be open in $\mathbb{R}^2 \setminus \mathcal{T}$. Furthermore, this component is also open in $\mathbb{R}^2$. Thus T_0 contains points which lie in the interior of a translate $\mathcal{T} + \Phi(v)$ for a certain $v \in \mathbb{Z}[\alpha]$. Of course, each of the sets

$$\mathbb{R}^2 \setminus \xi_{-1}(\mathcal{T} + \Phi(i)) \quad (0 \leq i \leq B - 1)$$

contains a bounded component $\xi_{-1}(T_0 + \Phi(i))$, which contains inner points of $\xi_{-1}(\mathcal{T} + \Phi(i) + \Phi(v))$. Since $v \neq 0$, by the inflation subdivision principle (2.3) there exists an $i_0 \in \{0, \dots, B-1\}$ for which

$$\xi_{-1}(\mathcal{T} + \Phi(i_0) + \Phi(v)) \cap \mathrm{Inn}(\mathcal{T}) = \emptyset.$$

Set $T_1 := \xi_{-1}(T_0 + \Phi(i_0))$. Again by (2.3) we conclude that T_1 is a component of $\mathbb{R}^2 \setminus \mathcal{T}$. Iterating this construction we can construct countably many components T_i $(i \geq 0)$ of $\mathbb{R}^2 \setminus \mathcal{T}$ which are all contained in a certain fixed disc $\{z \in \mathbb{R}^2 \mid |z| < c\}$ around the origin. By the compactness of $\mathcal{T}$ only a finite set F of translates of $\mathcal{T}$ has nonempty intersection with this circle. Thus each of these components contains points of the interior of one of the finitely many translates of $\mathcal{T}$ contained in F. It is clear that T_i is surrounded by $\partial(\mathcal{T})$, i.e., $\partial(T_i) \subset \partial(\mathcal{T})$ since we have shown that every component of $\mathbb{R}^2 \setminus \mathcal{T}$ is open in $\mathbb{R}^2$. Thus there exists a translate of $\mathcal{T}$ in F

whose interior contains infinitely many components separated by $\partial(\mathcal{T})$. Since our tiling is periodic, this holds also for $\mathcal{T}$. □

Remark 7.2. *(due to R. Okazaki) As each Jordan component contains inner points and, hence, rational points, the cardinality of connected components of* $\mathrm{Inn}(\mathcal{T})$ *is at most countable. Thus we can not produce "uncountably" many simple curves* C_i *by extending the first part of the above proof.*

8. Definition of the graphs which describe the vertices of a tile

Let $x^2 + Ax + B$ be the minimal polynomial of α which forms a CNS. From now on, the constant

$$C := 2A - B \tag{8.1}$$

will play an important rôle. In the sequel, we shall construct automata which recognize multiple points of our tiling.

We have already seen in the first part of Section 6 that for $A = 0$ the set $\mathcal{T}$ is a rectangle. Since this case is trivial we want to exclude it in the remaining part of the paper. As before we call a canonical number system *nontrivial* if the minimal polynomial of its base α has $A \neq 0$.

Let $\mathcal{G}_1(\mathbb{Z}[\alpha])$ be a labelled directed graph with set of states [6] $\mathbb{Z}[\alpha]$ and set of labels $\mathcal{N} \times \mathcal{N}$ whose elements are written as $a|a'$. The labelled edges connecting two states s_1 and s_2 are defined by

$$s_1 \xrightarrow{a|a'} s_2 \quad \text{if and only if} \quad \alpha s_2 - s_1 = a - a' \quad (s_1, s_2 \in \mathbb{Z}[\alpha], a, a' \in \mathcal{N}). \tag{8.2}$$

Since a' is uniquely determined by s_1, s_2 and a we sometimes omit it and write just $s_1 \xrightarrow{a} s_2$ for the edges in $\mathcal{G}_1(\mathbb{Z}[\alpha])$ and its subgraphs (i. e. we will identify the set of labels with $\mathcal{N}$). Note that the graph $\mathcal{G}_1(\mathbb{Z}[\alpha])$ and its subgraphs can also be regarded as so-called *transducer* automata (cf. for instance Berstel [6] for the definition of a transducer automaton; see also Scheicher-Thuswaldner [38], where the relations between subgraphs of $\mathcal{G}_1(\mathbb{Z}[\alpha])$ and transducer automata are discussed in detail).

Remark 8.1. *We avoid the matrix representation which is occasionally used to construct number systems, tilings, and graphs (see* [19], [31]*). Since we emphasize on quadratic number systems and the idea of the "generalized imaginary part" was important in previous sections, using matrices seems not so illuminating to illustrate the geometric nature of our tile* $\mathcal{T}$*.*

Definition 8.2. *For a graph* G *we denote by* $\mathrm{Red}(G)$ *the graph that emerges from* G *if all states of* G *which are not the endpoint of a walk of infinite length are removed. (i.e. one deletes successively all states which have no predecessor and the edges leading to it).*

Definition 8.3. *Let* G *be a subgraph of* $\mathcal{G}_1(\mathbb{Z}[\alpha])$*. We need two notions of "L-fold power of* G*". First define the graph* $G^{(L)}$ *in the following way.*

- *The states of* $G^{(L)}$ *are the sets* $\{s_1, \dots, s_L\}$ *consisting of pairwise different states* s_ℓ *of* G*.*
- *There exists an edge*

$$\{s_{11}, \dots, s_{1L}\} \xrightarrow{a} \{s_{21}, \dots, s_{2L}\}$$

in $G^{(L)}$ *if after a possible rearrangement of* $s_{21}, \dots, s_{2L}$ *there exist the edges*

$$s_{1\ell} \xrightarrow{a|a_\ell} s_{2\ell} \quad (1 \leq \ell \leq L)$$

in G *for certain* $a_1, \dots a_L \in \mathcal{N}$*.*

[6] Note that we adopt the notation "state" instead of the more common notation "vertex" in order to avoid confusions with the "vertex" of a tile.

Furthermore, let $G^L := \mathrm{Red}(G^{(L)})$.

Let $\mathcal{G}_1(S)$ be the restriction of $\mathcal{G}_1(\mathbb{Z}[\alpha])$ to S and consider the L-fold products

$$\begin{aligned} \mathcal{G}'_L(S) &:= \mathcal{G}_1(S)^{(L)}, \\ \mathcal{G}_L(S) &:= \mathcal{G}_1(S)^L. \end{aligned}$$

Before we show how $\mathcal{G}_L(S)$ corresponds to the sets $V_L(s_1,\dots,s_L)$ we recall a result on $\mathcal{G}_1(S)$, which has been shown in Müller et al. [31] (note that they employed the matrix representation of CNS in this paper). Similar results can be seen for instance in Indlekofer et al. [17] and Kátai [19].

Lemma 8.4. *Each infinite walk* $s_1 \xleftarrow{a_1} s_2 \xleftarrow{a_2} \cdots$ *in* $\mathcal{G}_1(S)$ *yields a point*

$$x = \sum_{j\geq 1} a_j \Phi(\alpha^{-j}) \in V_1(s_1)$$

and each point $x \in V_1(s_1)$ *can be constructed in this way.*

This lemma admits the following generalization.

Proposition 8.5. *Let* $L \geq 1$ *and let* $s_{01},\dots,s_{0L} \in S$ *be pairwise different. Then the following three assertions are equivalent.*

(i)
$$x = \sum_{j\geq 1} a_j \Phi(\alpha^{-j}) \in V_L(s_{01},\dots,s_{0L}).$$

(ii) *There exists an infinite walk of the shape*

$$\{s_{01},\dots,s_{0L}\} \xleftarrow{a_1} \{s_{11},\dots,s_{1L}\} \xleftarrow{a_2} \{s_{21},\dots,s_{2L}\} \xleftarrow{a_3} \cdots \tag{8.3}$$

in $\mathcal{G}_L(S)$.

(iii) *There exist the* L *infinite walks*

$$s_{0\ell} \xleftarrow{a_1} s_{1\ell} \xleftarrow{a_2} s_{2\ell} \xleftarrow{a_3} \cdots \quad (1 \leq \ell \leq L) \tag{8.4}$$

in $\mathcal{G}_1(S)$.

Proof. **(ii) ⇒ (iii):** Let (8.3) be a walk in $\mathcal{G}_L(S)$. Then it is also a walk in $\mathcal{G}'_L(S)$. By the definition of $\mathcal{G}'_L(S)$ this walk implies the existence of the L walks (8.4) in $\mathcal{G}_1(S)$ (note that the elements in the sets $\{s_{j1},\dots,s_{jL}\}$ may need some rearrangement).

(iii) ⇒ (ii): Suppose that the walks (8.4) exist in $\mathcal{G}_1(S)$. Note, that $s_{0\ell}$ $(1 \leq \ell \leq L)$ are pairwise different. By the definition of the edges in $\mathcal{G}_1(S)$ this implies that $s_{1\ell}$ $(1 \leq \ell \leq L)$ are pairwise different. By induction we derive that $s_{k\ell}$ $(1 \leq \ell \leq L)$ are pairwise different for each fixed k. Thus $\{s_{k1},\dots s_{kL}\}$ are states of $\mathcal{G}'_L(S)$. The walk (8.3) now exists in $\mathcal{G}'_L(S)$ by the definition of its edges in Definition 8.3. Since each state of (8.3) has infinitely many predecessors it is also a walk of $\mathcal{G}_L(S)$.

(i) ⇒ (iii): For $k \in \mathbb{N}$ suppose that

$$x_k = \sum_{j\geq 1} a_{j+k} \Phi(\alpha^{-j}) \in V_L(s_{k1},\dots,s_{kL}).$$

Then x_k admits the representations

$$x_k = \Phi(s_{k1}) + \sum_{j\geq 1} a_{k1j}\Phi(\alpha^{-j}) = \ldots = \Phi(s_{kL}) + \sum_{j\geq 1} a_{kLj}\Phi(\alpha^{-j})$$

for certain $s_{k\ell} \in S$ and $a_{k\ell j} \in \mathcal{N}$ $(1 \leq \ell \leq L,\ j \geq 1)$. This implies that

$$x_{k+1} = \xi_1(x_k) - \Phi(a_{k+1}) = \Phi(s_{k+1,\ell}) + \sum_{j\geq 1} a_{k+1,\ell j}\Phi(\alpha^{-j})$$

with

$$\alpha s_{k\ell} - s_{k+1,\ell} = a_{k+1} - a_{k1\ell} \quad (1 \leq \ell \leq L). \tag{8.5}$$

Furthermore, $x_{k+1} \in V_L(s_{k+1,1}, \dots, s_{k+1,L})$. (8.5) ensures that $\mathcal{G}_1(S)$ contains the edges

$$s_{k\ell} \xleftarrow{a_{k+1}} s_{k+1,\ell} \quad (1 \le \ell \le L).$$

Since k was arbitrary, and $x_0 = x \in V_L(s_{01}, \dots, s_{0L})$ we derive by induction that there exist the walks

$$s_{0\ell} \xleftarrow{a_1} s_{1\ell} \xleftarrow{a_2} s_{2\ell} \xleftarrow{a_3} \dots \quad (1 \le \ell \le L)$$

in $\mathcal{G}_1(S)$.

(iii) $\Rightarrow$ (i): Now suppose that the walks (8.4) exist in $\mathcal{G}_1(S)$. By Lemma 8.4 this yields $x \in V_1(s_{0\ell})$ for $1 \le \ell \le L$. Thus $x \in V_L(s_{01}, \dots, s_{0L})$. $\square$

The preceding results reduce the problem of determining the set S and the sets V_L to the construction of the graphs $\mathcal{G}_L(S)$. Since $\mathcal{G}_L(S)$ for $L \ge 2$ can be constructed easily from $\mathcal{G}_1(S)$ we want to find a way to construct $\mathcal{G}_1(S)$. For the construction of this graph there exist different algorithms. One algorithm starts with a large graph containing $\mathcal{G}_1(S)$. The construction of $\mathcal{G}_1(S)$ is then achieved by successively deleting appropriate states and edges (cf. Indlekofer et al. [17] and Strichartz-Wang [39]). This algorithm is well-suited for the determination of $\mathcal{G}_1(S)$ for a given base α. However, determining $\mathcal{G}_1(S)$ for a class of bases with help of this algorithm seems to be difficult, because the graph from which the algorithm starts becomes very large if one of the conjugates of α is close to one. Thus we will use another algorithm which has been established recently in Scheicher-Thuswaldner [38]. It starts from a so-called counting automaton $A_0(1)$, which is a transducer automaton that performs the addition of 1 on the α-adic expansions. The construction of $\mathcal{G}_1(S)$ is achieved by considering certain products of $A_0(1)$. Before we give a description of this algorithm we dwell upon this automaton. For the bases α considered in this paper the counting automaton $A_0(1)$ is well-known (cf. for instance Scheicher [37] and Thuswaldner [42]). For the bases α having $A > 0$ we depict it in Figure 2 (for $A = -1$ it is of a similar shape, cf. Thuswaldner [43]). Its states are defined by

$$\pm P_1 := \pm 1, \qquad \pm Q_1 := \pm(-1 + A + \alpha), \qquad \pm R := \pm(-A - \alpha). \tag{8.6}$$

"$\bullet$" denotes the "accepting state" 0. Furthermore, there exists a label from a state s_1 to a state s_2 in $A_0(1)$ labelled by $a|a'$ if and only if

$$s_1 + a = \alpha s_2 + a'.$$

Thus $A_0(1)$ (regarded as a graph) is a subgraph of $\mathcal{G}(\mathbb{Z}[\alpha])$. Moving along the edge $s_1 \xrightarrow{a|a'} s_2$ the automaton $A_0(1)$ reads the digit a and writes out the digit a'. Suppose, the automaton reads the α-adic digit string of $v \in \mathbb{Z}[\alpha]$ from right to left starting at P_1. Then, moving along the according edges, it writes out the α-adic digit string of $v + P_1$.

In order to present the desired algorithm for the construction of $\mathcal{G}_1(S)$ we need the following notion of product of two subgraphs of $\mathcal{G}_1(\mathbb{Z}[\alpha])$ (cf. Scheicher-Thuswaldner [38]).

Definition 8.6. *Let G_1 and G_2 be subgraphs of $\mathcal{G}_1(\mathbb{Z}[\alpha])$. Then the product graph $G := G_1 \otimes G_2$ is defined in the following way. Let s_{11}, s_{12} be states of G_1 and s_{21}, s_{22} be states of G_2. Furthermore, let $a_1, a_1', a_2 \in \mathcal{N}$.*

- *s_1 is a state of G if $s_1 = s_{11} + s_{21}$*
- *There exists an edge $s_1 \xrightarrow{a_1|a_2} s_2$ in G if there exist the edges*

$$s_{11} \xrightarrow{a_1|a_1'} s_{12} \in G_1 \quad \textit{and} \quad s_{21} \xrightarrow{a_1'|a_2} s_{22} \in G_2 \tag{8.7}$$

with $s_{11} + s_{21} = s_1$ and $s_{12} + s_{22} = s_2$ or there exist

$$s_{11} \xrightarrow{a_1'|a_2} s_{12} \in G_1 \quad \textit{and} \quad s_{21} \xrightarrow{a_1|a_1'} s_{22} \in G_2 \tag{8.8}$$

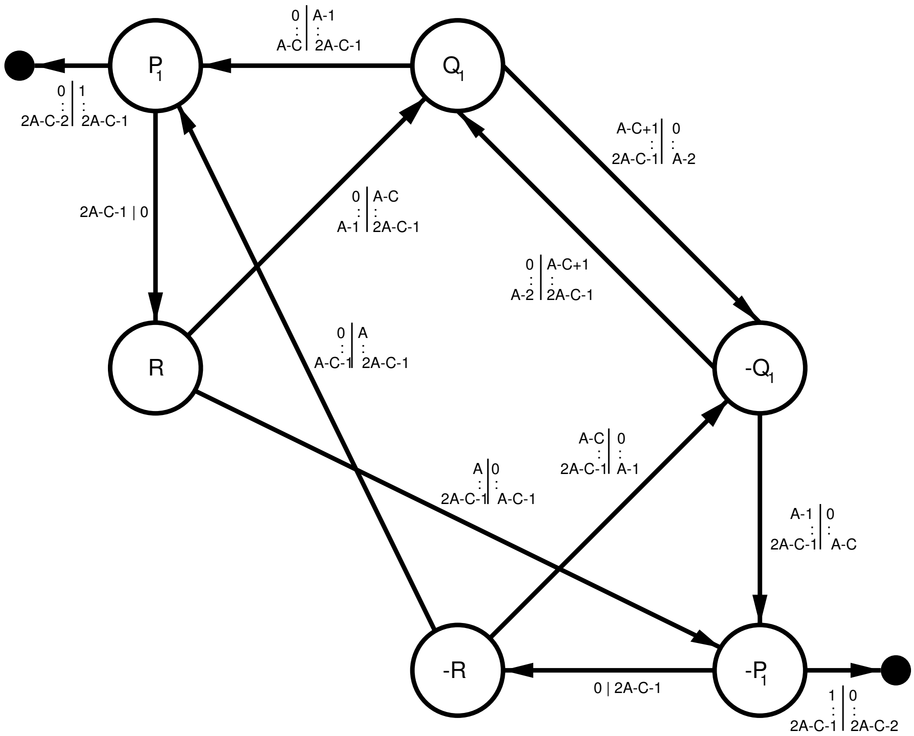

FIGURE 2. The counting automaton $A_0(1)$

with $s_{11} + s_{21} = s_1$ *and* $s_{12} + s_{22} = s_2$.

Definition 8.7. *We say that a subgraph G of $\mathcal{G}_1(\mathbb{Z}[\alpha])$ has property (C) if*

(C) *for each state s_1 and each $a \in \mathcal{N}$ there exists a unique state s_2 and a unique $a' \in \mathcal{N}$ such that $s_1 \xrightarrow{a|a'} s_2$ is an edge of G.*

Lemma 8.8. *Let G_1 and G_2 be subgraphs of $\mathcal{G}_1(\mathbb{Z}[\alpha])$ having property (C). Then there exists an edge between two states s_1 and s_2 of $G := G_1 \otimes G_2$ labelled by $a_1|a_2$, i.e. $s_1 \xrightarrow{a_1|a_2} s_2$, if and only if*

$$\alpha s_2 - s_1 = a_1 - a_2. \tag{8.9}$$

Proof. Suppose first that the edge $s_1 \xrightarrow{a_1|a_2} s_2$ exists in G. Then by the definition of $\otimes$ there exist states $s_{11}, s_{12} \in G_1$ and $s_{21}, s_{22} \in G_2$ such that (8.7) or (8.8) holds for a suitable $a'_1 \in \mathcal{N}$. W.l.o.g. we suppose, that (8.7) holds. Thus

$$\alpha s_{12} - s_{11} = a_1 - a'_1 \quad \text{and} \quad \alpha s_{22} - s_{21} = a'_1 - a_2$$

with $s_{11} + s_{21} = s_1$ and $s_{12} + s_{22} = s_2$. Adding these two equations yields (8.9).

To establish the other direction suppose that (8.9) holds for certain states $s_1, s_2 \in G$ and $a_1, a_2 \in \mathcal{N}$. Then there exist $s_{11} \in G_1$ and $s_{21} \in G_2$ such that $s_1 = s_{11} + s_{21}$. Since G_1 has property (C) we conclude that there exists a state $s_{12} \in G_1$ and an $a_1 \in \mathcal{N}$ such that G_1 contains the edge

$$s_{11} \xrightarrow{a_1|a'_1} s_{12} \quad \text{and, hence,} \quad \alpha s_{12} - s_{11} = a_1 - a'_1. \tag{8.10}$$

Set $s_{22} := s_2 - s_{12}$. Then (8.9) reads $\alpha(s_{12} + s_{22}) - s_{11} - s_{21} = a_1 - a'_1 + a'_1 - a_2$. Subtracting the identity at the left side of (8.10) yields

$$\alpha s_{22} - s_{21} = a'_1 - a_2. \tag{8.11}$$

On the other hand, since G_2 has property (C) we conclude that there exists a $s'_{22} \in G_2$ such that $s_{21} \xrightarrow{a'_1|a'_2} s'_{22}$, i.e. $\alpha s'_{22} - s_{21} = a'_1 - a'_2$ for a certain $a'_2 \in \mathcal{N}$. Subtracting this from (8.11) yields

$$\alpha(s_{22} - s'_{22}) = a'_2 - a_2.$$

Since $a'_2 - a_2$ can be of the shape αv for a $v \in \mathbb{Z}[\alpha]$ only if $v = 0$ we get $s_{22} = s'_{22}$ and thus $a_2 = a'_2$. Thus $s_{22} \in G_2$ and, hence, (8.11) implies

$$s_{21} \xrightarrow{a'_1|a_2} s_{22}. \tag{8.12}$$

Now the existence of the edge $s_1 \xrightarrow{a_1|a_2} s_2$ is a consequence of (8.10) and (8.12). □

After these preparations we are in a position to state the following algorithm.

Algorithm 8.9 (cf. Scheicher-Thuswaldner [38])**.** *The graph $\mathcal{G}_1(S)$, and with it the set S, can be determined by the following algorithm:*

$k := 1$
$A[1] := A_0(1)$
repeat
  $k := k + 1$
  $A[k] := \mathrm{Red}(A[k-1] \otimes A[1])$
until $A[k] = A[k-1]$
$\mathcal{G}_1(S) := A[k] \setminus \{0\}$

This algorithm always terminates after finitely many steps.

Algorithm 8.9 will allow us to construct the set S for each quadratic CNS. This will be done in the next section.

9. The construction of the set of neighbours of a tile

In this section we will construct the graph $\mathcal{G}_1(S)$ defined above. The number of states of $\mathcal{G}_1(S)$ — and therefore also the shape of the set S — mainly depends on the quantity J defined in (1.3). We define

$$P_n := n - (n-1)A - (n-1)\alpha, \quad Q_n := -n + nA + n\alpha, \quad R := -A - \alpha \quad (n \geq 1).$$

The main result of this section reads as follows.

Theorem 9.1. *Let α be the base of a nontrivial CNS with minimal polynomial $x^2 + Ax + B$ and let $\mathcal{T}$ be the associated central tile. If J is defined as in (1.3) then the set of "neighbours" of $\mathcal{T}$ is given by*

$$S = \{\pm P_1, \ldots, \pm P_J, \pm Q_1, \ldots, \pm Q_J, \pm R\}.$$

In particular, the central tile $\mathcal{T}$ has

$$\#S = 2 + 4J$$

"neighbours".

This theorem will be proved by determining explicitly the graph $\mathcal{G}_1(S)$ corresponding to a quadratic CNS, whose set of states is S. To this matter we need a family of graphs. Indeed, for $N \in \mathbb{N}$ let the graph $\mathcal{H}(N)$ be defined in the following way. The states of $\mathcal{H}(N)$ are

$$\pm P_1, \ldots, \pm P_N, \pm Q_1, \ldots, \pm Q_N, \pm R.$$

The edges of $\mathcal{H}(N)$ are defined by the following table. The greek letters in the column "name" are used later as abbreviation for the according set of labels.

edge	labels		name
$0 \to 0$	0 $\vdots$ $2A-C-1$	0 $\vdots$ $2A-C-1$	
$P_1 \to 0$	0 $\vdots$ $2A-C-2$	1 $\vdots$ $2A-C-1$	β
$P_1 \to R$	$2A-C-1$	0	γ
$R \to Q_1$	0 $\vdots$ $A-1$	$A-C$ $\vdots$ $2A-C-1$	δ
$R \to -P_1$	A $\vdots$ $2A-C-1$	0 $\vdots$ $A-C-1$	ε
$P_{n+1} \to Q_n$ $(1 \le n < N)$	0 $\vdots$ $A-3-(n-1)(A-C+1)$	$1+n(A-C+1)$ $\vdots$ $2A-C-1$	κ_n
$P_{n+1} \to -P_n$ $(1 \le n < N)$	$A-2-(n-1)(A-C+1)$ $\vdots$ $2A-C-1$	0 $\vdots$ $n(A-C+1)$	λ_n
$Q_n \to P_n$ $(1 \le n \le N)$	0 $\vdots$ $n(A-C+1)-1$	$A-1-(n-1)(A-C+1)$ $\vdots$ $2A-C-1$	μ_n
$Q_n \to -Q_n$ $(1 \le n \le N)$	$n(A-C+1)$ $\vdots$ $2A-C-1$	0 $\vdots$ $A-2-(n-1)(A-C+1)$	ν_n

TABLE 1

Furthermore, if there exists an edge $s_1 \xrightarrow{a|a'} s_2$ in $\mathcal{H}(N)$, then there also exists an edge $-s_1 \xrightarrow{a'|a} -s_2$. Thus for the labels of the edge $-P_{n+1} \to -Q_n$ we use the abbreviation $-\kappa_n$, the abbreviations $-\beta, -\gamma, -\delta, -\varepsilon, -\lambda_n, -\mu_n$ and $-\nu_n$ are defined in an analogous way.

Remark 9.2. *Clearly, $a|a' \in \nu_n$ means that the label $a|a'$ occurs in ν_n. If we write $a \in \nu_n$ we mean that there exists an $a' \in \mathcal{N}$ such that $a|a' \in \nu_n$.*

We need some results on this family of graphs.

Lemma 9.3. *There exists an edge between two states s_1 and s_2 of $\mathcal{H}(N)$ labelled by $a|a'$, i.e. $s_1 \xrightarrow{a|a'} s_2$ if and only if*

$$\alpha s_2 - s_1 = a - a'.$$

Moreover, $\mathcal{H}(N)$ has the property (C) defined in the previous section.

Proof. This follows from Table 1 by direct calculation. □

Lemma 9.4. *Let J be defined as in (1.3). Then*

$$\begin{aligned} \mathrm{Red}(\mathcal{H}(N)) &= \mathcal{H}(N) \quad \text{for} \quad N \le J, \\ \mathrm{Red}(\mathcal{H}(N)) &= \mathcal{H}(J) \quad \text{for} \quad N > J. \end{aligned}$$

Proof. Let first $N \le J$ It is easily seen from Table 1 that for $n \le N$ the sets of labels $\pm\nu_n$ are nonempty. Thus there exist cycles of the form $Q_n \to -Q_n \to Q_n$ $(1 \le n \le N)$. Since $\pm R$ has Q_1 as a predecessor and $\pm P_n$ $(1 \le n \le N)$ has $\pm Q_n$

as a predecessor we conclude that each state of $\mathcal{H}(N)$ is the endpoint of an infinite walk. Thus the first assertion is proved.

Let now $N > J$. For $n \le J$ all the arguments of the preceding paragraph remain valid, thus each of the states $\pm R, \pm P_n, \pm Q_n$ $(1 \le n \le J)$ is the endpoint of an infinite walk.

For $J < n \le N$ the set of labels $\pm\nu_n$ is empty, because in this case $n(A-C+1) > 2A - C - 1$. Thus there do not exist edges connecting Q_n and $-Q_n$. Looking at Table 1 this implies that $\pm Q_n$ has no predecessor in this case. It is now easily seen from Table 1 that the states $\pm P_n$ $(J < n \le N)$ have only finitely many predecessors. Namely, the longest walk with P_n as endpoint is $Q_N \to P_N \to P_{N-1} \to \cdots \to P_n$. Thus $\mathrm{Red}(\mathcal{H}(N))$ contains exactly the states of $\mathcal{H}(J)$ and the edges connecting them. This proves the second assertion. $\square$

Remark 9.5. *For $A > 0$ Theorem 9.1 follows immediately from the following characterization of the graph $\mathcal{G}_1(S)$. For $A = -1$ it was shown in* Akiyama-Thuswaldner [1] *that the corresponding tile is essentially the same as the tile for $A = 1$. In fact, these tiles are related to each other by reflection on the origin. This implies that all results of the present paper which are valid for $A = 1$ also hold for $A = -1$. In particular, this treats the exceptional case $A = -1$ in Theorem 9.1.*

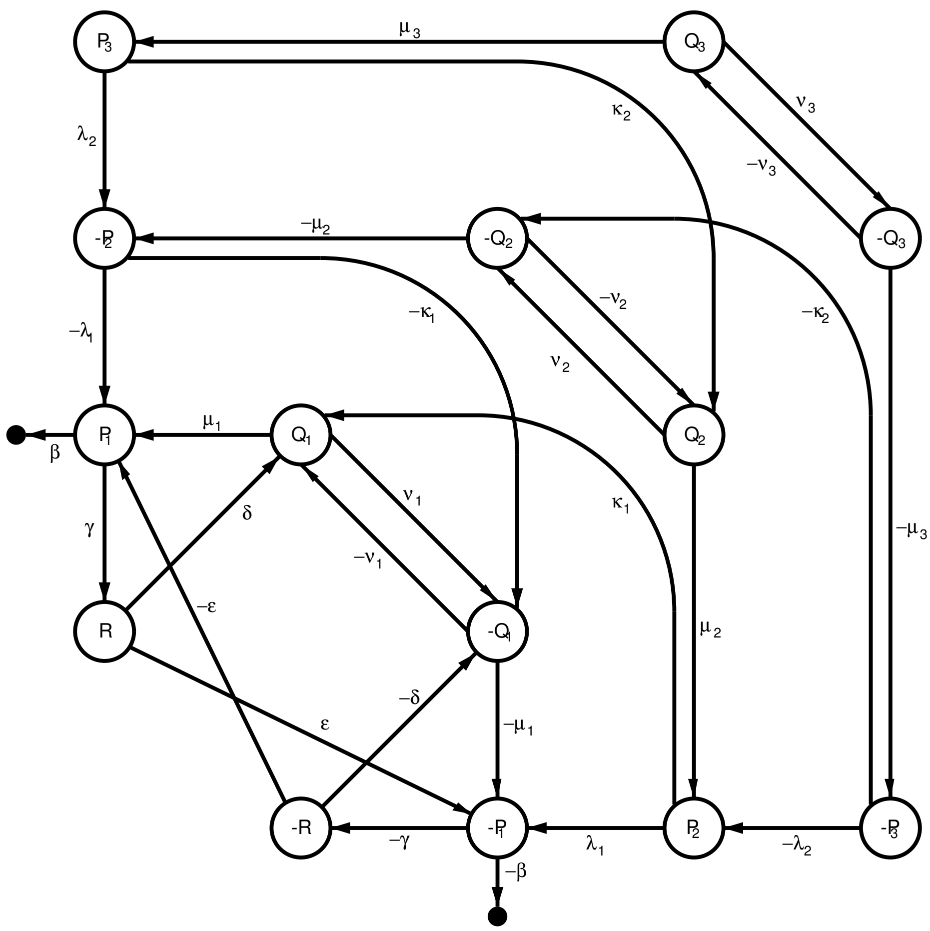

Figure 3. The graph $\mathcal{G}_1(S)$ for $j = 3$

Theorem 9.6. *Let α be the base of a nontrivial CNS whose minimal polynomial $x^2 + Ax + B$ has $A > 0$ and let $\mathcal{T}$ be the associated central tile. If J is defined as in (1.3) then*

$$\mathcal{G}_1(S) = \mathcal{H}(J) \setminus \{0\}$$

where $\mathcal{H}(J)$ is defined as above. In particular, the states of $\mathcal{G}_1(S)$ are

$$\pm P_1, \dots, \pm P_J, \pm Q_1, \dots, \pm Q_J, \pm R$$

and its edges are defined via Table 1.

Remark 9.7. *By the definition of J and C we always have*

$$\frac{(J-2)A + J + 1}{J-1} \le C < \frac{(J-1)A + J + 2}{J}. \tag{9.1}$$

This inequality will be needed frequently in the following calculations.

Remark 9.8. *Suppose that $A > 0$. Note that Figure 2 shows $\mathcal{G}_1(S)$ for $J = 1$. We depicted $\mathcal{G}_1(S)$ for $J = 3$ in Figure 3. From these figures one can easily imagine how the picture of $\mathcal{G}_1(S)$ looks for other values of J.*

Proof. In order to prove Theorem 9.6 we have to use Algorithm 8.9. If we could show that

$$A[N] = \mathrm{Red}(\mathcal{H}(N)) \quad (N \ge 1) \tag{9.2}$$

then this algorithm would yield the result by Lemma 9.4. Thus it remains to establish (9.2). We will do this by induction on N. The induction start is easy. In fact, note that $A[1] = A_0(1)$ for $A > 0$. Comparing Figure 2 with the definition of $\mathcal{H}(1)$ yields together with Lemma 9.4 that

$$A[1] = \mathcal{H}(1) = \mathrm{Red}(\mathcal{H}(1)).$$

Now we proceed to the induction argument. Suppose that for a certain $N \ge 2$ we have

$$A[N-1] = \mathrm{Red}(\mathcal{H}(N-1)).$$

First we assume that $N \le J + 1$. Then Lemma 9.4 yields

$$A[N-1] = \mathcal{H}(N-1).$$

In order to prove (9.2) we have to compare $A(N) = \mathrm{Red}(A[N-1] \otimes A[1])$ with $\mathcal{H}(N)$. Since the graph $\mathcal{H}(N)$ is explicitly known the problem consists in determining the graph $\mathrm{Red}(A[N-1] \otimes A[1])$.

The states of the graph $\mathrm{Red}(A[N-1] \otimes A[1])$. In order to determine the states of this graph we start with considering $A[N-1] \otimes A[1]$. In particular, we will determine the states of this graph. By the definition of $\otimes$ they are given by the following list (in this list we set $P_0 = -R$ and $Q_0 = 0$).

(V01) All states of $A[N-1]$ (because 0 is a state of $A[1]$)
(V02) $\pm(P_n + P_1) = n + 1 - (n-1)A - (n-1)\alpha$ with $1 \le n \le N-1$
(V03) $\pm(P_n - P_1) = \mp Q_{n-1}$ with $1 \le n \le N-1$
(V04) $\pm(Q_n + P_1) = -(n-1) + nA + n\alpha$ with $1 \le n \le N-1$
(V05) $\pm(Q_n - P_1) = \mp P_{n+1}$ with $1 \le n \le N-1$
(V06) $\pm(R + P_1) = \mp Q_1$
(V07) $\pm(R - P_1) = -1 - A - \alpha$
(V08) $\pm(P_n + Q_1) = \pm P_{n-1}$ with $1 \le n \le N-1$
(V09) $\pm(P_n - Q_1) = \pm P_{n+1}$ with $1 \le n \le N-1$
(V10) $\pm(Q_n + Q_1) = \pm Q_{n+1}$ with $1 \le n \le N-1$
(V11) $\pm(Q_n - Q_1) = \pm Q_{n-1}$ with $1 \le n \le N-1$
(V12) $\pm(R + Q_1) = \mp P_1 = -1 + 2A + 2\alpha$
(V13) $\pm(R - Q_1)$
(V14) $\pm(P_n + R) = \mp Q_n$ with $1 \le n \le N-1$

(V15) $\pm(P_n - R) = n - (n-2)A - (n-2)\alpha$ with $1 \le n \le N-1$
(V16) $\pm(Q_n + R) = \mp P_n$ with $1 \le n \le N-1$
(V17) $\pm(Q_n - R) = -n + (n+1)A + (n+1)\alpha$ with $1 \le n \le N-1$
(V18) $\pm(R + R) = -2A - 2\alpha$

Now we have to determine the edges connecting these states. First we deal with the states in (V01), (V03), (V05), (V06), (V08) – (V12), (V14) and (V16). Note that these are exactly the states of $\mathcal{H}(N)$. We claim that the edges between these states are exactly the same as the edges of $\mathcal{H}(N)$. In fact, since by the induction hypothesis $\mathcal{H}(N-1) = A[N-1]$ and $\mathcal{H}(1) = A[1]$ the graphs $A[N-1]$ and $A[1]$ have property (C) by Lemma 9.3. Thus Lemma 8.8 implies that there is an edge between two states s_1 and s_2 of $A[N-1] \otimes A[1]$ labelled by $a|a'$ if and only if $\alpha s_2 - s_1 = a - a'$. Since by Lemma 9.3 the edges of $\mathcal{H}(N)$ are defined in the same way the claim is proved. In particular, we have shown the following result:

(9.3) The restriction of $A[N-1] \otimes A[1]$ to the states of $\mathcal{H}(N)$ is equal to $\mathcal{H}(N)$.

This result guarantees that each of the states of $\mathcal{H}(N)$ is the endpoint of an infinite walk in $A[N-1] \otimes A[1]$ and is thus contained in $\mathrm{Red}(A[N-1] \otimes A[1])$.

Thus in order to show (9.2) it remains to prove that all the other states of $A[N-1] \otimes A[1]$ can not deserve as end point of a walk of infinite length. This has to be done for the states in (V02), (V04), (V07), (V13), (V15), (V17) and (V18). To this matter we will determine the direct predecessors of the states in these classes. Note that for instance $P_n + P_1 = P_{n+1} - R$ $(1 \le n \le N-1)$, i.e. the classes (V02) and (V14) contain (apart from two states) the same states. Despite of this fact two different representations of a state of $A[N-1] \otimes A[1]$ as the sum of a state of $A[N-1]$ and a state of $A[1]$ have to be treated separately. This is due to the fact that by the definition of $\otimes$ each of these representations contributes different predecessors.

The direct predecessors of the states in (V02). Let $1 \le n \le N-1$. We confine ourselves to the states $P_n + P_1$, since it is easily seen that by symmetry the existence of $s_1 \xrightarrow{a|a'} s_2$ implies the existence of $-s_1 \xrightarrow{a'|a} -s_2$. By the definition of $\otimes$ the candidates for the direct predecessors of $P_n + P_1$ contributed by this representation are $-P_{n+1} - R$, $-P_{n+1} - P_2$, $-P_{n+1} + Q_1$, $Q_n - R$, $Q_n - P_2$ and $Q_n + Q_1$ for $1 \le n \le N-1$ and $-R - R$.

We will now examine, if there exist edges leading from one of these states to $P_n + P_1$.

(i) $-P_{n+1} - R$: Looking at Table 1 by the definition of the graphs $A[1]$ and $A[N-1]$ we have the edges $-P_{n+1} \xrightarrow{-\lambda_n} P_n$ in $A[N-1]$ and $-R \xrightarrow{-\varepsilon} P_1$ in $A[1]$. In order to get an edge connecting $(-P_{n+1}-R)$ with (P_n+P_1) we need either $a_1|a_1' \in -\varepsilon$ and $a_1'|a_2 \in -\lambda_n$ or $a_1|a_1' \in -\lambda_n$ and $a_1'|a_2 \in -\varepsilon$. We consider the first of these two possibilities (the second one leads to the same result). From Table 1 we see that $a_1|a_1' \in -\varepsilon$ implies $A \le a_1' \le 2A - C - 1$ and $a_1'|a_2 \in -\lambda_n$ implies $0 \le a_1' \le n(A - C + 1)$. We need an integer a_1' that fulfills both of these conditions. But such an a_1' can exist only if $A \le n(A - C + 1)$, which is equivalent to $C \le ((n-1)A + n)/n$. If $n \le N - 1 < J$ this contradicts (9.1). Thus there do not exist edges of the shape $(-P_n - R) \to (P_n + P_1)$ contributed by this representation for $1 \le n \le N-1$. But if $n = N - 1 = J$ then there exists an edge

$$Q_{J+1} = (-P_{J+1} - R) \to (P_J + P_1). \tag{9.4}$$

Note that we are not interested in the labelling of this edge. Thus we leave it away.

(ii) $-P_{n+1} - P_2$: We have $-P_{n+1} \xrightarrow{-\lambda_n} P_n$ in $A[N-1]$ and $-P_2 \xrightarrow{-\lambda_1} P_1$ in $A[1]$. As before we need a_1' such that $a_1|a_1' \in -\lambda_n$ and $a_1'|a_2 \in -\lambda_1$ or vice versa. Again it suffices to deal with the first alternative. This leads to the conditions $A-2-(n-1)(A-C+1) \le a_1' \le 2A-C-1$ and $0 \le a_1' \le A-C+1$. If there exists an integer a_1' which fulfills both of these conditions, this implies $C \le ((n-1)A+n+2)/n$. Thus by (9.1) there exists an edge $(-P_{J+1}-P_2) \to (P_J+P_1)$ only for $n=J$. But $-P_{J+1}-P_2$ is not an element of the set of states of $A[N-1] \otimes A[1]$. Hence, this edge does not belong to this graph and the present case does not contribute any edge.

(iii) $-P_{n+1}+Q_1$: We have $-P_{n+1} \xrightarrow{-\lambda_n} P_n$ and $Q_1 \xrightarrow{\mu_1} P_1$. As before we need a_1' such that $a_1|a_1' \in -\lambda_n$ and $a_1'|a_2 \in \mu_1$. This leads to the condition $C \le ((n-1)A+n+1)/n$ and thus there can exist an edge only for $n=J$. Since $-P_{J+1}+Q_1$ is not a state of $A[N-1] \otimes A[1]$ this case does not contribute any edge.

(iv) Q_n-R: We have $Q_n \xrightarrow{\mu_n} P_n$ and $-R \xrightarrow{-\varepsilon} P_1$. This yields the condition $C \le ((n-1)A+n-1)/n$ and thus there exists an edge

$$(Q_J-R) \to (P_J+P_1) \tag{9.5}$$

for $n=J$.

(v) Q_n-P_2: We have $Q_n \xrightarrow{\mu_n} P_n$ and $-P_2 \xrightarrow{-\lambda_1} P_1$. This yields $C \le ((n-1)A+n+1)/n$. Thus there can exist an edge only for $n=J$. But since Q_J-P_2 is not a state of $A[N-1] \otimes A[1]$ this case does not contribute any edge.

(vi) Q_n+Q_1: We have $Q_n \xrightarrow{\mu_n} P_n$ and $Q_1 \xrightarrow{\mu_1} P_1$, hence, $C \le ((n-1)A+n)/n$. Thus for $n=J$ there exists the edge

$$(Q_J+Q_1) \to (P_J+P_1). \tag{9.6}$$

(vii) $-R-R$: We have $-R \xrightarrow{-\varepsilon} P_1$ and $-R \xrightarrow{-\varepsilon} P_1$. In this case the condition on a_1' is not fulfilable. Thus this case does not contribute an edge.

The direct predecessors of the states in (V04). The candidates for direct predecessors of Q_n+P_1 $(1 \le n \le N-1)$ are $-Q_n-R$, $-Q_n-P_2$, $-Q_n+Q_1$, $P_{n+1}-R$, $P_{n+1}-P_2$ and $P_{n+1}+Q_1$ for $1 \le n \le N-1$ and $+R-R$.

(i) $-Q_n-R$: We have $-Q_n \xrightarrow{-\nu_n} Q_n$ and $-R \xrightarrow{-\varepsilon} P_1$. This yields to the condition $A<C$, which is never fulfilled.

(ii) $-Q_n-P_2$: We have $-Q_n \xrightarrow{-\nu_n} Q_n$ and $-P_2 \xrightarrow{-\lambda_1} P_1$. This yields the condition $n(A-C+1) \le (A-C+1)$ which by (9.1) implies $n=1$. For $n=1$ we have $Q_1+P_1=R$, which belongs to $\mathcal{H}(N)$. Thus its predecessors have been determined before, and so this case does not contribute new edges.

(iii) $-Q_n+Q_1$: We have $-Q_n \xrightarrow{-\nu_n} Q_n$ and $Q_1 \xrightarrow{\mu_1} P_1$. The required conditions can not be fulfilled in this case. Thus it contributes no edge.

(iv) $P_{n+1}-R$: We have $P_{n+1} \xrightarrow{\kappa_n} Q_n$ and $-R \xrightarrow{-\varepsilon} P_1$. The required conditions can not be fulfilled in this case. Thus it contributes no edge.

(v) $P_{n+1}-P_2$: We have $P_{n+1} \xrightarrow{\kappa_n} Q_n$ and $-P_2 \xrightarrow{-\lambda_1} P_1$. The required conditions can not be fulfilled in this case. Thus it contributes no edge.

(vi) $P_{n+1}+Q_1$: We have $P_{n+1} \xrightarrow{\kappa_n} Q_n$ and $Q_1 \xrightarrow{\mu_1} P_1$. The required conditions can not be fulfilled in this case. Thus it contributes no edge.

(vii) $+R-R$: We have $R \xrightarrow{\delta} Q_1$ and $-R \xrightarrow{-\varepsilon} P_1$. The required conditions can not be fulfilled in this case. Thus it contributes no edge.

The direct predecessors of the states in (V07), (V13), (V15) and (V17). It remains to deal with the predecessors of $R-P_1$, $R-Q_1$, P_n-R and Q_n-R. In these cases one argues in the same way as before. It turns out, that in any case the appropriate conditions can not be fulfilled. This means, that no additional edge is contributed.

Conclusion. Let first $N \leq J$. In this case there are no edges leading away from states of the classes (V02), (V04) (V07), (V13), (V15) and (V17). Thus none of these states can belong to $\mathrm{Red}(A[N-1] \otimes A[1])$. This implies that the only states, that can belong to $\mathrm{Red}(A[N-1] \otimes A[1])$ are the states of the classes (V01), (V03), (V05), (V06), (V08) – (V12), (V14) and (V16). As mentioned above these states are exactly the states of $\mathcal{H}(N)$. Thus (9.3) implies together with Lemma 9.4

$$A[N] = \mathrm{Red}(A[N-1] \otimes A[1]) = \mathrm{Red}(\mathcal{H}(N)) = \mathcal{H}(N). \tag{9.7}$$

Now let $N = J+1$. In this case the classes (V02), (V04) (V07), (V13), (V15) and (V17) contribute (apart from symmetry) the edges (9.4), (9.5) and (9.6). The state Q_{J+1} in (9.4) has no predecessor. In fact, as one can see from Table 1 by an easy calculation, $-\nu_{J+1}$ and κ_{J+1} are empty. The state $Q_J - R$ in (9.5) and the state $Q_J + Q_1$ in (9.6) have no predecessor, as can be seen from the above calculations. Thus, as before, the only states that can belong to $\mathrm{Red}(A[N-1] \otimes A[1])$ are the states of the classes (V01), (V03), (V05), (V06), (V08) – (V12), (V14) and (V16). Now (9.3) implies together with Lemma 9.4

$$A[J+1] = \mathrm{Red}(A[J] \otimes A[1]) = \mathrm{Red}(\mathcal{H}(J+1)) = \mathcal{H}(J). \tag{9.8}$$

Up to now we proved the result for $N \leq J+1$. The remaining cases can be shown easily in the following way. From (9.7) and (9.8) we get $A[J+1] = A[J]$, which implies $A[J+2] = \mathrm{Red}(A[J+1] \otimes A[1]) = \mathrm{Red}(A[J] \otimes A[1]) = A[J+1]$ and, hence, for all $d \in \mathbb{N}$ we have

$$A[J+d] = A[J] = \mathcal{H}(J) = \mathrm{Red}(\mathcal{H}(J+d)) \tag{9.9}$$

From (9.7), (9.8) and (9.9) we obtain (9.2) for each $N \geq 1$ and the theorem is proved. □

10. The structure of the set of L-vertices of a tile

In the previous section we constructed the graph $\mathcal{G}_1(S)$ for each base of a nontrivial canonical number system in a quadratic number field. This makes it possible to determine the graphs $\mathcal{G}_L$ for $L \geq 2$ via Definition 8.3 in a rather easy way. Proposition 8.5 then yields a characterization of the elements of V_L by means of $\mathcal{G}_L$. This easy algorithm allows one to characterize the set V_L for a given positive integer L and a given canonical number system. The purpose of this section is to give an easy criterion which decides whether the set V_L is empty, finite, countable infinite, or uncountable infinite. As in the previous section the value of C defined in (8.1) plays an important rôle. In particular, we shall prove the following theorem.

Theorem 10.1. *Let α be the base of a nontrivial CNS with minimal polynomial $x^2 + Ax + B$ and let $\mathcal{T}$ be the associated central tile. Let C be defined by (8.1) and let J be as in (1.3).*

- *If $J = 1$ then*

$$\begin{array}{lcl} V_1 & & \textit{is uncountable infinite}, \\ V_2 & & \textit{contains six elements}, \\ V_L & = & \emptyset \quad (L \geq 3). \end{array}$$

- *If $J = 2$ and $C = 3$ then*

$$\begin{array}{lcl} V_1 & & \textit{is uncountable infinite}, \\ V_2 & & \textit{is countable infinite}, \\ V_L & = & \emptyset \quad (L \geq 3). \end{array}$$

- *If* $J \geq 2$ *and* $C \neq ((J-2)A+J+1)/(J-1)$ *then*

$$\begin{array}{lcl} V_L & & \textit{is uncountable infinite } (1 \leq L \leq J), \\ V_L & = & \emptyset \quad (L \geq J+1). \end{array}$$

- *If* $J > 2$ *and* $C = ((J-2)A+J+1)/(J-1)$ *then*

$$\begin{array}{lcl} V_L & & \textit{is uncountable infinite } (1 \leq L \leq J-1), \\ V_J & & \textit{finite}, \\ V_L & = & \emptyset \quad (L \geq J+1). \end{array}$$

In the remaining part of this section we always assume that $A > 0$. In particular, we will prove Theorem 10.1 for all CNS with $A > 0$. By Remark 9.5 it is then valid also in the exceptional case $A = -1$.

To prove Theorem 10.1 we need a series of preliminary results. We want to avoid the construction of the product graphs $\mathcal{G}_L(S)$ because their complexity increases very fast with L. Thus we try to extract the desired results from $\mathcal{G}_1(S)$. By Proposition 8.5 a point $x = \sum_{j \geq 1} a_j \Phi(\alpha^{-j})$ belongs to V_L if and only if there exist L pairwise different states $s_{01}, \dots, s_{0L} \in S$ such that in each $s_{0\ell}$ $(1 \leq \ell \leq L)$ there ends an infinite walk $W_\ell \in P(\mathcal{G}_1(S))$ with labelling $(\dots a_3 a_2 a_1)$. Here $P(G)$ denotes the set of walks in the graph G. Thus in order to estimate the cardinality of V_L we have to estimate the number of sets $\{W_1, \dots, W_L\}$ consisting of L infinite walks, which have all the same labelling and end up in pairwise different states.

To this end we need the following partition on the set of states S of the graph $\mathcal{G}_1(S)$ in J levels:

1st level: The states $\pm P_1$, $\pm Q_1$ and $\pm R$.
n-th level: The states $\pm P_n$, $\pm Q_n$ $\qquad (2 \leq n \leq J)$.

If a state s lies in the n-th level we write $L(s) = n$ for short. With these definitions we can state the following auxiliary results.

Lemma 10.2. *Let* $W \in P(\mathcal{G}_1(S))$ *be an infinite walk ending in a state* s_0. *If we trace backwards through the walk* W *the function* $L(s)$ *of its states increases monotonically. I.e., if we arrive at a state* s' *with* $L(s') = n$, *then the levels of all its predecessors are greater than or equal to* n.

Proof. This is an immediate consequence of the definition of $\mathcal{G}_1(S)$ and can be checked by Table 1. □

Proposition 10.3. *Let* $W \in P(\mathcal{G}_1(S))$ *be an infinite walk ending in a state* s_0. *Then one of the following possibilities occurs:*

(i) *All states of* W *belong to level* 1.
(ii) *If we trace backwards through* W *we end up in one of the cycles* $\pm Q_n \leftarrow \mp Q_n \leftarrow \pm Q_n$ $(1 \leq n \leq J)$.

Proof. If the first possibility occurs, we are ready. Thus suppose that there exist states in W which have level greater than 1. Since $L(\cdot)$ can take only finitely many values, there exists a state $s_{\max}$ with maximal level $n_0 \geq 2$. Thus by Lemma 10.2 all predecessors of $s_{\max}$ must have level n_0. Since (apart from the labelling) the only possibility for an infinite walk at level n_0 is the cycle $\pm Q_{n_0} \leftarrow \mp Q_{n_0} \leftarrow \pm Q_{n_0}$ we are ready. □

Definition 10.4. *Let* $W \in P(\mathcal{G}_1(S))$ *be a walk of infinite length. We say that* W *has property (P1) if it fulfils the first possibility of Proposition 10.3 and that it has property (P2) if it fulfils the second possibility of Proposition 10.3.*

In the next step we shall look more closely to the walks in W that have property (P1), i.e. the walks, which remain totally in the subgraph of level 1. Let $A(1)$ be the graph that emerges from $A_0(1)$ if we leave away the state 0 and all the edges leading to it. Then the above mentioned subgraph of level 1 is isomorphic to $A(1)$. This is easy to see by comparing the definition of $\mathcal{G}_1(S)$ with Figure 2. Thus $A(1)$ contains all walks which have property (P1).

Let $A(2) := A(1)^2$. $A(2)$ is depicted in Figure 4. This graph will be used in the following lemma.

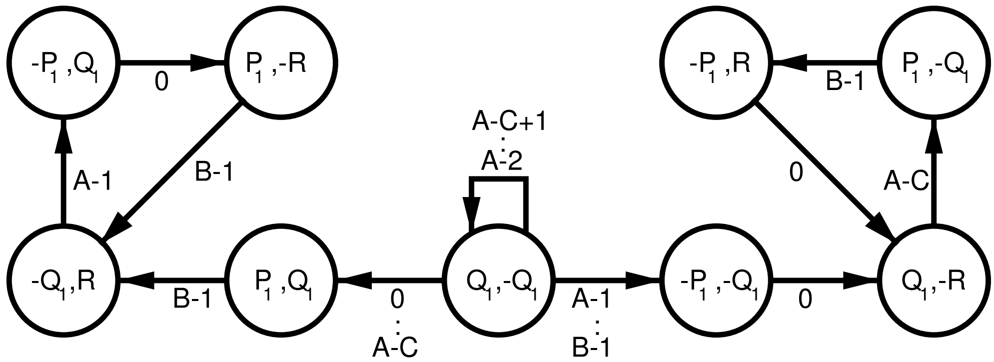

FIGURE 4. The graph $A(2)$

Lemma 10.5. *The following two assertions are equivalent.*

- *There exist two walks W_1 and W_2 with property (P1) which start in different states and have both the same labelling $(\dots a_2 a_1)$.*
- *There exists a walk in $A(2)$ with the labelling $(\dots a_2 a_1)$.*

Proof. The proof is the same as the proof for the equivalence of assertions (ii) and (iii) of Proposition 8.5. □

In the proof of Theorem 7.1 we need the following result which follows immediately from the shape of $A(2)$ together with Proposition 8.5.

Lemma 10.6. *Let $2A \geq B+3$. Since in this case there exist infinitely many walks in $A(2)$ starting in $(-P_1, Q_1)$ the intersection $\mathcal{T} \cap (\mathcal{T} - P_1) \cap (\mathcal{T} + Q_1)$ has infinitely many elements.*

Lemma 10.7. *Let W be a walk in $A(2)$. If one traces backwards through W one ends either in the cycle $(Q_1, -Q_1)$ or in the cycle $(-Q_1, R) \leftarrow (P_1, -R) \leftarrow (-P_1, Q_1)$ or in the cycle $(Q_1, -R) \leftarrow (-P_1, R) \leftarrow (P_1, -Q_1)$. The corresponding labellings are*

$$([a]^\infty) \quad \textit{and} \quad ([(B-1)0(A-1)]^\infty) \quad \textit{and} \quad ([0(B-1)(A-C)]^\infty)$$

where $a \in \{A-C+1, \dots, A-2\}$. Note that the first circle only exists for $J \geq 2$.

Proof. This can immediately be seen from Figure 4. □

If we compute $A(3) = A(1)^3$ we see that it is the empty graph. Thus there do not exist 3 walks with property (P1) having the same labelling which end in pairwise different states.

Now we dwell upon the class of walks which fulfil condition (ii) of Proposition 10.3, i.e. the walks having property (P2). If we trace backwards through a walk W of this class we end up in one of the cycles $\pm Q_n \leftarrow \mp Q_n \leftarrow \pm Q_n$.

Let $L \in \mathbb{N}$ and let $A = (\dots a_2 a_1)$ be a labelling. Since we want to determine the cardinality of V_L by the equivalence of assertion (i) and (iii) of Proposition 8.5 we search for sets of L walks $\{W_1, \dots, W_L\}$ which fulfil

$$\begin{aligned} &W_\ell \text{ has property (P2) } (1 \le \ell \le L),\\ &W_1, \dots, W_L \text{ end up in pairwise different states,}\\ &W_1, \dots, W_L \text{ have all the same labelling } A. \end{aligned} \tag{10.1}$$

Suppose we found a set of L walks $\{W_1, \dots, W_L\}$ which fulfils (10.1). If we cut off the last r edges of these walks we arrive at L walks $W_1^{(r)}, \dots, W_L^{(r)}$. These walks again fulfil (10.1) with the labelling $A_r = (\dots a_{r+2} a_{r+1})$. If we choose r large enough then by Proposition 10.3 each walk $W_i^{(r)}$ is of the shape

$$W_i^{(r)} : \pm Q_{n_i} \xleftarrow{a_{r+1}} \mp Q_{n_i} \xleftarrow{a_{r+2}} \pm Q_{n_i} \xleftarrow{a_{r+3}} \dots .$$

for a certain $n_i \in \{1, \dots, J\}$ Thus if we search for the maximal number L such that there exists a set of L walks that fulfils (10.1) it suffices to search among the cycles $\pm Q_n \leftarrow \mp Q_n \leftarrow \pm Q_n$.

Lemma 10.8. *For a given labelling A there exist at most $J-1$ different cycles of the shape*

$$\pm Q_n \leftarrow \mp Q_n \leftarrow \pm Q_n \leftarrow \dots \quad (2 \le n \le J) \tag{10.2}$$

which have labelling A. The possible labellings, which occur in $J-1$ different cycles of this shape are the following ones: The labelling $(\dots b_2 a_2 b_1 a_1)$ with $a_i \in \{0, \dots, A-2-(J-1)(A-K+1)\}$ and $b_i \in \{J(A-K+1), \dots, 2A-K-1\}$ occurs in the $J-1$ cycles

$$Q_n \leftarrow -Q_n \leftarrow Q_n \leftarrow \dots \quad (2 \le n \le J). \tag{10.3}$$

The labelling $(\dots b_2 a_2 b_1 a_1)$ with $a_i \in \{J(A-C+1), \dots, 2A-C-1\}$ and $b_i \in \{0, \dots, A-2-(J-1)(A-C+1)\}$ occurs in the $J-1$ cycles

$$-Q_n \leftarrow Q_n \leftarrow -Q_n \leftarrow \dots \quad (2 \le n \le J). \tag{10.4}$$

The possible labellings which occur in $J-2$ different cycles are characterized as follows: Let $1 \le r \le J-1$. Then the labeling $(\dots b_2 b_1 a_2 a_1)$ with $a_i \in \{r(A-C+1), \dots, A-2-(J-r-1)(A-C+1)\}$ and $b_i \in \{(J-r)(A-C-1), \dots, A-2-(r-1)(A-C+1)\}$ occurs in the $J-2$ cycles

$$\begin{aligned} Q_n \leftarrow -Q_n \leftarrow Q_n \leftarrow \dots \quad (2 \le n \le r),\\ -Q_n \leftarrow Q_n \leftarrow -Q_n \leftarrow \dots \quad (2 \le n \le J-r). \end{aligned} \tag{10.5}$$

All the other labellings are contained in less than $J-2$ cycles of the shape (10.2).

Proof. Note that for a given labelling we have to check only the $2J-2$ different walks

$$\pm Q_n \xleftarrow{c_1} \mp Q_n \xleftarrow{c_2} \pm Q_n \xleftarrow{c_3} \mp Q_n \xleftarrow{c_4} \dots \quad (2 \le n \le J). \tag{10.6}$$

Let r be the largest number such that there exists a cycle

$$W_r : Q_r \xleftarrow{c_1} -Q_r \xleftarrow{c_2} Q_r \xleftarrow{c_3} -Q_r \xleftarrow{c_4} \dots$$

(if there does not exist any such cycle set $r = 0$). Then by the definition of the labels $\pm\nu_n$ $(1 \le n \le J)$ in Table 1 this implies the existence of the walks

$$W_\rho : Q_\rho \xleftarrow{c_1} -Q_\rho \xleftarrow{c_2} Q_\rho \xleftarrow{c_3} -Q_\rho \xleftarrow{c_4} \dots \quad (2 \le \rho \le r).$$

We claim that the existence of W_r implies that the walks

$$-W_\sigma : -Q_\sigma \xleftarrow{c_1} Q_\sigma \xleftarrow{c_2} -Q_\sigma \xleftarrow{c_3} Q_\sigma \xleftarrow{c_4} \dots$$

do not exist if $\sigma > J-r$. In fact the existence of W_r implies by the definition of $-\nu_r$ in Table 1 that $c_1 \in \{0, \dots, A-2-(r-1)(A-K+1)\}$. If $-W_\sigma$ exists, this implies by the definition of ν_σ that $c_1 \in \{\sigma(A-C+1), \dots, 2A-C-1\}$. Thus $-W_\sigma$ can exist only if

$$\sigma(A-C+1) \le A-2-(r-1)(A-C+1)$$

which is equivalent to

$$\frac{(\sigma+r-2)A+\sigma+r+1}{\sigma+r-1} \le C.$$

By (9.1) the last inequality can be true only for $\sigma \le J-r$ which proves the claim. Thus for $r=J$ there can exist at most $J-1$ different cycles of the shape (10.6) for a given labelling A. Namely, these cycles are the $J-1$ cycles in (10.3). By the definition of the set of labels $\pm\nu_n$ these $J-1$ cycles do exist for the labels characterized in the statement of the present lemma. By the same reasoning one obtains that for $r=0$ there exist the $J-1$ cycles in (10.4) for the labels indicated above. Finally, if $1 \le r \le J-1$ there exist the $J-2$ cycles from (10.5) for the labellings indicated above. □

Now we will put together the cycles of level 1 and the cycles of higher level. This yields the following result.

Proposition 10.9. *Let C and $J \ge 2$ be defined as in Theorem 10.1. Then for a given labelling A there exist at most J cycles with different starting states in $\mathcal{G}_1(S)$ having labelling A. The labellings which occur in J cycles with different starting states can be characterized as follows: Let $0 \le r \le J$ be fixed. Then the labeling $(\dots b_2a_2b_1a_1)$*

$$\begin{array}{lcl} a_i & \in & \{r(A-C+1), \dots, A-2-(J-r-1)(A-C+1)\} \quad \textit{and} \\ b_i & \in & \{(J-r)(A-C+1), \dots, A-2-(r-1)(A-C+1)\} \end{array} \tag{10.7}$$

occurs in the J cycles

$$\begin{array}{ll} Q_n \leftarrow -Q_n \leftarrow Q_n \leftarrow \dots & (1 \le n \le r), \\ -Q_n \leftarrow Q_n \leftarrow -Q_n \leftarrow \dots & (1 \le n \le J-r). \end{array}$$

If $J=2$ then also the labellings

$$([a]^\infty) \quad \textit{and} \quad ([(B-1)0(A-1)]^\infty) \quad \textit{and} \quad ([0(B-1)(A-C)]^\infty) \tag{10.8}$$

with $a \in \{A-C+1, \dots, A-2\}$ occur in 2 cycles. There exists no labelling which occurs in $J+1$ different cycles.

Proof. First of all, consider the labellings (10.7) corresponding to the choices $r=0$ and $r=J$. By Lemma 10.8 these labellings occur in $J-1$ cycles of the shape (10.2). On the other hand, this labelling yields no path in $A(2)$, thus by Lemma 10.5 it occurs for at most one cycle W with property (P1). In fact, it occurs for the cycle $Q_1 \leftarrow -Q_1 \leftarrow Q_1$ if $r=0$ and for the cycle $-Q_1 \leftarrow Q_1 \leftarrow -Q_1$ if $r=J$. Summing up we conclude that in these cases there exist J cycles starting at different states with labelling (10.7).

For the cases $1 \le r \le J-1$ we see that $J-2$ cycles of the shape (10.2) are contributed by Lemma 10.8. These labellings yield the walk $(Q_1, -Q_1) \leftarrow (Q_1, -Q_1) \leftarrow \dots$ in $A(2)$ by Lemma 10.7. Thus by Lemma 10.5 there exist two cycles in level 1 with this labelling. Namely, these are the two cycles $\pm Q_1 \leftarrow \mp Q_1 \leftarrow \pm Q_1$. Summing up we conclude that also in these cases there exist J cycles starting at different states with labelling (10.7).

The fact that the labellings in (10.8) occur in 2 cycles follows immediately from Lemma 10.7.

Thus the cycles which yield $J-1$ and $J-2$ walks of the shape (10.2) yield exactly J walks in $\mathcal{G}_1(S)$. Since $A(3)=0$ the remaining labellings can yield at most $J-1$ walks ending in pairwise different states. □

Lemma 10.10. *Suppose that $J > 2$ and let $W_1, \ldots, W_J$ be walks in $\mathcal{G}_1(S)$ which have all the same labelling $(\ldots c_4 c_3 c_2 c_1)$ and which end in pairwise different states. Then they all start in a cycle of the shape*

$$\pm Q_n \leftarrow \mp Q_n \leftarrow \pm Q_n \ldots. \tag{10.9}$$

Proof. If each of the walks $W_1, \ldots, W_J$ would have property (P1) there would exist at least 3 walks with the same labelling which end in different states and have property (P1). This is impossible because it would imply the existence of a walk in $A(3)$ which is the empty graph. Thus we may assume that there exists a walk among $W_1, \ldots, W_J$ which has property (P2). W.l.o.g. we assume that this walk is W_1. W_1 starts in a cycle of the shape $\pm Q_n \leftarrow \mp Q_n \leftarrow \pm Q_n \ldots$ for a certain $n \in \{1, \ldots, J\}$. Thus its labelling has to end up with a string $(\ldots b_2 a_2 b_1 a_1)$ with $a_i \in \pm \nu_n$ and $b_i \in \mp \nu_n$. But the only walks in $A(1)$ which allow this labelling are the cycles $\pm Q_1 \leftarrow \mp Q_1 \leftarrow \pm Q_1 \ldots$. This is easy to see because each walk which leaves these cycles passes one of the edges $\pm R \leftarrow \mp P_1$ which is impossible with this labelling. So each of the walks $W_1, \ldots W_J$ ends up in a cycle of the shape (10.9) and we are done. □

After this list of auxiliary results we are in a position to give a proof of Theorem 10.1.

Proof of the theorem. By the remarks after the statement of Theorem 10.1 we may confine ourselves to $A > 0$. We split the proof in several parts.

The case $J = 1$. In this case $\mathcal{G}_1(S)$ is equal to $A(1)$. Since this graph has uncountably many walks having all different labellings Proposition 8.5 yields that V_1 has uncountably many elements. The automata $\mathcal{G}_2(S)$ and $\mathcal{G}_3(S)$ can easily be constructed with help of Definition 8.3. $\mathcal{G}_2(S)$ is a graph consisting of 6 states. It has exactly 6 different walks which yields $\#V_2 = 6$ by Proposition 8.5. Since $\mathcal{G}_3(S)$ is the empty graph we conclude that $V_3 = \emptyset$. Moreover $V_L \subset V_3$ for $L \geq 3$, which yields $V_L = \emptyset$ for $L \geq 3$.

The case $J = 2$ **and** $C = 3$. Also this case can be treated by explicitly computing $\mathcal{G}_L$ for $L \geq 1$. This is done in the next section.

The emptiness argument for the case $J \geq 2$. By Proposition 10.9 there do not exist $J + 1$ walks in $\mathcal{G}_1(S)$ which end in pairwise different states and have all the same labelling. By Proposition 8.5 this implies $V_L = \emptyset$ for $L \geq J + 1$.

The uncountability argument for the case $J \geq 2$. Suppose first that $C \neq ((J-2)A + J + 1)/(J-1)$. If we construct a labelling of a walk which occurs in the J different cycles

$$Q_n \leftarrow -Q_n \leftarrow Q_n \leftarrow \ldots \quad (1 \leq n \leq J) \tag{10.10}$$

it is possible by Proposition 10.9 to select at least between two different labels for each edge. Thus the number of different labellings which occur these cycles is at least as large as the number of all $\{0, 1\}$-sequences of infinite length. Thus there exist uncountably many labellings which are allowed for the J different walks (10.10). So in this case Proposition 8.5 yields that V_L has uncountably many elements for $L \leq J$.

For $C = ((J-2)A + J + 1)/(J-1)$ the same arguments as above yield that each labelling $(\ldots b_2 a_2 b_1 a_1)$ with

$$\begin{array}{rcl} a_i & \in & \{(J-1)(A-C+1), \ldots, 2A-C-1\} \quad \text{and} \\ b_i & \in & \{0, \ldots, A-2-(J-2)(A-C+1)\} \end{array} \tag{10.11}$$

occurs in the $J - 1$ cycles

$$Q_n \leftarrow -Q_n \leftarrow Q_n \leftarrow \ldots \quad (1 \leq n \leq J-1).$$

Since these are uncountably many different labels, we conclude that V_L has uncountably many elements for $L \le J-1$ in this case.

The finiteness argument for $J > 2$ and $C = ((J-2)A + J + 1)/(J-1)$. Let $r \in \{0, \dots, J\}$ be fixed and assume that $r \ge J - r$ (the contrary case $r < J - r$ can be treated in the same way). In this case we get from (10.7)

$$\begin{aligned} a &:= r(A - C + 1) = A - 2 - (J - r - 1)(A - C + 1) \quad \text{and} \\ b &:= (J - r)(A - C + 1) = A - 2 - (r - 1)(A - C + 1) \end{aligned}$$

Thus the only labelling which occurs in each of the J cycles

$$\begin{aligned} Q_n \xleftarrow{a} -Q_n \xleftarrow{b} Q_n \xleftarrow{a} \dots \quad (1 \le n \le r), \\ -Q_n \xleftarrow{a} Q_n \xleftarrow{b} -Q_n \xleftarrow{a} \dots \quad (1 \le n \le J - r). \end{aligned} \tag{10.12}$$

is the labelling $([ba]^\infty)$. Hence, for each constellation of J cycles there exist only two elements of V_L. The first of these elements is contributed by the J cycles in (10.12) with labelling $([ba]^\infty)$. The second element comes from the set of J cycles which emerges from (10.12) by adding the edge $-Q \xleftarrow{b}$ to the first r cycles in (10.12) and the edge $Q \xleftarrow{b}$ to the remaining cycles in (10.12). Let $\{W_1 \dots, W_J\}$ be a set of J walks in $\mathcal{G}_1(S)$ which have the following properties

$$\begin{aligned} &W_1, \dots, W_J \text{ have all the same labelling,} \\ &W_1, \dots, W_J \text{ end up in pairwise different states.} \end{aligned} \tag{10.13}$$

By Lemma 10.10 for $J > 2$ each of these walks ends up in a cycle of the shape (10.9). Thus we can construct each set of J walks which fulfils (10.13) by adding edges at the beginning of J cycles which fulfil (10.13) in a way such that the resulting walks again fulfil (10.13). By Proposition 10.9 the cycles which fulfil (10.13) are characterized by (10.12). We will show that to each constellation in (10.12) there correspond only finitely many sets of J walks which fulfil (10.13). This will imply that there are only finitely many sets of J walks fulfilling (10.13) and we are ready.

Let (10.12) be a given constellation of cycles. In order to get all walks which end up in this constellation we distinguish four cases.

Case 1: We add edges in a way such that all W_J remain in the J cycles given in (10.12). Thus by the above considerations this case contributes only 2 sets of J different walks.

In the following cases we have to keep track of the following fact. If we choose a labelling such that the walk W_ρ leaves the cycle $Q_{n_0} \leftarrow -Q_{n_0} \leftarrow Q_{n_0}$ then all walks which end up in cycles of the shape $Q_n \leftarrow -Q_n \leftarrow Q_n$ for $n > n_0$ also leave these cycles. This is due to the fact that $\pm\nu_n \subset \pm\nu_{n_0}$ for $n > n_0$ (cf. Table 1).

Case 2: Let $n_0 \ge 3$. We add the edges in a way that each W_ρ which ends up in $Q_n \leftarrow -Q_n \leftarrow Q_n$ leaves its cycle if $n \ge n_0$ and remains in it if $n < n_0$.

Suppose w.l.o.g. that for $0 \le i \le 2$ the walk W_{i+1} ends up in the cycle $Q_{n_0-i} \leftarrow -Q_{n_0-i} \leftarrow Q_{n_0-i}$. Then W_2 and W_3 remain in their cycles and W_1 leaves its cycle. Thus if we add the labelling $(\dots c_2 d_2 c_1 d_1)$ we must fulfil $c_s \in -\nu_{n_0-1}$ and $d_s \in \nu_{n_0-1}$. With these constraints we get the following possibilities for W_1: Either

$$W_1 : Q_{n_0-1} \leftarrow P_{n_0} \leftarrow Q_{n_0} \leftarrow -Q_{n_0} \leftarrow Q_{n_0} \leftarrow \dots$$

or

$$W_1 : -Q_{n_0-2} \leftarrow P_{n_0-1} \leftarrow P_{n_0} \leftarrow Q_{n_0} \leftarrow -Q_{n_0} \leftarrow Q_{n_0} \leftarrow \dots .$$

In the first case it ends up in the same state as

$$W_2 : Q_{n_0-1} \leftarrow -Q_{n_0-1} \leftarrow Q_{n_0-1} \leftarrow -Q_{n_0-1} \leftarrow Q_{n_0-1} \leftarrow \dots$$

in the second case it ends up in the same state as

$$W_3 : -Q_{n_0-2} \leftarrow Q_{n_0-2} \leftarrow -Q_{n_0-2} \leftarrow Q_{n_0-2} \leftarrow -Q_{n_0-2} \leftarrow Q_{n_0-2} \leftarrow \dots .$$

Anyway, we can not add more than two edges if we want to construct walks which meet condition (10.13).

Case 3: Let $n_0 = 2$. We add the edges in a way that each W_ρ which ends up in $Q_n \leftarrow -Q_n \leftarrow Q_n$ leaves its cycle for $n \geq 2$. and remains in it for $n = 1$.

Again there are two subcases to distinguish. The first subcase corresponds to the first subcase of Case 2, in the second subcase one gets three walks which end up in level 1. It is easily seen from the structure of $A(1)$ that at least two of them will end up in the same state after finitely many steps.

Case 4: Let $n_0 = 1$, i.e. we add the edges in a way that each W_ρ which ends up in $Q_n \leftarrow -Q_n \leftarrow Q_n$ leaves its cycle.

In this case three walks end up in level 1. Thus again at least two of them end up in the same state after finitely many steps.

Summing up we see that from each of these four cases there are contributed only finitely many walks. Thus each constellation (10.12) contributes only finitely many sets of J walks which fulfil (10.13). Since there are $J+1$ constellations we conclude that for $J > 2$ and $C = ((J-2)A+J+1)/(J-1)$ there exist only finitely many sets of J walks fulfilling (10.13). By Proposition 8.5 this implies that V_J is finite. $\square$

11. Some remarks on the case $2A = B + 3$

Let α be the base of a canonical number system in a quadratic number field, whose minimal polynomial $x^2 + Ax + B$ fulfills $C = 3$ where $C = 2A - B$. In this section we explicitly determine the sets V_L $(L \geq 2)$ related to those tiles $\mathcal{T}$, which correspond to this class of bases. We will prove the following result (note that $C = 3$ implies $A > 0$; thus we have no exceptional cases).

Theorem 11.1. *Let α be the base of a CNS with minimal polynomial x^2+Ax+B. Assume that $2A = B+3$ and let $\mathcal{T}$ be the tile corresponding to this number system. Then*

$$V_2 = \bigcup_{s_1 \neq s_2} V_2(s_1, s_2),$$

contains countably many elements. More precisely, the sets $V_2(s_1, s_2)$ contain the following elements.

$$\begin{aligned}
V_2(P_1,Q_1) &:= \{\Phi\,(0.d[A-2]^\infty) \mid 0 \leq d \leq A-3\},\\
V_2(P_1,-Q_1) &:= \{\Phi\,(0.[(A-3)0(B-1)]^r(A-2)d[(B-1)0]^\infty) \mid 1 \leq d \leq A-2\}\\
&\quad\cup\ \{\Phi\,(0.[(A-3)0(B-1)]^r(A-3)0d[A-2]^\infty) \mid A-1 \leq d \leq B-1\}\\
&\quad\cup\ \Phi\,(\{0.[(A-3)0(B-1)]^r(A-3)0(B-1)d[0(B-1)]^\infty) \mid 0 \leq d \leq A-3\},\\
V_2(P_1,-R) &:= \{\Phi\,(0.[0(A-1)(B-1)]^r0d[(B-1)0]^\infty) \mid A-1 \leq d \leq B-1\}\\
&\quad\cup\ \{\Phi\,(0.[0(A-1)(B-1)]^r0(A-2)d[0(B-1)]^\infty) \mid A-2 \leq d \leq B-2\}\\
&\quad\cup\ \{\Phi\,(0.[0(A-1)(B-1)]^r0(A-1)(B-1)d[A-2]^\infty) \mid 0 \leq d \leq A-3\},\\
V_2(P_1,P_2) &:= \{\Phi\,(0.d[0(B-1)]^\infty) \mid 0 \leq d \leq A-3\},\\
V_2(Q_1,-Q_1) &:= \{\Phi\,(0.[A-2]^\infty)\},\\
V_2(Q_1,-R) &:= \{\Phi\,(0.[0(B-1)(A-3)]^r0d[A-2]^\infty) \mid A-1 \leq d \leq B-1\}\\
&\quad\cup\ \{\Phi\,(0.[0(B-1)(A-3)]^r0(B-1)d[0(B-1)]^\infty) \mid 0 \leq d \leq A-3\}\\
&\quad\cup\ \Phi\,(\{0.[0(B-1)(A-3)]^r0(B-1)(A-2)d[(B-1)0]^\infty) \mid 1 \leq d \leq A-2\},\\
V_2(Q_1,-P_2) &:= \{\Phi\,(0.d[(B-1)0]^\infty) \mid 1 \leq d \leq A-2\},\\
V_2(Q_1,Q_2) &:= \{\Phi\,(0.[0(B-1)]^\infty)\}.
\end{aligned}$$

The sets $V_2(-P_1,-Q_1)$, $V_2(-P_1,Q_1)$, $V_2(-P_1,R)$, $V_2(-P_1,-P_2)$, $V_2(-Q_1,R)$, $V_2(-Q_1,P_2)$ and $V_2(-Q_1,-Q_2)$ are defined via the identity

$$V_2(s_1,s_2) = \{\Phi(0.[B-1]^\infty) - x \mid x \in V(-s_1,-s_2)\}.$$

Furthermore it is clear, that $V_2(s_1, s_2) = V_2(s_2, s_1)$*. For the remaining pairs* $s_1, s_2 \in S$*,* $s_1 \neq s_2$*, the corresponding set* $V_2(s_1, s_2) = \emptyset$*.*

Proof. The proof of this theorem is done in the following way. First we need the graph $\mathcal{G}_1(S)$, which has been constructed in the previous section. It is easy to construct the product graph $\mathcal{G}_2(S)$ using Definition 8.3. A lengthy, but easy calculation yields the graph depicted in Figure 5.

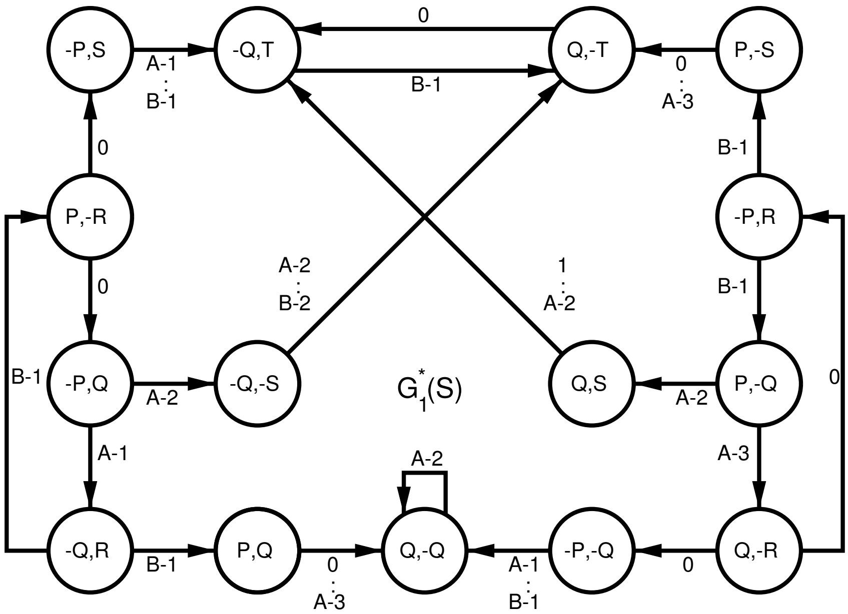

FIGURE 5. The graph $\mathcal{G}_2(S)$

By Proposition 8.5, the labeling of each infinite walk in $\mathcal{G}_2(S)$ starting at a node (s_1, s_2) yields an element of $V_2(s_1, s_2)$. Thus the characterization of the sets $V_2(s_1, s_2)$ claimed in the theorem can easily be extracted from Figure 5. □

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