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On a generalization of the radix representation - a survey

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Dedicated to Hugh C. Williams on the occasion of his 60th Birthday

1 Introduction

Let $P(X) = p_d X^d + \cdots + p_0 \in \mathbb{Z}[X]$ with $p_0 \geq 2$ and $\mathcal{N} = \{0, 1, \ldots, p_0 - 1\}$. If $p_d = 1$ then $P(X)$ is called a CNS[1]-polynomial, whenever every non-zero element of $R := \mathbb{Z}[X]/P\mathbb{Z}[X]$ can be written uniquely in the form

$$a_0 + a_1 x + \cdots + a_\ell x^\ell \tag{1.1}$$

with $a_0, ..., a_\ell \in \mathcal{N}, a_\ell \neq 0$; here x denotes the image of X under the canonical epimorphism from $\mathbb{Z}[X]$ to R. This means that every coset $Q + P\mathbb{Z}[X]$ $(Q \in \mathbb{Z}[X])$

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[1]CNS is the abbreviation for Canonical Number System.

contains a polynomial with coefficients belonging to $\mathcal{N}$. The polynomial (1) will be called the CNS-representation of the coset. The set of CNS-polynomials will be denoted by $\mathcal{C}$.

This concept was introduced by the fourth author [23] as a natural generalization of bases of canonical number systems or radix representations in algebraic number fields, which were defined in [10] and [6]. A complex number α is the base of a canonical number system in the algebraic number field $\mathbb{K}$ if and only if α is a zero of an irreducible CNS-polynomial and $1, \alpha, \ldots, \alpha^{d-1}$ is an integral basis of $\mathbb{Z}_{\mathbb{K}}$, where $\mathbb{Z}_{\mathbb{K}}$ denotes the maximal order of $\mathbb{K}$.

In this paper we give a survey on results on canonical number systems in algebraic number fields and on CNS-polynomials. The "backward" division of polynomials, which will be defined in Section 2, plays a special rôle. Changing the bases $1, X, X^2, \ldots$ appropriately one obtains a mapping $\tau_P : \mathbb{Z}^d \mapsto \mathbb{Z}^d$, which enables one to decide quite efficiently whether $P \in \mathcal{C}$ or not. The properties and applications of τ_P will be described in Sections 4 and 5. This mapping can be generalized further and one obtains a decomposition of $\mathbb{R}^d$ into convex sets.

The CNS-concept was generalized to simultaneous representations of tuples of integers in [9] and studied recently in [24]. The generalization for polynomials over finite fields can be found in [27], where the complete characterization of CNS-polynomials over finite fields is given. Because of lack of space we will not deal with these generalizations.

2 "Backward" division of polynomials

If $p_d = 1$ then it is clear that every coset of $\mathbb{Z}[X]/P(X)\mathbb{Z}[X]$ has an element of degree at most $d-1$ with coefficients, which can be arbitrarily large, say

$$A(x) = A_0 + A_1 x + \cdots + A_{d-1} x^{d-1}. \tag{2.1}$$

To transform $A(x)$ into the form (1) it is straightforward to use the following "backward" division process. Let $\mathbb{Z}'[X] = \{A(X) \in \mathbb{Z}[X] \ : \ \deg A < d\}$ and

$$T_P(A) = \sum_{i=0}^{d-1} (A_{i+1} - q p_{i+1}) X^i,$$

where $A_d = 0$ and $q = [A_0/p_0]$. Then $T_P \ : \ \mathbb{Z}'[X] \to \mathbb{Z}'[X]$ and

$$A(X) = a_0 + X T_P(A), \text{ with } a_0 = A_0 - q p_0.$$

If it causes no confusion we omit the subscript P.

Thus, to obtain the CNS representation of $A(X)$ one has to compute the iterates $T(A), T^2(A), \ldots$ until $T^\ell(A) = 0$ for some $\ell > 0$. This "backward" division process can become divergent (e.g. $A(X) = -1$ for $P(X) = X^2 + 4X + 2$) or ultimately periodic (e.g. $A(X) = -1$ for $P(X) = X^2 - 2X + 2$) or can terminate after finitely many steps (e.g. $A(X) = -1$ for $P(X) = X^2 + 2X + 2$). This means that $\mathcal{C}$ is a proper subset of $\mathbb{Z}[X]$.

Let

$$\Pi(P) = \{A \ : \ T_P^\ell(A) = A \text{ for some } \ell > 0\}$$

denote the set of periodic elements with respect to the mapping T_P. It is clear that we always have $0 \in \Pi(P)$. Moreover $P(X) \in \mathcal{C}$ if and only if $\Pi(P) = \{0\}$. Hence, it is enough to study the map $T_P \ : \ \mathbb{Z}^d \to \mathbb{Z}^d$ defined as

$$T_P((A_0, \ldots, A_{d-1})) = (A_1 - q p_1, \ldots, A_{d-1} - q p_{d-1}, -q p_d), \ q = [A_0/p_0].$$

The following theorem is easy to prove.

Theorem 2.1 (Analytical conditions) *If $P(X) \in \mathcal{C}$ then*

- *all roots of $P(X)$ are lying outside the closed unit circle, and*
- *all real roots of $P(X)$ are less than -1.*

This theorem implies that if $P \in \mathcal{C}$ then $p_0 \geq 2$.
For monic $P(X) \in \mathbb{Z}[X]$ and for $c > 0$ let

$$P_c = \{A(X) = \sum_{i=0}^{d-1} A_i X^i \in \mathbb{Z}'[X] \ : \ |A_i| \leq c, \ 0 \leq i \leq d-1\}.$$

The next theorem was proved for irreducible polynomials in [20], for square-free polynomials in [23] and in the general case in [3] and [24].

Theorem 2.2 *Assume that for $P(X) \in \mathbb{Z}[X]$ the conditions of Theorem 2.1 hold. Then there exists a computable constant $c > 0$ such that $P(X) \in \mathcal{C}$ if and only if every $A(X) \in P_c$ has a CNS-representation.*

As the set P_c is finite for all $c > 0$, the CNS property is algorithmically decidable. Unfortunately the constant c in Theorem 2.2 is usually large, therefore it is hard to apply it (cf. [20]). However, there are important special cases of CNS-polynomials. The first was discovered by B. Kovács [19] and proved in the general case in [23].

Theorem 2.3 *Let $P(X) = X^d + p_{d-1}X^{d-1} + \cdots + p_0$. If $p_0 \geq 2$ and $p_{d-1} \leq \cdots \leq p_1 \leq p_0$ and $P(X)$ is not divisible by a cyclotomic polynomial then $P(X) \in \mathcal{C}$.*

The second special case appeared in [2] and has been generalized slightly in [26] and [3]:

Theorem 2.4 *Assume that $p_2 \geq 0, \ldots, p_{d-1} \geq 0$, $\sum_{i=1}^d p_i \geq 0$ and $p_0 > \sum_{i=1}^d |p_i|$, then $P(X) \in \mathcal{C}$.*

3 CNS in algebraic number fields

By the remark after Theorem 2.1 the bases of radix representations in $\mathbb{Q}$ correspond to the roots of $X + p_0$ with $p_0 \geq 2$, i.e., they are negative integers. The negative base representations were studied for the first time in [7]. The radix representations in the Gaussian integers $\mathbb{Z}[\sqrt{-1}]$ were studied by Knuth [13] (see also [14]) and by Penney [22]. In [10] all CNS in Gaussian integers were characterized. This characterization has been extended to algebraic integers in real and imaginary quadratic number fields in [11, 12]. The same characterization was established independently in [6]. Brunotte [4] gave a new proof without assuming irreducibility of the quadratic polynomials.

Theorem 3.1 *We have $P(X) = X^2 + p_1X + p_0 \in \mathcal{C}$ if and only if $-1 \leq p_1 \leq p_0$ and $p_0 \geq 2$.*

A. Kovács [15, 17] considered the possible length of periods and the size of $\Pi(P)$ corresponding to irreducible quadratic polynomials $P(X)$ with two complex roots. Let θ be a root of $P(X)$ and assume that its representation in the canonical integral basis $1, \omega$ of $\mathbb{Q}(\theta)$ is $\theta = a + b\omega, b > 0$. Then he proved in [17] among others that the cardinality of $\Pi(P)$ can only be $b, b+1$ or $b+2$. A full characterization of $\Pi(P)$ for quadratic polynomials can be found in Thuswaldner [28].

For cubic number fields much less is known. Körmendi [21] described up to one possible exception all bases of CNS in $\mathbb{Q}(\sqrt[3]{2})$ and B. Kovács and Pethő [20] in all but one totally real fields with discriminant at most 564. In the field defined by one root of the polynomial $X^3 + 1749X^2 + 5975X + 5108$ their result was not complete, because the constant appearing in Theorem 2.2 was to large. These gaps were filled in [4]. In [1] all CNS in infinite parametric families of number rings were established.

After some computation and proving that the conditions are necessary, Gilbert [6] proposed the following conjecture for irreducible cubic polynomials.

Conjecture 3.2 Let $P = X^3 + p_2X^2 + p_1X + p_0$. Then $P \in \mathcal{C}$ if and only if

(i) $p_0 \geq 2$,
(ii) $p_2 \geq 0$,
(iii) $p_1 + p_2 \geq -1$,
(iv) $p_1 - p_2 \leq p_0 - 2$,
(v) $p_2 \leq \begin{cases} p_0 - 2, & \text{if} \quad p_1 \leq 0, \\ p_0 - 1, & \text{if} \quad 1 \leq p_1 \leq p_0 - 1, \\ p_0, & \text{if} \quad p_1 \geq p_0. \end{cases}$

We will come back later to this conjecture, but will mention already here that the situation is much more complicated if $p_0 \geq 6$.

For higher degree fields nearly nothing is known. The only general result is due to B. Kovács [19].

Theorem 3.3 *There exists in $\mathbb{Z}_{\mathbb{K}}$ a CNS if and only if $\mathbb{Z}_{\mathbb{K}}$ is monogenic, i.e., there exists an $\alpha \in \mathbb{Z}_{\mathbb{K}}$ such that $\{1, \alpha, \ldots, \alpha^{d-1}\}$ is an integral basis in $\mathbb{Z}_{\mathbb{K}}$.*

Combining this theorem with a result of Győry [8] we obtain that up to translation by integers there exist only finitely many CNS in $\mathbb{Z}_{\mathbb{K}}$.

4 Brunotte's mapping

As we mentioned before, Theorem 2.2 is not efficient enough to decide the CNS property of a polynomial. Brunotte [4] observed that the basis transformation

$$\begin{aligned} \{1, x, \ldots, x^{d-1}\} &\rightarrow \{w_1, \ldots, w_d\}, \\ w_j &= \sum_{i=j}^{d} p_i x^{i-j}, \ j = 1, \ldots, d \end{aligned}$$

of R implies a nicer and much better applicable transformation as T_P is. Indeed, if

$$\begin{aligned} A(x) &= \sum_{j=1}^{d} \bar{A}_j w_j, \quad \text{then} \\ T_P(A) &= -tw_1 + \sum_{j=2}^{d} \bar{A}_{j-1} w_j, \text{ where } t = \left[\frac{p_1\bar{A}_1 + \cdots + p_d\bar{A}_d}{p_0}\right]. \end{aligned}$$

Hence T_P implies the mapping $\tau_P : \mathbb{Z}^d \rightarrow \mathbb{Z}^d$

$$\tau_P(\underline{A}) = \left(-\left[\frac{p_1A_1 + \cdots + p_dA_d}{p_0}\right], A_1, \ldots, A_{d-1}\right),$$

where $\underline{A} = (A_1, \ldots, A_d)$. The mapping τ_P will be called **Brunotte's mapping**. Scheicher and Thuswaldner [26] made the same discovery independently.

Brunotte's mapping is easy to implement and it is immediately clear that $P \notin \mathcal{C}$ if either the analytical conditions of Theorem 2.1 do not hold or there exists $0 \neq \underline{A} \in \mathbb{Z}^d$ and $\ell > 0$ such that $\tau_P^\ell(\underline{A}) = \underline{A}$. Its importance relies on the following theorem, which is in some sense the converse of the last statement and makes it possible to decide the CNS property. Moreover it enables a far reaching generalization. It was proved originally in [4] and refined in [3]. The present version was published in [1].

Theorem 4.1 *Suppose that $E \in \mathbb{Z}^d$ has the following properties:*

- $(1, 0, \ldots, 0) \in E$,
- $-E \subseteq E$,
- $\tau_P(E) \subseteq E$,
- *for each $e \in E$ there exists some $\ell > 0$ with $\tau_P^\ell(e) = 0$.*

Then $P(X) \in \mathcal{C}$.

Such a set E will be called the *set of witnesses* of $P \in \mathcal{C}$.

5 Applications of Brunotte's mapping

Applying Theorem 4.1 Brunotte was able to characterize all CNS trinomials [5].

Theorem 5.1 *If $d > 2$ then the following assertions hold:*

(i) *$X^d + bX + c$ belongs to $\mathcal{C}$ if and only if $-1 \leq b \leq c - 2$,*
(ii) *if $1 < q < d$ and $q \not| d$ then $X^d + bX^q + c \in \mathcal{C}$ if and only if $0 \leq b \leq c - 2$.*

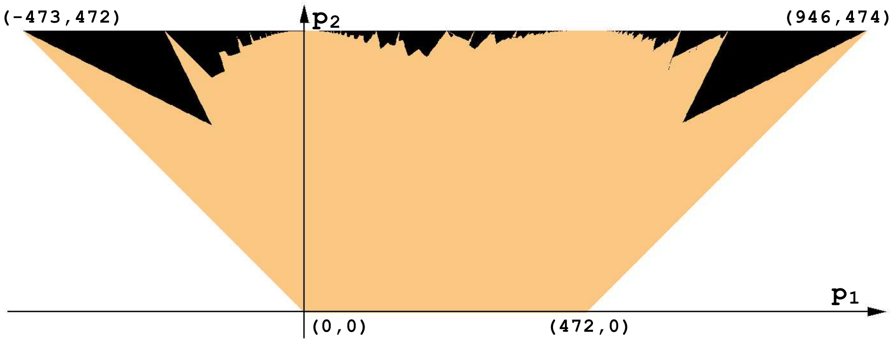
Akiyama et al. [1] examined Gilbert's Conjecture 3.2 and proved it in some cases, e.g. if

- $p_1 \leq -1, p_2 \leq p_0 - 2$ and $-1 \leq p_1 + p_2 \leq 0$,
- $p_1 \leq -1$, $0 \leq p_2 < \min\{p_0 - 1, 2p_0/3\}$ and $1 + p_1 + p_2 \geq 0$,
- $\leq p_1 \leq p_0 - 1$ and $0 \leq p_2 \leq (2p_0 - 1)/3$.

On the other hand they found that Gilbert's Conjecture does not hold if $p_0 \geq 6$. We present some counterexamples:

(i) $\mathbf{p_1 \leq 0}$. Let $2 \leq p_1 + p_2 \leq -p_1$ and $p_0 \leq \min\{p_2 - p_1, p_1 + 2p_2 + 1\}$ then $(1, -1, -1)$ is a periodic element whose period is $(1, -1, -1); 2, 1, -1, -1$. Here and in the sequel we present a period by giving a vector and the first coordinate of the following vectors.
(ii) $\mathbf{1 \leq p_1 \leq p_0 - 1}$. Let $p_0 \geq 28$ and $\frac{7p_0 - 5p_2}{6} + 1 \leq p_1 \leq -p_0 + \frac{3}{2}p_2$. Then the element $(1, -3, 1)$ is periodic with period $(1, -3, 1); 3, -2, -2, 3, 1, -3$.
(iii) $\mathbf{p_1 > p_0}$. Let $p_0 + \frac{1}{2}p_2 + 1 \leq p_1 < p_0 + \frac{2}{3}p_2 - \frac{1}{3}$. Then the element $(3, -2, 1)$ is periodic with period $(3, -2, 1); -2, 1, 1, -2$.

We visualize the situation with the example $p_0 = 474$. In this case there are $396,830$ CNS-polynomials and $52,046$ polynomials that violate Gilbert's conjecture. The point (p_1, p_2) on Picture 1 corresponds to the polynomial $X^3 + p_2X^2 + p_1X + p_0$. The displayed region is defined by the inequalities from Conjecture 3.2. The gray points correspond to members of $\mathcal{C}$ and the black ones to those, which violate Gilbert's conjecture. From this picture it is to be expected that the set of cubic CNS polynomials has a complicated structure.



Picture 1. CNS polynomials for $p_0 = 474$.

Theorem 2.1 implies that for fixed degree and given $p_0 \geq 2$ there exist only finitely many CNS-polynomials. Especially interesting is the case $p_0 = 2$, i.e. the generalizations of the binary expansion. Using Brunotte's algorithm A. Kovács [18] computed all binary CNS polynomials of degree $d \leq 8$. The result of his computation is displayed in the next table.

Degree	1	2	3	4	5	6	7	8
Number of CNS-polynomials	1	3	4	12	7	25	12	20

To show how hard it is to decide whether a polynomial belongs to $\mathcal{C}$ we give two examples: For $X^8 + 2X^7 + 3X^6 + 3X^5 + 3X^4 + 3X^3 + 3X^2 + 3X + 2$ the smallest set of witnesses has $241,719$ elements, while for $X^3 + 317X^2 + 632X + 317$ has $1,308,322$ elements.

A natural question is whether the CNS-property belongs to the NP or to the coNP class? The above examples indicate that the CNS-property cannot be decided in polynomial time. In the other direction Scheicher and Thuswaldner [26] noticed that if $P(X) = X^3 + 196X^2 + 341X + 199$ then the length of the period of $(-11, 10, -6)$ is 84. They conjecture that already cubic polynomials can have arbitrarily long cycles. Note that $(3, -2, 1)$ is another periodic point of τ_P, but with period length 7.

6 Generalization of Brunotte's mapping

In an earlier stage of our investigations we tried to prove algebraic properties of the set $\mathcal{C}$. It turned out that $\mathcal{C}$ is not closed under addition, multiplication and incrementation by 1. However, some of these algebraic properties are valid for large subsets of $\mathcal{C}$. Especially, the examples (e.g. $x^3 + 80x^2 + 117x + 89$) where $P(x) \in \mathcal{C}$ but $P(x) + 1 \notin \mathcal{C}$ seem to be rather exceptional.

There are of course trivial algebraic results (which do not show anything new) if one appropriately restricts to subsets of $\mathcal{C}$. Let for example M be the set of CNS polynomials of degree 2 or of degree 3 which satisfy the assumptions of Proposition 3.3 in [1]. Then, if $Q = X + k, k \geq 2$ and $P \in M$ then $P + Q \in \mathcal{C}$.

The only non-trivial algebraic result was proved in [5]. It asserts that if $P(X) \in \mathcal{C}$ and $k \geq 1$ then $P(X^k) \in \mathcal{C}$.

A closer look at $\mathcal{C}$ showed that it (or a related set) has to be the union of convex bodies. To show this property we followed Paul Erdős instruction: "If you cannot solve a problem, then try to generalize it and solve the more general problem." Brunotte's mapping allows such a generalization.

Let $r=(r_1,\dots,r_d)\in\mathbb{R}^d, r_d\neq 0$. With r we associate the mapping $\tau_r\ :\ \mathbb{Z}^d\to\mathbb{Z}^d$ by the following way: if $a=(a_1,\dots,a_d)\in\mathbb{Z}^d$ then let

$$\tau_r(a)=(-[ra],a_1,\dots,a_{d-1}),$$

where $ra=r_1a_1+\cdots+r_da_d$. Obviously this is a generalization of Brunotte's mapping by taking $r=(\frac{p_1}{p_0},\dots,\frac{p_d}{p_0})$.

Let

$$\mathcal{C}_d=\{r\ :\ \text{for all } a\in\mathbb{Z}^d \text{ there exists } \ell>0 \text{ such that } \tau_r^\ell(a)=0\}.$$

The next theorem shows that the set of mappings τ_r has some convexity property.

Theorem 6.1 *Let $r_1,\dots,r_k\in\mathbb{R}^d$ and $a\in\mathbb{Z}^d$ be such that $\tau_{r_1}(a)=\cdots=\tau_{r_k}(a)$. Let s be any convex linear combination of $r_1,\dots,r_k$. Then we have $\tau_s(a)=\tau_{r_1}(a)=\cdots=\tau_{r_k}(a)$.*

This theorem implies immediately the following corollary

Corollary 6.2 *Let $r_1,\dots,r_k\in\mathbb{R}^d$ have the same period, i.e. $\tau_{r_1}^\ell(a)=\cdots=\tau_{r_k}^\ell(a), \ell=0,\dots,v$ and $a=\tau_{r_1}^v(a)$. Then if s lies in the convex hull of $r_1,\dots,r_k$ the mapping τ_s is periodic and has the same period as τ_{r_1}.*

For example, it is easy to check that for the plane vectors $r_1=\left(\frac{381}{254},\frac{253}{254}\right), r_2=\left(\frac{421}{254},\frac{253}{254}\right)$ and $r_3=\left(\frac{344}{254},\frac{176}{254}\right)$ the corresponding mappings have the same period $(-2,1);3,-2,1,1,-2$, hence, the corresponding mapping for any point lying in the triangle r_1,r_2,r_3 has this period too.

A three-dimensional example is: $r_1=\left(\frac{382}{254},\frac{253}{254},\frac{1}{254}\right), r_2=\left(\frac{421}{254},\frac{253}{254},\frac{1}{254}\right)$ and $r_3=\left(\frac{344}{254},\frac{176}{254},\frac{1}{254}\right)$. Here is the period $(3,-2,1);-2,1,1,-2,3$.

Theorem 4.1 can be generalized for this setting.

Theorem 6.3 *Let $r_1,\dots,r_k\in\mathbb{R}^d$ and denote by H the convex hull of $r_1,\dots,r_k$. For $z\in\mathbb{Z}^d$ take $m(z)=\min_{1\le i\le k}\{-[r_iz]\}$ and $M(z)=\max_{1\le i\le k}\{-[r_iz]\}$. Suppose that there exists a finite set E, which satisfies the following conditions:*

- *$\pm e\in E$ for all d-dimensional unit vectors e,*
- *for each $z=(z_1,\dots,z_d)\in E$ and*

  $$j\in[\min\{m(z),-M(-z)\},\max\{-m(-z),M(z)\}]\cap\mathbb{Z}$$

  we have $(j,z_1,\dots,z_{d-1})\in E$,
- *$\cap_{j=1}^\infty\tau_{r_i}^j(E)=\{0\}$ for each $i\in\{1,\dots,k\}$.*

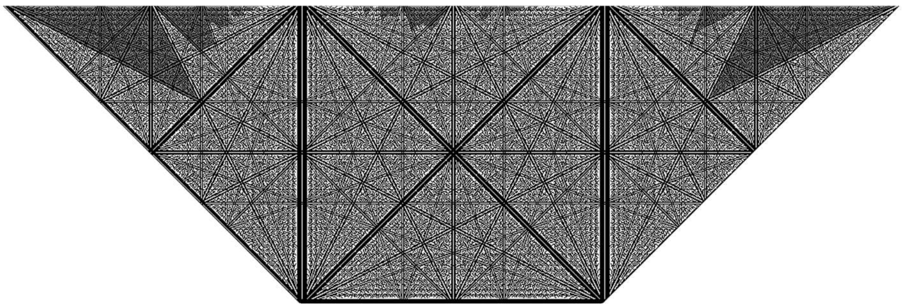
Then $H\subseteq\mathcal{C}_d$.

For example the square with vertices $\left(\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{3},\frac{2}{3}\right),\left(\frac{2}{3},\frac{1}{3}\right),\left(\frac{2}{3},\frac{1}{3}\right)$ is a subset of $\mathcal{C}_2$ as one can show by using the set of witnesses $E=E_1\cup(-E_1)\cup\{(0,0)\}$, where $E_1=\{(1,0),(0,1),(1,-1),(1,1),(2,-1),(1,-2),(2,0),(0,2),(1,2),(2,-2)\}$.

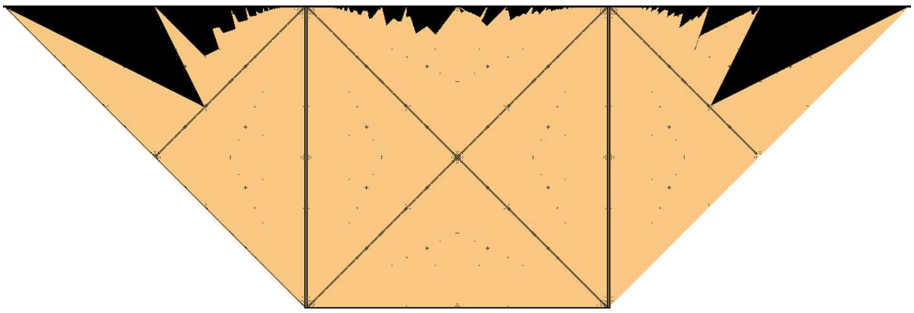
It is easy to see that $\mathcal{C}_2$ is a subset of the region

$$R_2=\{(\gamma_1,\gamma_2)\ :\ -1\le\gamma_1<2, 0\le\gamma_2<1,-\gamma_1\le\gamma_2<\gamma_1+1\}.$$

On Pictures 2 and 3 we present two approximations of $\mathcal{C}_2$. We displayed there all $(\gamma_1,\gamma_2)=\left(\frac{p_1}{p_0},\frac{p_2}{p_0}\right)\in R_2$ with $p_0,p_1,p_2\in\mathbb{Z}$. For Picture 2 we have chosen $p_0=60$, and for Picture 3 we took $p_0=174$. The light-gray points belong and the dark-gray points do not belong to $\mathcal{C}_2$. The status of the points lying on the black lines could not be decided for the chosen precision. However, it can be shown that a considerable part of the black points does indeed belong to $\mathcal{C}_2$.



Picture 2. An approximation of $\mathcal{C}_2$, $p_0 = 60$.



Picture 3. Better approximation of $\mathcal{C}_2$, $p_0 = 174$.

The top boundary of Pictures 1 and 3 seems to be very similar. Unfortunately we do not understand yet the relation between the two sets. By the last theorem $\mathcal{C}_d$ is the union of convex sets, but it is not clear whether finite or countably many sets appear in this union.

References

[1] S. Akiyama, H. Brunotte and A. Pethő, *Cubic CNS polynomials, notes on a conjecture of W.J. Gilbert*, J. Math. Anal. and Appl., to appear

[2] S. Akiyama and A. Pethő, *On canonical number systems*, Theoret. Comput. Sci., **270** (2002), 921 - 933.

[3] S. Akiyama and H. Rao, *New criteria for canonical number systems*, preprint

[4] H. Brunotte, *On trinomial bases of radix representations of algebraic integers*, Acta Sci. Math. (Szeged), **67** (2001), 407-413.

[5] H. Brunotte, *Characterization of CNS trinomials*, Acta Sci. Math. (Szeged), **68** (2002), 673-679.

[6] W.J. Gilbert, *Radix representations of quadratic fields*, J. Math. Anal. Appl., **83** (1981), 264-274.

[7] V. Grünwald, *Intorno all'aritmetica dei sistemi numerici a base negativa con particolare riguardo al sistema numerico a base negativo-decimale per lo studio delle sue analogie coll'aritmetica ordinaria (decimale)*, Giornale di matematiche di Battaglini, **23** (1885), 203-221, 367.

[8] K. Győry, *Sur les polynômes à coefficients entiers et de discriminant donné, III*, Publ. Math. Debrecen, **23** (1976), 141-165.

[9] K.-H. Indlekofer, I. Kátai and P. Racskó, *Number systems and fractal geometry*, Probability theory and applications, Essays to the Mem. of J. Mogyoródi, Math. Appl., **80** (1992), 319 -334.

[10] I. Kátai and J. Szabó, *Canonical number systems for complex integers*, Acta Sci. Math. (Szeged), **37** (1975), 255-260.

[11] I. Kátai and B. Kovács, *Kanonische Zahlensysteme in der Theorie der quadratischen algebraischen Zahlen*, Acta Sci. Math. (Szeged), **42** (1980), 99-107.

[12] I. Kátai and B. Kovács, *Canonical number systems in imaginary quadratic fields*, Acta Math. Acad. Sci. Hungar., **37** (1981), 159-164.

[13] D. E. Knuth *An imaginary number system*, Comm. ACM, **3** (1960), 245-247.

[14] D. E. Knuth *The Art of Computer Programming, Vol. 2 Semi-numerical Algorithms*, Addison Wesley (1998) London 3rd-edition.

[15] A. Kovács, *On expansions of Gaussian integers with non-negative digits*, Math. Pannon., **10** (1999), 177-191.

[16] A. Kovács, *On the computation of attractors for invertible expanding linear operators in* $\mathbb{Z}^k$, Publ. Math. Debrecen, **56** (2000), 97-120.

[17] A. Kovács, *On canonical expansions of integers in imaginary quadratic fields*, Acta Math. Hung., **93** (2001), 347-357.

[18] A. Kovács, *Generalized binary number systems*, Ann. Univ. Sci. Budap. Rolando Etvös, Sect. Comput., **20** (2001), 195-206.

[19] B. Kovács, *Canonical number systems in algebraic number fields*, Acta Math. Acad. Sci. Hungar., **37** (1981), 405-407.

[20] B. Kovács and A. Pethő, *Number systems in integral domains, especially in orders of algebraic number fields,* Acta Sci. Math. (Szeged), **55** (1991) 287-299.

[21] S. Körmendi, *Canonical number systems in* $\mathbf{Q}(\sqrt[3]{2})$, Acta Sci. Math. (Szeged), **50** (1986), 351 - 357.

[22] W. Penney, *A 'binary' system for complex numbers*, J. Assoc. Comput. Mach., **12** (1965), 247-248.

[23] A. Pethő, *On a polynomial transformation and its application to the construction of a public key cryptosystem, Computational Number Theory*, Proc., Walter de Gruyter Publ. Comp. Eds.: A. Pethő, M. Pohst, H.G. Zimmer and H.C. Williams, 1991, pp 31-43.

[24] A. Pethő *Notes on CNS polynomials and integral interpolation*, to appear.

[25] K. Scheicher, *Kanonische Ziffernsysteme und Automaten,* Grazer Math. Ber., **333** (1997), 1-17.

[26] K. Scheicher and J.M. Thuswaldner, *On the characterization of Canonical Number Systems*, Osaka J. Math., to appear.

[27] K. Scheicher and J.M. Thuswaldner, *Digit systems in polynomial rings over finite fields*, Finite Fields Appl., to appear.

[28] J.M. Thuswaldner, *Attractors for invertible expanding linear operators and number systems in* $\mathbb{Z}^2$, Publ. Math. Debrecen, **58** (2001), 423-440.