A SURVEY ON TOPOLOGICAL PROPERTIES OF TILES RELATED TO NUMBER SYSTEMS

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Abstract. In the present paper we give an overview of topological properties of self-affine tiles. After reviewing some basic results on self-affine tiles and their boundary we give criteria for their local connectivity and connectivity. Furthermore, we study the connectivity of the interior of a family of tiles associated to quadratic number systems and give results on their fundamental group. If a self-affine tile tessellates the space the structure of the set of its "neighbors" is discussed.

1. Introduction and basic definitions

Let X be a complete metric space and let $f_i : X \to X$ $(1 \le i \le m)$ be injective contractions. In [31] it is proved that there is a unique compact non-empty set K satisfying

$$K = f_1(K) \cup \ldots \cup f_m(K).$$

$\{f_i\}_{1 \le i \le m}$ is called *iterated function system* (IFS for short). K is called the *attractor* of this IFS.

Let $\mathbf{A}$ be an expanding $d \times d$ matrix (i.e. a matrix each of whose eigenvalues is strictly greater than 1) and suppose that $|\det(\mathbf{A})| = m$ for some integer $m > 1$. Let $\mathcal{D} := \{a_1, \ldots, a_m\} \subset \mathbb{R}^d$ be a finite set of vectors. Then the non-empty compact set T which satisfies

$$T = \bigcup_{j=1}^{m} \mathbf{A}^{-1}(T + a_j) \tag{1.1}$$

is the attractor of the IFS $\{\mathbf{A}^{-1}(x + a_j)\}_{1 \le i \le m}$. The union on the right hand side of (1.1) will be denoted by $F(T)$. If the d-dimensional Lebesgue measure μ_d satisfies $\mu_d(T) > 0$ and $\mu_d((T + a_i) \cap (T + a_j)) = 0$ for $i \ne j$, we call $T = T(\mathbf{A}, \mathcal{D})$ a *self-affine tile*. $\mathcal{D}$ is often called the *digit set* of T.

Let $\mathbf{A}$ be an expanding $d \times d$ integer matrix and $\mathcal{D} \subset \mathbb{Z}^d$. Then we call $T = T(\mathbf{A}, \mathcal{D})$ an *integral self-affine tile*. If $\mathcal{D}$ is even a complete set of coset representatives for $\mathbb{Z}^d / A\mathbb{Z}^d$ then we say that $T = T(\mathbf{A}, \mathcal{D})$ is an integral *digit tile*.

Fundamental properties of self-affine tiles have been proved for instance in [9, 25, 37, 42, 43, 44, 62]. We also mention the survey papers [65, 66] and the monograph [27].

Self-affine tiles have been studied from many viewpoints. There is a natural connection to radix representations which has been explored for instance in [23, 32, 33, 34, 53] (here especially integral digit tiles play a role). Several authors study the fractal structure of their

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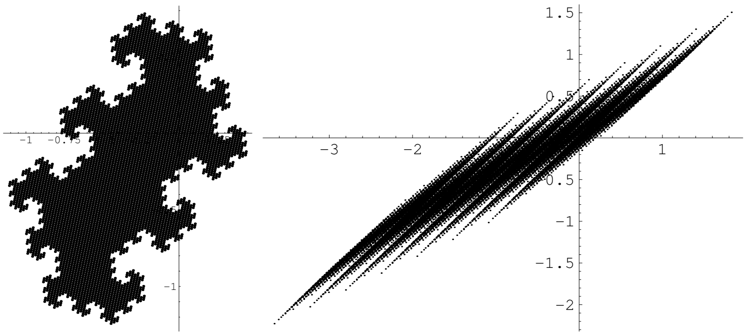

FIGURE 1. Examples of CNS tiles

boundary ([19, 24, 56, 63, 65, 66]) and their dynamical properties (cf. [58]). Furthermore, their connection to wavelets has been studied for instance in [13, 26, 67].

The present paper is mainly devoted to topological properties of self-affine tiles. As reviewed later, there are considerable results on the topology of a fixed self-affine tile. Our main goal is to show topological results for *families* of tiles. Especially we study integral digit tiles associated to quadratic *canonical number systems* (CNS for short). We recall the definition of quadratic CNS. Let $p(x) = x^2+Ax+B \in \mathbb{Z}[x]$ with $B \geq 2$ and $\mathcal{N} = \{0,\dots,B-1\}$. Then $(p,\mathcal{N})$ is a quadratic CNS if each $q \in \mathbb{Z}[x]/p(x)\mathbb{Z}[x]$ admits a representation of the shape

$$q(x) = \sum_{j=0}^{n} c_j x^j$$

with $c_j \in \mathcal{N}$. It was shown in [14, 22, 35, 36] that $(p,\mathcal{N})$ is a quadratic CNS if and only if

$$-1 \leq A \leq B, \quad B \geq 2.$$

To each quadratic CNS we can associate a self-affine tile. This tile is defined by $T = T(\mathbf{A},\mathcal{D})$ with

$$\mathbf{A} = \begin{pmatrix} 0 & -B \\ 1 & -A \end{pmatrix} \quad \text{and} \quad \mathcal{D} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} B-1 \\ 0 \end{pmatrix} \right\} \tag{1.2}$$

and can be interpreted as the set of all "α^{-1}-adic expansions" where α is a root of $p(x)$. (cf. for instance [6, 24, 34, 56]). In what follows we just call the tiles in (1.2) *quadratic CNS tiles*. We shall show later that the quantity

$$J := \max\left\{1, \left\lfloor \frac{B-1}{B-A+1} \right\rfloor \right\} \tag{1.3}$$

has great influence on the topological behavior of a quadratic CNS tile.

The choice $A = B = 2$ corresponds to Knuth's well-known "twin-dragon" (cf. [39, p. 206]). Pictures of the quadratic CNS tiles with parameters $A = 2, B = 3$ and $A = 6, B = 7$ are shown in Figure 1.

Of course it is possible to define CNS and their associated tiles of arbitrary degree in a similar way. For further details on CNS and their associated tiles we refer the reader to

[2, 4, 52, 55, 56]. In higher dimensional cases, topological study of such a family of tiles will be an interesting question.

2. Geometric properties of tiles and their boundary

An important tool for obtaining results on the topology of tiles is the so-called *contact matrix* (cf. [25, p. 134]) which is associated to an integral digit tile $T(\mathbf{A},\mathcal{D})$. Let $\{e_1,\dots,e_d\}$ be the canonical basis of the lattice $\mathbb{Z}^d$ and set $C_0 := \{0,\pm e_1,\dots,\pm e_d\}$. Recursively define the sets

$$C_n := \{k \in \mathbb{Z}^d \,|\, (\mathbf{A}k+\mathcal{D}) \cap (l+\mathcal{D}) \neq \emptyset \text{ for some } l \in C_{n-1}\}.$$

Set $R := \bigcup_{n\ge 0} C_n$ which is easily seen to be a finite set (cf. [25, Lemma 4.5]). Define the $|R|\times|R|$ matrix $\mathcal{C}$, called the *contact matrix*, by

$$c_{kl} := |(\mathbf{A}k+\mathcal{D}) \cap (l+\mathcal{D})| \quad (k,l \in R).$$

To this matrix we associate the *contact graph* $G(R)$. The vertices of $G(R)$ are the elements of the set R. There is a directed edge from k to l labeled by $a|a'$ $(a,a' \in \mathcal{D})$ if $\mathbf{A}k+a' = l+a$.

The following theorem gives criteria which are equivalent to the fact that a given integral digit tile "tiles" the space without overlap. Recall that a digit set of an integral digit tile is called *primitive*, if it is not contained in any proper $\mathbf{A}$-invariant sublattice of $\mathbb{Z}^d$.

Theorem 2.1 (cf. [25, 42, 43, 44, 64, 65])**.** *Let $T = T(\mathbf{A},\mathcal{D})$ be an integral digit tile with primitive digit set. Let T_0 be the d-dimensional unit cube centered at the origin and set $T_n := F^n(T_0)$. Then the following statements are equivalent*

- $\{T+x \,|\, x \in \mathbb{Z}^d\}$ *is a tiling of* $\mathbb{R}^d$.
- $\mu_d(T) = 1$.
- $\lim_{n\to\infty} \partial T_n = \partial T$ *(Hausdorff metric).*
- $\lim_{n\to\infty} \partial T_n$ *(Hausdorff metric) is not space filling.*
- *Let λ be the spectral radius of the contact matrix $\mathcal{C}$. Then $\lambda < |\det(\mathbf{A})|$.*

We mention that the last criterion concerning the spectral radius of the contact matrix can be checked algorithmically.

An integral digit tile which satisfies the equivalent statements of Theorem 2.1 will be called a *$\mathbb{Z}^d$-tile.*

Let $T = T(\mathbf{A},\mathcal{D})$ be a $\mathbb{Z}^d$-tile. We will derive a representation for ∂T (cf. [57, 60]). To this matter we first define the "set of neighbors" of T, i.e.,

$$S := \{k \in \mathbb{Z}^d \setminus \{0\} \,|\, T \cap (T+k) \neq \emptyset\}.$$

Since the $\mathbb{Z}^d$-translates of T tile $\mathbb{R}^d$ we have

$$\partial T = \bigcup_{k\in S} B_k$$

where we set $B_k := T \cap (T+k)$ for convenience. By (1.1) we have $\mathbf{A}B_k = (T+\mathcal{D}) \cap (T+\mathcal{D}+\mathbf{A}k)$ and thus

$$\mathbf{A}B_k := \bigcup_{a,a'\in\mathcal{D}} \left(B_{\mathbf{A}k+a'-a} + a\right).$$

In order to remove all elements B_k which are empty, define a graph $G(S)$ with set of vertices S. Furthermore, there is a label $a|a'$ from k to l if

$$\mathbf{A}k + a' = l + a \tag{2.1}$$

holds. It is now easy to see that

$$\mathbf{A}B_k = \bigcup_{k \xrightarrow{a|a'} l} (B_l + a) \tag{2.2}$$

(the union is extended over all edges in $G(S)$ leading away from k). Thus the sets B_k ($k \in S$) are the solutions of the *graph directed construction* in (2.2). It is known that they are even defined uniquely by this construction (cf. [20, Chapter 3]). For general IFS attractors with non-empty interior it is not so easy to get information on their boundary (cf. for instance [18, 45, 47, 61]).

In what follows, for each set $M \subset \mathbb{Z}^d$ we will denote by $G(M)$ the graph with vertices M and edges between these vertices defined by (2.1). In general, $G(R \setminus \{0\}) \subset G(S)$ holds. Also $G(R)$ can be used instead of $G(S)$ in order to describe the boundary. This is a bit more difficult to show (cf. [57, Section 2]) but has the advantage that $G(R)$ is in general easier to determine than $G(S)$. Both graphs have been used for instance in order to determine the dimension of the boundary of certain classes of tiles (cf. for instance [19, 56, 63, 65, 66]).

3. Connectivity and homeomorphy to a closed disc

In [38, Theorem 4.3] the following criterion was given (cf. also [29, Theorem 4.6] where a more general result is shown; also the local connectivity assertion is contained in [29, Theorem 4.6]; in [25, Theorem 2.5] a two-dimensional version can be found).

Theorem 3.1. *Let $T(\mathbf{A}, \mathcal{D})$ be a self-affine tile. Define*

$$\mathcal{E} := \{(a_i, a_j) \in \mathcal{D} \times \mathcal{D} \,|\, (T + a_i) \cap (T + a_j) \neq \emptyset\}.$$

Then T is a locally connected continuum if and only if for any two distinct $a_i, a_j \in \mathcal{D}$ there exists a subset $\{a_{i_1}, \dots, a_{i_k}\} \subseteq \mathcal{D}$ such that $a_{i_1} = a_i$, $a_{i_k} = a_j$ and $(a_{i_\ell}, a_{i_{\ell+1}}) \in \mathcal{E}$ for ($1 \leq \ell \leq k-1$).

Proof. The "only if" part is obvious. Thus we turn to the "if" part. By repeated use of (1.1) and the application of the condition, we see that any two points in T are ε-chain connected. As T is compact, this implies the result. □

Remark 3.2. *Note that if an integral self-affine tile T is connected, it is even arcwise connected and there exists a continuous mapping from* $[0, 1]$ *onto T (cf.* [29, Remark 2]*).*

We have the following consequence of Theorem 3.1. (cf. [6]).

Corollary 3.3. *Each quadratic CNS tile T is a locally connected continuum.*

The only thing to show is that T and $T + \begin{pmatrix}1\\0\end{pmatrix}$ have non-empty intersection in order to meet the conditions of Theorem 3.1. For tiles corresponding to CNS of degree three and four the connectivity has been shown in [3]. For tiles of degree higher than four it is not known whether they are always connected.

We also mention that in [28] it was shown that any integral self-affine tile having $|\mathcal{D}| = 2$ is connected (see also [38, Proposition 4.1]). Furthermore, in [38, Theorem 5.5] it was proved that to any expanding 2×2 integer matrix $\mathbf{A}$ there exists a digit set $\mathcal{D}$ such that $T(\mathbf{A}, \mathcal{D})$ is a connected integral self-affine tile.

A more difficult problem is to decide whether a self-affine tile T is homeomorphic to a closed disc (disclike, for short). We start with a result which holds for arbitrary IFS in the plane satisfying the following condition (cf. [20, p. 35]): An IFS $\{f_j\}_{1 \le j \le m}$ is said to fulfill the *open set condition* if there exists a non-empty bounded open set U such that

$$U \supset \bigcup_{j=1}^{m} f_j(U)$$

where the union is disjoint.

Theorem 3.4 (cf. [48])**.** *Let $\{f_i\}_{1 \le i \le m}$ with $f_i : \mathbb{R}^2 \to \mathbb{R}^2$ be an IFS satisfying the open set condition. Let K be the attractor of this IFS. If* $\mathrm{int}(K)$ *is connected then K is homeomorphic to a closed disc.*

Sketch of the proof. We confine ourselves to prove the result for a connected $\mathbb{Z}^2$-tile T having connected interior. In the general case one has to use the fact that any plane IFS with open set condition induces a “tiling” of $\mathbb{R}^2$ (cf. for instance [48] or [51, Lemma 2.3]).

Since T is the closure of its interior (cf. for instance [66, Theorem 2.1]) it is easy to see that the connectivity of $\mathrm{int}(T)$ implies that T has no cut point. Thus T is a locally connected continuum without cut points. From [41, Chap.10, §61 II, Theorem 4] that the boundary of each component of $\mathbb{R}^2 \setminus T$ is a simple closed curve.

We will now prove that $\mathbb{R}^2 \setminus T$ is connected. Suppose not. Then it has a bounded component C. Since T is a $\mathbb{Z}^2$-tile there has to exist at least one $v \in \mathbb{Z}^2$ such that $C \cap \mathrm{int}(T + v) \neq \emptyset$ (note that C is open and $T + v$ is the closure of its interior). Now there are two possibilities: firstly, $T + v$ is contained in $\bar{C}$. By iterating this argumentation we derive that there are infinitely many translates $v' \in \mathbb{Z}^2$ such that $T + v' \subset \bar{C}$, which is impossible because C is bounded. Secondly, $T + v$ is not contained in $\bar{C}$. This easily leads to a contradiction to the fact that $\mathrm{int}(T) + v$ is connected.

Thus $\mathbb{R}^2 \setminus T$ is connected and, hence, the only component of $\mathbb{R}^2 \setminus T$. This implies that $\partial(\mathbb{R}^2 \setminus T) = \partial T$ is a simple closed curve. This curve separates $\mathrm{int}(T)$ and $\mathrm{int}(\mathbb{R}^2 \setminus T)$. Since T is bounded it is a closed disc by Jordan's theorem. □

In general, it is not easy to decide whether $\mathrm{int}(K)$ is connected or not. In [5] connectivity of $\mathrm{int}(T)$ is shown for a class of quadratic CNS tiles. This leads to the following result.

Theorem 3.5. *Let T be a quadratic CNS tile. Then T is homeomorphic to a closed disc if and only if $J = 1$, i.e. $2A < B + 3$.*

Rough sketch of the proof. It is easy to check that T fulfills the open set condition with $U = \mathrm{int}(T)$. Thus in view of Theorem 3.4 we only have to show that $\mathrm{int}(T)$ is connected. The proof of this claim is rather involved. One constructs a “skeleton”, which is a connected subset of $\text{int}(T)$ consisting of infinitely many line segments. This skeleton can be shown to be dense in T which yields the connectivity of $\text{int}(T)$. □

This result gives a definite answer to the topological structure of quadratic CNS tiles with $J = 1$. Observe that the tile on the left hand side of Figure 1 belongs to this class.

Another approach for deciding whether T is homeomorphic to a closed disc is pursued in [10, 11]. In these papers criteria for disclikeness of self-affine tiles are established, which are easier to check algorithmically. Let T be a $\mathbb{Z}^2$-tile and let S be as in Section 2. Let $k \in S$. If $T \cap (T + k)$ is a single point, we call $T + k$ a *vertex neighbor*. If this set is uncountable, we call it an *edge neighbor*.

The following result is a special case of [10, Lemma 5.1]. We give it in the form occurring in [11, Proposition 1.1].

Theorem 3.6. *Let T be a $\mathbb{Z}^2$-tile which is homeomorphic to a closed disc. Then one of the following alternatives holds.*

- *T has no vertex neighbors and 6 edge neighbors $T \pm k$, $T \pm l$, $T \pm (k + l)$ with $\mathbb{Z}k + \mathbb{Z}l = \mathbb{Z}^2$.*
- *T has 4 edge neighbors $T \pm k$, $T \pm l$ and 4 vertex neighbors $T \pm k \pm l$ with $\mathbb{Z}k + \mathbb{Z}l = \mathbb{Z}^2$.*

A partial converse of this result was shown in [11, Theorems 2.1 and 2.2]. Let F be a finite subset of $\mathbb{Z}^d$. We say that a set $M \subset \mathbb{Z}^d$ is F-connected if for any $u, v \in M$ there exists $\{u_1, u_2 \ldots, u_n\} \subset M$ with $u = u_1$ and $v = u_n$ such that $u_{i+1} - u_i \in F$.

Theorem 3.7. *Let $T = T(\mathbf{A}, \mathcal{D})$ be a $\mathbb{Z}^2$-tile. Then the following holds.*

- *If $|S| \leq 6$ then T is homeomorphic to a disc if and only if $\mathcal{D}$ is S-connected.*
- *If $S = \{\pm k, \pm l, \pm k \pm l\}$ ($|S| = 8$) then T is homeomorphic to a disc if and only if T is $\{\pm k, \pm l\}$-connected.*

Thus it remains to determine the set of neighbors of T in order to apply these criteria. There exist algorithms which allow to determine the neighbors of a given tile (cf. [57, 60]). We will discuss these algorithms in Section 6.

All the above theorems are valid only in the plane. In $\mathbb{R}^3$ neither Theorem 3.4 nor Theorem 3.7 remains valid (cf. [51, Example 5.2] for a counterexample).

Let $T = T(\mathbf{A}, \mathcal{D})$ be a $\mathbb{Z}^2$-tile. For $|\mathcal{D}| \leq 4$ up to affine conjugacy all disclike tiles have been characterized (cf. [10, 21, 59]). In fact, for $|\mathcal{D}| = 2$ there are three, for $|\mathcal{D}| = 3$ there are seven and for $|\mathcal{D}| = 4$ there are 29 conjugacy classes of disclike tiles.

We conclude this section by mentioning the sets which do not fit totally in our context: the Rauzy Fractal and the Thurston tiling which has connection to substitutive dynamics and Pisot number systems (cf. [1, 7, 54]). It can be defined with help of a graph directed construction. For the smallest Pisot number, the corresponding tile is shown to be homeomorphic to a closed disc in [46].

4. Tiles which are not homeomorphic to a disc

In this section we deal with connected self-affine tiles whose interior is disconnected. In this setting the situation becomes very complicated and only few general results exist. However, many examples and even classes of tiles have been studied in some detail.

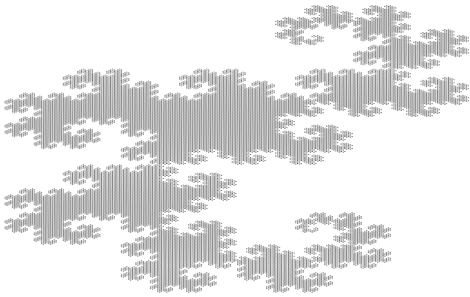

FIGURE 2. The Heighway dragon

We start with two results due to [51], where the structure of the components of $\mathrm{int}(T)$ is investigated.

Theorem 4.1 (cf. [51, Theorem 1.1, Theorem 1.2 and Lemma 2.2])**.** *Let K be the attractor of an IFS in $\mathbb{R}^2$ satisfying the open set condition and assume that K is connected. Then the closure of each component of* $\mathrm{int}(K)$ *is a locally connected continuum. If the number of components is finite, then the closure of at least one component is homeomorphic to a closed disc.*

A remark on the proof. The proof is very complicated. When the number of components is infinite, it relies on the following fact: Assume that $\overline{U_i}$ is not locally connected. Then $\overline{U_i}$ contains a *convergence continuum* (cf. [41, §49, VI, Definition and Theorem I]). From this fact one derives in a very tricky way, that also K is not locally connected. Because K is connected, this is a contradiction to [29, Theorem 4.6]. □

Under some additional conditions which look not so easy to check in general, in [51] it is shown that $\overline{U_i}$ is homeomorphic to a disc for each i.

Let T be a connected $\mathbb{Z}^2$-tile and let U be a component of its interior. If $\overline{U}$ has a hole (i.e. its complement has a bounded component) then some neighbor $T+k$ of T ($k \in S$) has to fill this hole at least partially. In [50, Proposition 3.1] it is shown that this implies that $T+k$, and hence also T, has a cut point. In other words, if T has no cut point then the closure of each interior component is homeomorphic to a disc. A criterion for cut points is given in [50, Theorem 1.1].

In [5] the following result was shown.

Theorem 4.2. *If T is a quadratic CNS tile with $J > 1$, i.e. $2A \geq B + 3$ then* $\mathrm{int}(T)$ *is disconnected. It even has infinitely many components.*

From Theorem 4.1 and 4.2 we conclude that the closure of each component of the interior of these quadratic CNS tiles is a locally connected continuum. (Observe that the right tile in Figure 1 belongs to this class of tiles.)

For many examples and even classes of tiles more information on the topological structure has been obtained.

Example.(The Heighway Dragon) The Heighway dragon (see Figure 2) is the attractor of the IFS

$$f_1(x) := \frac{1}{\sqrt{2}} R\left(\frac{\pi}{4}\right) x, \quad f_2(x) := \frac{1}{\sqrt{2}} R\left(\frac{3\pi}{4}\right) x + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

where $R(\theta)$ denotes the counterclockwise rotation by θ. References to this set are [12, 16, 17, 49].

Its topological structure was studied thoroughly in [49, Theorem 1.1]. In particular, it is proved that the Heighway dragon is the closure of a countable union of geometrically similar sets each of which is homeomorphic to a closed disc. Any two of these sets intersect in at most one point and for each three of them there are at least two with empty intersection.

Example. The quadratic CNS tile corresponding to the choice $A = 4, B = 5$ depicted in Figure 3 has been studied in [50]. It has been shown that this tile has no cutpoint and that the closure of every component of its interior is homeomorphic to a closed disc.

Also for the Eisenstein set and the Lévy dragon several topological properties have been deduced (cf. [8, 50, 51]).

Let T be a $\mathbb{Z}^2$-tile. The next question we want to address is the shape of the components of $\mathrm{int}(T)$. To our knowledge there exists no general result in this direction. Apart from the above-mentioned result on the Heighway dragon, for the Lévy dragon it has been shown that the interior components have at least 16 different shapes (cf. [8]).

Concerning quadratic CNS tiles we announce the following result which will appear in a forthcoming paper.

Theorem 4.3. *Let T be a quadratic CNS tile with $J = 2$. Let M be the closure of the component of* $\mathrm{int}(T)$ *containing zero. M is the attractor of a graph-directed construction with affine functions which is defined by means of an explicit graph with* 18 *vertices.*

This result is shown with help of a very lengthy but elementary calculation. In Figure 3 we depicted the quadratic CNS tile corresponding to the parameters $A = 4, B = 5$ together with the component of its interior which contains the origin.

5. The fundamental group of a tile

In this section we are concerned with the fundamental group of a self-affine tile. The fundamental group of a set K will be denoted by $\pi_1(K)$. $\mathbb{S}^1$ and $\mathbb{D}^2$ denote the unit circle and closed unit disc, respectively. Furthermore, $\mathbb{S}^2$ denotes the one point compactification of $\mathbb{R}^2$. For an open ball with radius r centered at x we will write $B_r(x)$. We will use freely definitions from [40, 41] and [30].

First we prove the following theorem.

Theorem 5.1. *Let $T \subset \mathbb{S}^2$ be a connected $\mathbb{Z}^2$-tile.*

- *If $\mathbb{S}^2 \setminus T$ is connected then $\pi_1(T)$ is trivial. If, moreover, T contains no cut points, it is homeomorphic to a closed disc.*

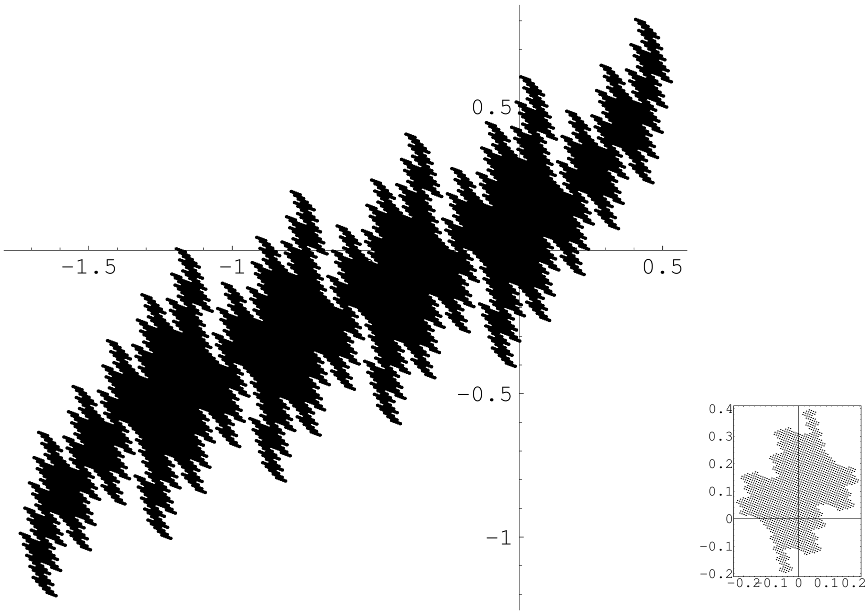

FIGURE 3. The quadratic CNS tile with $A = 4, B = 5$ and the component of its interior containing the origin.

- *If $\mathbb{S}^2 \setminus T$ is disconnected then $\pi_1(T)$ is uncountable and not free. Furthermore, T is not locally simply connected and has no universal cover.*

In particular, the fundamental group of a $\mathbb{Z}^2$-tile is either trivial or uncountable.

Proposition 5.2. *Let $K \subset \mathbb{S}^2$ be a locally arcwise connected set. If $\mathbb{S}^2 \setminus K$ is disconnected then K contains a non-trivial loop.*

Proof. Let U_1 and U_2 be two components of $\mathbb{S}^2 \setminus K$ and select $p \in U_1$ and $q \in U_2$. Then K cuts $\mathbb{S}^2$ between p and q. By [41, §62, VI, Theorem 1] this implies the existence of a simple closed curve $C \subset K$ which also cuts $\mathbb{S}^2$ between p and q. From this follows by [41, §59, IV, Theorem 4] that every set obtained from C by deformation in $\mathbb{S}^2 \setminus \{p, q\}$ can not be a single point. Since $K \subset \mathbb{S}^2 \setminus \{p, q\}$ this holds also for each deformation of C in K. Thus C is a non-trivial loop in K. This proves the result. □

Proposition 5.3. *Let $K \subset \mathbb{S}^2$ be a connected, locally arcwise connected continuum. Suppose that $\mathbb{S}^2 \setminus K$ has infinitely many components. Then the following assertions hold: (i) $\pi_1(K)$ is not free, (ii) $\pi_1(K)$ is uncountable, (iii) K is not locally simply connected and (iv) K has no universal cover.*

Proof. We show assertion (iii). Then [15, Theorem 3.1] yields the other assertions. Let $\{U_i\}_{i \geq 0}$ be the components of $\mathbb{S}^2 \setminus K$ and select $x_i \in U_i$. Let x be an accumulation point of the sequence x_i and fix $\varepsilon > 0$. Since K is a locally connected continuum there exists an $\eta > 0$ such that every pair of points with distance less than η can be connected by an arc of diameter ε (cf. [41, §50, II, Theorem 4]).

By a theorem of Schönflies (cf. [41, §61, II, Theorem 10]) we have $\lim_{i \to \infty} \operatorname{diam}(U_i) = 0$. The definition of x implies that there is an $m \in \mathbb{N}$ such that $U_m \subset B_{\eta/2}(x)$. Let $K_r :=$

$K \cap B_r(y)$. Then $K_{\eta/2}$ is a locally arcwise connected set with disconnected complement. Applying Proposition 5.2 yields a nontrivial loop $C_\eta \subset K_{\eta/2}$. This loop can be connected with x by an arc contained in K_ε. By [30, Proposition 1.5] this yields a nontrivial loop in $K_{\max(\varepsilon,\eta)}$ which is based in x. Thus K is not locally simply connected at x. This proves (iii). □

Lemma 5.4. *Let $T \subset \mathbb{S}^2$ be a connected $\mathbb{Z}^2$-tile with disconnected complement $\mathbb{S}^2 \setminus T$. Then $\mathbb{S}^2 \setminus T$ has infinitely many components.*

Proof. This can be proved by imitating part (ii) of the proof of [5, Theorem 7.1]. □

Proof of Theorem 5.1. Note that T is a locally arcwise connected continuum by Theorem 3.1 and the remark after it. Thus the first part follows from Kuratowski [41, §61, IV, Theorem 11], which implies that T is an absolute retract. Just note that for a set M which is an absolute retract we can extend each continuous function $f : \mathbb{S}^1 \to M$ to a continuous function $\tilde{f} : \mathbb{D}^2 \to M$. This implies that each loop in M is trivial. The second part follows from Lemma 5.4 together with Proposition 5.3. □

In [57, Section 7] an easy example of a self-affine tile whose complement is connected and which is not homeomorphic to a disc is discussed.

It is not trivial to decide whether the complement of a given self-affine tile is connected or not. We state the following result on quadratic CNS tiles.

Theorem 5.5. *Let T be a quadratic CNS tile and let $J > 1$, i.e. $2A \geq B + 3$. Then $\mathbb{S}^2 \setminus T$ is not connected.*

The proof of this result is rathor technical and will appear in a forthcoming paper. Theorem 5.5 and Theorem 5.1 imply the following result.

Theorem 5.6. *Let T be a quadratic CNS tile. If $J > 1$ then $\pi(T)$ is not free has uncountably many elements. Furthermore, T is not locally simply connected and and has no universal cover.*

6. Neighbors of tiles and points where many tiles coincide

Before we discuss algorithms which allow to determine the set of neighbors of a given tile, we want to illustrate the connection between neighbors and the topology of T with help of two figures (this connection can be seen for instance in Theorem 3.7).

In Figure 4 we see a tiling with tiles being homeomorphic to a disc. One can see that the tile in the center has six neighbors and that there are six points in which three different tiles meet.

However, in Figure 5 things are totally different. The interior of the tile T is disconnected in this case. So there are points in which the tile becomes very "thin". In some of these points two neighbors of T meet. On the other hand, because of these points T has neighbors which it meets in points, where some of the other tiles are "thin". This heuristic shows that the fact that the interior of a tile is disconnected increases the number of its neighbors.

The first algorithm for the determination of the neighbors of a $\mathbb{Z}^d$-tile $T(\mathbf{A}, \mathcal{D})$ was established in [60]. It starts from a large graph G which contains $G(S)$. $G(S)$ is constructed

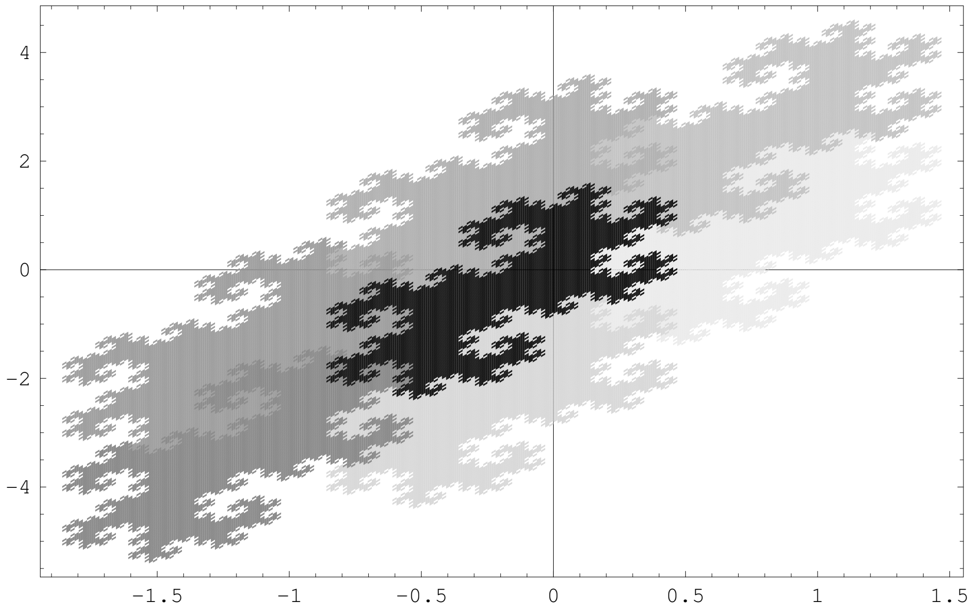

FIGURE 4. A tiling with tiles being homeomorphic to a disc.

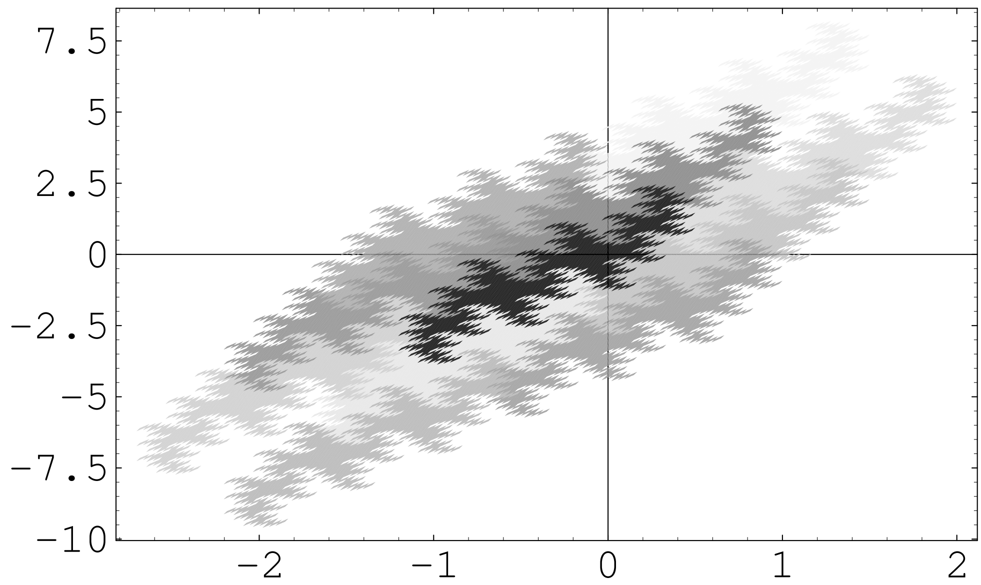

FIGURE 5. A tiling with tiles whose interiors have infinitely many components.

from G by successive pruning of vertices. This algorithm is easy to apply, however, if one of the eigenvalues of $\mathbf{A}$ is close to one in modulus the graph G becomes large. It does not easily apply to a family of tiles.

In [57] another algorithm is presented. It starts from the contact graph $G(R)$ which can be shown to be a subgraph of $G(S)$ (cf. [57, Corollary 4.3]). $G(S)$ is calculated by a certain graph product of $G(R)$ with itself. We start with the definition of this graph product.

Definition 6.1. *Let G_1 and G_1' be subgraphs of $G(\mathbb{Z}^d)$. The product $G_2 := G_1 \otimes G_1'$ is defined in the following way. Let r_1, s_1 be vertices of G_1 and r_1', s_1' be vertices of G_1'. Furthermore, let $\ell_1, \ell_1', \ell_2 \in \mathcal{D}$.*

- *r_2 is a vertex of G_2 if $r_2 = r_1 + r_1'$*
- *There exists an edge $r_2 \xrightarrow{\ell_1|\ell_2} s_2$ in G_2 if there exist*

$$r_1 \xrightarrow{\ell_1|\ell_1'} s_1 \in E(G_1) \quad \text{and} \quad r_1' \xrightarrow{\ell_1'|\ell_2} s_1' \in E(G_1')$$

with $r_1 + r_1' = r_2$ and $s_1 + s_1' = s_2$ or there exist

$$r_1 \xrightarrow{\ell_1'|\ell_2} s_1 \in E(G_1) \quad \text{and} \quad r_1' \xrightarrow{\ell_1|\ell_1'} s_1' \in E(G_1')$$

with $r_1 + r_1' = r_2$ and $s_1 + s_1' = s_2$.

Before we give the algorithm, we introduce the following notation. Let G be a graph. We denote by $\mathrm{Red}(G)$ the graph that emerges from G if all vertices of G, which are not the starting point of a walk of infinite length, are removed.

Algorithm 6.2. *The graph $G(S)$, and with it the set S, can be determined by the following algorithm starting from the graph $G(R)$.*

$p := 1$
$A[1] := \mathrm{Red}(G(R))$
repeat
  $p := p + 1$
  $A[p] := \mathrm{Red}(A[p-1] \otimes A[1])$
until $A[p] = A[p-1]$
$G(S) := A[p] \setminus \{0\}$

This algorithm always terminates after finitely many steps.

Since the contact matrix is of an easy shape, this algorithm turns out to be suitable in order to get the set of neighbors even for a class of tiles. For each quadratic CNS tile the contact graph has only $|R| = 7$ vertices. Thus it was possible to derive the following result (cf. [5]).

Theorem 6.3. *Let T be a quadratic CNS tile (with $A \neq 0$ to avoid trivialities). If J is defined as in (1.3) then the set of neighbors S of T has $2 + 4J$ elements, i.e. T has $2 + 4J$ neighbors.*

Using certain product graphs, it is also possible to characterize points of a tiling, in which more tiles coincide. Let T be a $\mathbb{Z}^d$-tile. We call a point $v \in T$ an *L-vertex*, if v is contained in at least L pairwise disjoint tiles different from T. More precisely, for pairwise disjoint $s_1, \dots, s_L \in S$ we set

$$V_L(s_1, \dots, s_L) := \left\{ x \in \mathbb{R}^2 \,\middle|\, x \in T \cap \bigcap_{j=1}^{L} (T + s_j) \right\}.$$

The set of L-vertices of T is then defined by

$$V_L = \bigcup_{\{s_1,\dots,s_L\} \subset S} V_L(s_1, \dots, s_L)$$

where the union is extended over all subsets of S containing L elements. A 2-vertex is sometimes simply called vertex.

For the set of vertices of a quadratic CNS tile T we have the following characterization (cf. [5]).

Theorem 6.4. *Let T be a quadratic CNS tile (with $A \neq 0$ to avoid trivialities) and J be as in (1.3).*

- *If $J = 1$ then*

$$\begin{aligned} V_1 &\quad \textit{is uncountable infinite}, \\ V_2 &\quad \textit{contains six elements}, \\ V_L &= \emptyset \quad (L \geq 3). \end{aligned}$$

- *If $J = 2$ and $2A = B + 3$ then*

$$\begin{aligned} V_1 &\quad \textit{is uncountable infinite}, \\ V_2 &\quad \textit{is countable infinite}, \\ V_L &= \emptyset \quad (L \geq 3). \end{aligned}$$

- *If $J \geq 2$ and $B - A + 1$ does not divide $B - 1$ then*

$$\begin{aligned} V_L &\quad \textit{is uncountable infinite } (1 \leq L \leq J), \\ V_L &= \emptyset \quad (L \geq J + 1). \end{aligned}$$

- *If $J > 2$ and $B - A + 1$ divides $B - 1$ then*

$$\begin{aligned} V_L &\quad \textit{is uncountable infinite } (1 \leq L \leq J - 1), \\ V_J &\quad \textit{finite}, \\ V_L &= \emptyset \quad (L \geq J + 1). \end{aligned}$$

The proof mainly depends on Theorem 6.3 and a certain graph product.

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References

[1] S. Akiyama. On the boundary of self affine tilings generated by Pisot numbers. *J. Math. Soc. Japan*, 54(2):283–308, 2002.

[2] S. Akiyama, T. Borbély, H. Brunotte, A. Pethő, and J. M. Thuswaldner. Generalized radix representations and dynamical systems I. *submitted.*

[3] S. Akiyama and N. Gjini. Connectedness of number theoretic tilings. *submitted.*

[4] S. Akiyama and H. Rao. New criteria for canonical number systems. *Acta Arith.*, 111(1):5–25, 2004.

[5] S. Akiyama and J. M. Thuswaldner. Topological structure of fractal tilings generated by quadratic number systems. *submitted.*

[6] S. Akiyama and J. M. Thuswaldner. Topological properties of two-dimensional number systems. *J. Theor. Nomb. Bordx.*, 12:69–79, 2000.

[7] P. Arnoux and S. Ito. Pisot substitutions and Rauzy fractals. *Bull. Belg. Math. Soc. Simon Stevin*, 8(2):181–207, 2001. Journées Montoises d'Informatique Théorique (Marne-la-Vallée, 2000).

[8] S. Bailey, T. Kim, and R. S. Strichartz. Inside the Lévy dragon. *Amer. Math. Monthly*, 109(8):689–703, 2002.

[9] C. Bandt. Self-similar sets 5. *Proc. Amer. Math. Soc.*, 112:549–562, 1991.

[10] C. Bandt and G. Gelbrich. Classification of self-affine lattice tilings. *J. London Math. Soc. (2)*, 50:581–593, 1994.

[11] C. Bandt and Y. Wang. Disk-like self-affine tiles in $\mathbb{R}^2$. *Discrete Comput. Geom.*, 26:591–601, 2001.

[12] A. Benedek and R. Panzone. On some notable plane sets. II. Dragons. *Rev. Un. Mat. Argentina*, 39(1-2):76–89, 1994.

[13] O. Bratteli and P.E.T. Jorgensen. Iterated function systems and permutation representations of the Cuntz algebra. *Mem. Amer. Math. Soc.*, 139(663):x+89, 1999.

[14] H. Brunotte. On trinomial bases of radix representations of algebraic integers. *Acta Sci. Math. (Szeged)*, 67(3-4):521–527, 2001.

[15] G. R. Conner and J. W. Lamoreaux. On the existence of universal covering spaces for metric spaces and subsets of the euclidean plane. *preprint.*

[16] C. Davis and D. Knuth. Number representations and dragon curves I. *J. Recreational Math.*, 3:66–81, 1970.

[17] C. Davis and D. Knuth. Number representations and dragon curves II. *J. Recreational Math.*, 3:133–149, 1970.

[18] F. M. Dekking and P. van der Wal. The boundary of the attractor of a recurrent iterated function system. *Fractals*, 10:77–89, 2002.

[19] P. Duvall, J. Keesling, and A. Vince. The Hausdorff dimension of the boundary of a self-similar tile. *J. London Math. Soc. (2)*, 61:748–760, 2000.

[20] K. J. Falconer. *Techniques in Fractal Geometry.* John Wiley and Sons, Chichester, New York, Weinheim, Brisbane, Singapore, Toronto, 1997.

[21] G. Gelbrich. Crystallographic reptiles. *Geom. Dedicata*, 51:235–256, 1994.

[22] W. J. Gilbert. Radix representations of quadratic fields. *J. Math. Anal. Appl.*, 83:264–274, 1981.

[23] W. J. Gilbert. Complex numbers with three radix representations. *Canadian J. Math.*, 34:1335–1348, 1982.

[24] W. J. Gilbert. Complex bases and fractal similarity. *Ann. sc. math. Quebec*, 11(1):65–77, 1987.

[25] K. Gröchenig and A. Haas. Self-similar lattice tilings. *J. Fourier Anal. Appl.*, 1:131–170, 1994.

[26] K. Gröchenig and W. R. Madych. Multiresolution analysis, Haar bases, and self-similar tilings of $\mathbf{R}^n$. *IEEE Trans. Inform. Theory*, 38(2, part 2):556–568, 1992.

[27] B. Grünbaum and G. C. Shephard. *Tilings and Patterns.* W. H. Freeman and Company, New York, 1987.

[28] D. Hacon, N. Saldanha, and J. J. P. Veerman. Remarks on self-affine tilings. *Exp. Math.*, 3(4):317–327, 1994.

[29] M. Hata. On the sturcture of self-similar sets. *Japan J. Appl. Math.*, 2:381–414, 1985.

[30] A. Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[31] J. E. Hutchinson. Fractals and self-similarity. *Indiana Univ. Math. J.*, 30:713–747, 1981.

[32] K.-H. Indlekofer, I. Kátai, and P. Racsko. Some remarks on generalized number systems. *Acta Sci. Math. (Szeged)*, 57:543–553, 1993.

[33] I. Kátai. Number systems and fractal geometry. *preprint.*

[34] I. Kátai and I. Kőrnyei. On number systems in algebraic number fields. *Publ. Math. Debrecen*, 41(3–4):289–294, 1992.

[35] I. Kátai and B. Kovács. Kanonische Zahlensysteme in der Theorie der Quadratischen Zahlen. *Acta Sci. Math. (Szeged)*, 42:99–107, 1980.

[36] I. Kátai and B. Kovács. Canonical number systems in imaginary quadratic fields. *Acta Math. Hungar.*, 37:159–164, 1981.

[37] R. Kenyon. Self-replicating tilings. In *Symbolic dynamics and its applications (New Haven, CT, 1991)*, volume 135 of *Contemp. Math.*, pages 239–263. Amer. Math. Soc., Providence, RI, 1992.

[38] I. Kirat and K.-S. Lau. On the connectedness of self-affine tiles. *J. London Math. Soc. (2)*, 62:291–304, 2000.

[39] D. E. Knuth. *The Art of Computer Programming, Vol 2: Seminumerical Algorithms.* Addison Wesley, London, 3rd edition, 1998.

[40] K. Kuratowski. *Topology*, volume I. Academic Press, Polish Scientific Publishers, New York, London, Warszawa, 1966.

[41] K. Kuratowski. *Topology*, volume II. Academic Press, Polish Scientific Publishers, New York, London, Warszawa, 1968.

[42] J. Lagarias and Y. Wang. Integral self-affine tiles in $\mathbb{R}^n$ I. standard and nonstandard digit sets. *J. London Math. Soc.*, 54(2):161–179, 1996.

[43] J. Lagarias and Y. Wang. Self-affine tiles in $\mathbb{R}^n$. *Adv. Math.*, 121:21–49, 1996.

[44] J. Lagarias and Y. Wang. Integral self-affine tiles in $\mathbb{R}^n$ II. lattice tilings. *J. Fourier Anal. Appl.*, 3:83–102, 1997.

[45] J. Luo. A connected tile has a locall connected boundary. *preprint.*

[46] J. Luo. A note on a self-similar tiling generated by the minimal Pisot number. *Fractals*, 10(3):335–339, 2002.

[47] J. Luo, S. Akiyama, and J. M. Thuswaldner. On the boundary connectedness of connected tiles. *preprint.*

[48] J. Luo, H. Rao, and B. Tan. Topological structure of self-similar sets. *Fractals*, 10:223–227, 2002.

[49] S.-M. Ngai and N. Nguyen. The Heighway dragon revisited. *Discrete Comput. Geom.*, 29(4):603–623, 2003.

[50] S.-M. Ngai and T.-M. Tang. A technique in the topology of connected self-similar tiles. *preprint.*

[51] S.-M. Ngai and T.-M. Tang. Topology of connected self-similar tiles in the plane with disconnected interiors. *preprint.*

[52] A. Pethö. On a polynomial transformation and its application to the construction of a public key cryptosystem. In *Computational number theory (Debrecen, 1989)*, pages 31–43. de Gruyter, Berlin, 1991.

[53] B. Praggastis. Numeration systems and Markov partitions from self-similar tilings. *Trans. Amer. Math. Soc.*, 351(8):3315–3349, 1999.

[54] G. Rauzy. Nombres algébriques et substitutions. *Bull. Soc. Math. France*, 110(2):147–178, 1982.

[55] K. Scheicher and J. M. Thuswaldner. On the characterization of canonical number systems. *Osaka Math. J.*, to appear.

[56] K. Scheicher and J. M. Thuswaldner. Canonical number systems, counting automata and fractals. *Math. Proc. Cambridge Philos. Soc.*, 133(1):163–182, 2002.

[57] K. Scheicher and J. M. Thuswaldner. Neighbours of self-affine tiles in lattice tilings. In P. Grabner and W. Woess, editors, *Proceedings of the Conference "Fractals in Graz"*, pages 241–262, 2002.

[58] B. Solomyak. Dynamics of self-similar tilings. *Ergodic Theory Dynam. Systems*, 17(3):695–738, 1997.

[59] H. J. Song and B. S. Kang. Disclike lattice reptiles induced by exact polyominoes. *Fractals*, 7:9–22, 1998.

[60] R. Strichartz and Y. Wang. Geometry of self-affine tiles I. *Indiana Univ. Math. J.*, 48:1–23, 1999.
[61] T.-M. Tang. Arcwise connectedness of the boundaries of connected self-similar sets. *preprint.*
[62] W. Thurston. Groups, tilings and finite state automata. AMS Colloquium Lecture Notes, 1989.
[63] J. J. P. Veerman. Hausdorff dimension of boundaries of self-affine tiles in $\mathbb{R}^n$. *Bol. Mex. Mat.*, 3(4):1–24, 1998.
[64] A. Vince. Self-replicating tiles and their boundary. *Discrete Comput. Geom.*, 21:463–476, 1999.
[65] A. Vince. Digit tiling of euclidean space. In *Directions in Mathematical Quasicrystals*, pages 329–370, Providence, RI, 2000. Amer. Math. Soc.
[66] Y. Wang. Self-affine tiles. In K. S. Lau, editor, *Advances in Wavelet*, pages 261–285. Springer, 1998.
[67] Y. Wang. Wavelets, tiling, and spectral sets. *Duke Math. J.*, 114(1):43–57, 2002.

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