BROWDER'S CONVERGENCE FOR ONE-PARAMETER NONEXPANSIVE SEMIGROUPS

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Abstract. We give the sufficient and necessary condition of Browder's convergence theorem for one-parameter nonexpansive semigroups which was proved in [T. Suzuki, Browder's type convergence theorems for one-parameter semigroups of nonexpansive mappings in Banach spaces, Israel J. Math., 157 (2007), 239–257]. We also discuss the perfect kernels of topological spaces.

1. Introduction

Let C be a closed convex subset of a Banach space E. A family of mappings $\{T(t) : t \geq 0\}$ is called a *one-parameter strongly continuous semigroup of nonexpansive mappings* (*one-parameter nonexpansive semigroup*, for short) on C if the following are satisfied:

(i) For each $t \geq 0$, $T(t)$ is a nonexpansive mapping on C, that is,

$$\|T(t)x - T(t)y\| \leq \|x - y\|$$

holds for all $x, y \in C$.

(ii) $T(s+t) = T(s) \circ T(t)$ for all $s, t \geq 0$.

(iii) For each $x \in C$, the mapping $t \mapsto T(t)x$ from $[0, \infty)$ into C is strongly continuous.

There are six papers concerning the existence of common fixed points of $\{T(t) : t \geq 0\}$; see [1, 2, 4, 5, 9, 11]. Recently, Suzuki [11] proved that $\bigcap_{t \geq 0} F\big(T(t)\big)$ is nonempty provided every nonexpansive mapping on C has a fixed point, where $F\big(T(t)\big)$ is the set of all fixed points of $T(t)$. He also proved a semigroup version of Browder's [3] convergence theorem in [10, 12].

Theorem 1 ([12]). *Let τ be a nonnegative real number. Let $\{\alpha_n\}$ and $\{t_n\}$ be real sequences satisfying*

(i) $0 < \alpha_n < 1$ *and* $0 < t_n$ *for* $n \in \mathbb{N}$;

(ii) $\lim_n t_n = \tau$;

(iii) $t_n \neq \tau$ *for* $n \in \mathbb{N}$ *and* $\lim_n \alpha_n / (t_n - \tau) = 0$.

Let C be a weakly compact convex subset of a Banach space E. Assume that either of the following holds:

- *E is uniformly convex with uniformly Gâteaux differentiable norm.*
- *E is uniformly smooth.*
- *E is a smooth Banach space with the Opial property and the duality mapping J of E is weakly sequentially continuous at zero.*

2000 *Mathematics Subject Classification.* 47H20.

Key words and phrases. Nonexpansive semigroup, common fixed point, Browder's convergence, perfect kernel.

The authors are supported in part by Grants-in-Aid for Scientific Research from the Japanese Ministry of Education, Culture, Sports, Science and Technology (No. 18540020, No. 18740075).

Let $\{T(t) : t \geq 0\}$ be a one-parameter nonexpansive semigroup on C. Fix $u \in C$ and define a sequence $\{u_n\}$ in C by

$$u_n = (1 - \alpha_n)\, T(t_n) u_n + \alpha_n\, u \tag{1}$$

for $n \in \mathbb{N}$. Then $\{u_n\}$ converges strongly to Pu, where P is the unique sunny nonexpansive retraction from C onto $\bigcap_{t \geq 0} F\big(T(t)\big)$.

See [6, 7, 15] for the notions such as 'Opial property', etc.

In this paper, we give the sufficient and necessary condition on $\{\alpha_n\}$ and $\{t_n\}$.

2. Sufficiency

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers.

In this section, we generalize Theorem 1.

Theorem 2. *Let $\{\alpha_n\}$ and $\{t_n\}$ be real sequences satisfying*

(i) $0 < \alpha_n < 1$ *and* $0 \leq t_n$ *for* $n \in \mathbb{N}$;
(ii) $\{t_n\}$ *is bounded*;
(iii) $\lim_n \alpha_n / (t_n - \tau) = 0$ *for all* $\tau \in [0, \infty)$, *where* $1/0 = \infty$.

Let E, C, $\{T(t) : t \geq 0\}$, P, u and $\{u_n\}$ be as in Theorem 1. Then $\{u_n\}$ converges strongly to Pu.

Proof. Let $\{f(n)\}$ be an arbitrary subsequence of $\{n\}$. Since $\{t_n\}$ is bounded, so is $\{t_{f(n)}\}$. Hence there exists a cluster point $\tau \in [0, \infty)$ of $\{t_{f(n)}\}$. From (iii), there exists $\nu \in \mathbb{N}$ such that $t_{f(n)} \neq \tau$ and $t_{f(n)} \neq 0$ for $n \in \mathbb{N}$ with $n \geq \nu$. We choose a subsequence $\{g(n)\}$ of $\{n\}$ such that $g(1) \geq \nu$ and $\{t_{f \circ g(n)}\}$ converges to τ. From (iii) again, we have

$$\lim_{n \to \infty} \frac{\alpha_{f \circ g(n)}}{t_{f \circ g(n)} - \tau} = 0.$$

By Theorem 1, $\{u_{f \circ g(n)}\}$ converges strongly to Pu. Since $\{f(n)\}$ is arbitrary, we obtain that $\{u_n\}$ converges strongly to Pu. □

As a direct consequence of Theorem 2, we obtain the following.

Corollary 1. *Let $\{\alpha_n\}$ and $\{t_n\}$ be real sequences satisfying Conditions* (i)–(iii) *of Theorem 2. Let $\{T(t) : t \geq 0\}$ be a one-parameter nonexpansive semigroup on a bounded closed convex subset C of a Hilbert space E. Let P be the metric projection from C onto $\bigcap_{t \geq 0} F\big(T(t)\big)$. Fix $u \in C$ and define a sequence $\{u_n\}$ in C by* (1). *Then $\{u_n\}$ converges strongly to Pu.*

We note that we need Condition (i) in order to define $\{u_n\}$. In the remainder of this paper, we discuss Conditions (ii) and (iii).

3. Necessity

In this section, we shall show that Conditions (ii) and (iii) of Theorem 2 are best possible, in a sense that we cannot relax these conditions on $\{\alpha_n\}$ and $\{t_n\}$ any more.

For real numbers s and t with $t > 0$, we define 'mod' by

$$s \bmod t = s - [s/t]\, t,$$

where $[s/t]$ is the maximum integer not exceeding s/t.

Lemma 1. *Let* $\{\alpha_n\}$ *and* $\{t_n\}$ *be real sequences satisfying Condition* (i) *of Theorem 2. Assume* $\limsup_n t_n = \infty$*. Then for every nonnegative real number* υ*, there exists a positive real number* τ *such that*

$$\limsup_{n\to\infty} \frac{\alpha_n}{(t_n \bmod \tau) - \upsilon} = \infty.$$

Proof. We shall define two real sequences $\{\varepsilon_n\}$ and $\{\tau_n\}$ and a subsequence $\{f(n)\}$ of $\{n\}$ satisfying the following.

(i) $0 < \varepsilon_n < 1$ and $\upsilon + 1 + \varepsilon_n < \tau_n$.
(ii) $\alpha_{f(n)}/\big((t_{f(n)} \bmod \tau) - \upsilon\big) \geq n$ for $\tau \in [\tau_n - \varepsilon_n, \tau_n]$.
(iii) $[\tau_n - \varepsilon_n, \tau_n] \supset [\tau_{n+1} - \varepsilon_{n+1}, \tau_{n+1}]$.

We denote $t_{f(n)}$ by s_n and $\alpha_{f(n)}$ by β_n for $n \in \mathbb{N}$. We choose $f(1)$ satisfying $s_1 > 2\,\upsilon + 2$. We put

$$\varepsilon_1 := \beta_1/2 \in (0,1) \quad \text{and} \quad \tau_1 := s_1 - \upsilon > \upsilon + 2 > \upsilon + 1 + \varepsilon_1.$$

If $\tau \in [\tau_1 - \varepsilon_1, \tau_1]$, then since

$$1 \leq \frac{s_1}{s_1 - \upsilon} = \frac{s_1}{\tau_1} \leq \frac{s_1}{\tau} \leq \frac{s_1}{\tau_1 - \varepsilon_1} < \frac{s_1}{s_1/2} = 2,$$

we have

$$0 \leq (s_1 \bmod \tau) - \upsilon = s_1 - \tau - \upsilon = \tau_1 - \tau \leq \varepsilon_1 \leq \beta_1,$$

which implies (ii). We assume that ε_n, τ_n and $f(n)$ are defined for some $n \in \mathbb{N}$. We choose $f(n+1)$ satisfying $f(n+1) > f(n)$ and $s_{n+1} \geq 2\,\tau_n\,(\tau_n - \varepsilon_n)/\varepsilon_n$. Then we have

$$s_{n+1} > \tau_n \quad \text{and} \quad \frac{s_{n+1}}{\tau_n - \varepsilon_n} \geq \frac{s_{n+1}}{\tau_n} + 2.$$

Hence there exist real numbers p and q such that

$$\tau_n - \varepsilon_n \leq p < q \leq \tau_n \quad \text{and} \quad \frac{s_{n+1}}{p} = \frac{s_{n+1}}{q} + 1 \in \mathbb{N}.$$

We put

$$\tau_{n+1} = \frac{(s_{n+1} - \upsilon)\,q}{s_{n+1}} \quad \text{and} \quad \varepsilon_{n+1} = \frac{\beta_{n+1}\,q}{(n+1)\,s_{n+1}}.$$

Then it is obvious that $\tau_{n+1} \leq q$. Since

$$p - \upsilon - \beta_{n+1}/(n+1) \geq p - \upsilon - 1 \geq \tau_n - \varepsilon_n - \upsilon - 1 > 0,$$

We have

$$\tau_{n+1} - \varepsilon_{n+1} = q\,\frac{s_{n+1} - \upsilon - \beta_{n+1}/(n+1)}{s_{n+1}} = \frac{s_{n+1}\,p}{s_{n+1} - p}\,\frac{s_{n+1} - \upsilon - \beta_{n+1}/(n+1)}{s_{n+1}}$$
$$= p\,\frac{s_{n+1} - \upsilon - \beta_{n+1}/(n+1)}{s_{n+1} - p} > p.$$

Therefore

$$\tau_n - \varepsilon_n \leq p < \tau_{n+1} - \varepsilon_{n+1} < \tau_{n+1} \leq q \leq \tau_n.$$

So we note

$$(s_{n+1} \bmod \tau) - \upsilon = s_{n+1} - \tau\, s_{n+1}/q - \upsilon$$

for $\tau \in [\tau_{n+1} - \varepsilon_{n+1}, \tau_{n+1}]$. Since

$$(s_{n+1} \bmod \tau_{n+1}) - \upsilon = 0 \quad \text{and} \quad \big(s_{n+1} \bmod (\tau_{n+1} - \varepsilon_{n+1})\big) - \upsilon = \beta_{n+1}/(n+1),$$

we have

$$0 \leq (s_{n+1} \bmod \tau) - \upsilon \leq \beta_{n+1}/(n+1)$$

for $\tau \in [\tau_{n+1} - \varepsilon_{n+1}, \tau_{n+1}]$. Therefore we have defined $\{\varepsilon_n\}$, $\{\tau_n\}$ and $\{f(n)\}$ which satisfy (i)–(iii). Cantor's intersection theorem yields that there exists $\tau \in \mathbb{R}$ such that $\tau \in \bigcap_{n=1}^{\infty}[\tau_n - \varepsilon_n, \tau_n]$. By (ii), we have

$$\limsup_{n\to\infty} \frac{\alpha_n}{(t_n \bmod \tau) - \upsilon} \geq \limsup_{n\to\infty} \frac{\beta_n}{(s_n \bmod \tau) - \upsilon} \geq \lim_{n\to\infty} n = \infty.$$

This completes the proof. □

Lemma 2. *Let $\{\alpha_n\}$ and $\{t_n\}$ be real sequences with Condition* (i) *of Theorem 2. Assume*

$$\lim_{n\to\infty} \frac{\alpha_n}{|t_n - \tau|} > 0 \quad \textit{and} \quad \lim_{n\to\infty} t_n = \tau \tag{2}$$

for some $\tau \in (0, \infty)$. Then there exists a subsequence $\{f(n)\}$ of $\{n\}$ such that either

$$\lim_{n\to\infty} \frac{\alpha_{f(n)}}{t_{f(n)} \bmod \tau} > 0 \quad \textit{and} \quad \lim_{n\to\infty} (t_{f(n)} \bmod \tau) = 0 \tag{3}$$

or

$$\lim_{n\to\infty} \frac{\alpha_{f(n)}}{(t_{f(n)} \bmod \tau) - \tau} < 0 \quad \textit{and} \quad \lim_{n\to\infty} (t_{f(n)} \bmod \tau) = \tau \tag{4}$$

holds.

Proof. If there exists a subsequence $\{f(n)\}$ of $\{n\}$ such that $t_{f(n)} \geq \tau$ for all $n \in \mathbb{N}$, then

$$t_{f(n)} \bmod \tau = t_{f(n)} - \tau = |t_{f(n)} - \tau|$$

for sufficiently large $n \in \mathbb{N}$. Thus (3) holds. If there exists a subsequence $\{f(n)\}$ of $\{n\}$ such that $t_{f(n)} < \tau$ for all $n \in \mathbb{N}$, then

$$(t_{f(n)} \bmod \tau) - \tau = t_{f(n)} - \tau = -|t_{f(n)} - \tau|$$

for all $n \in \mathbb{N}$. Thus (4) holds. □

Lemma 3. *Let $\{\alpha_n\}$ and $\{t_n\}$ be real sequences with Condition* (i) *of Theorem 2. Assume*

$$\lim_{n\to\infty} \frac{\alpha_n}{t_n} > 0 \quad \textit{and} \quad \lim_{n\to\infty} t_n = 0.$$

Then (3) *holds for every positive real number τ and every subsequence $\{f(n)\}$ of $\{n\}$.*

Proof. Obvious. □

Lemma 4. *Let $\{\alpha_n\}$ and $\{t_n\}$ be real sequences with Condition* (i) *of Theorem 2. Assume that the conjunction of Condition* (ii) *and Condition* (iii) *of Theorem 2 does not hold. Then there exist a positive real number τ and a subsequence $\{f(n)\}$ of $\{n\}$ such that either* (3) *or* (4) *holds.*

Proof. We consider the following four cases:

- $\limsup_n t_n = \infty$
- $\limsup_n t_n < \infty$ and $\limsup_n \alpha_n > 0$
- $\limsup_n t_n < \infty$, $\lim_n \alpha_n = 0$ and $\limsup_n \alpha_n/|t_n - \tau| > 0$ for some $\tau \in (0, \infty)$
- $\limsup_n t_n < \infty$, $\lim_n \alpha_n = 0$ and $\limsup_n \alpha_n/t_n > 0$.

In the first case, using Lemma 1, there exist a positive real number τ and a subsequence $\{f(n)\}$ of $\{n\}$ such that

$$\lim_{n\to\infty} \frac{\alpha_{f(n)}}{t_{f(n)} \bmod \tau} = \infty.$$

It is obvious that $\lim_n (t_{f(n)} \bmod \tau) = 0$. Thus (3) holds. Next, we note that it is sufficient to show the existence of a subsequence $\{g(n)\}$ of $\{n\}$ such that we can apply either Lemma 2 or Lemma 3. In the second case, we can choose a subsequence $\{g(n)\}$ of $\{n\}$ such that $\lim_n \alpha_{g(n)} > 0$ and $\{t_{g(n)}\}$ converges to some nonnegative real number τ. Then $\{\alpha_{g(n)}\}$ and $\{t_{g(n)}\}$ satisfy (2). So we can apply either Lemma 2 or Lemma 3. In the third case, we can choose a subsequence $\{g(n)\}$ of $\{n\}$ such that $\lim_n \alpha_{g(n)}/|t_{g(n)} - \tau| > 0$. Then $\lim_n |t_{g(n)} - \tau| = 0$ holds. Hence we can apply Lemmas 2. Similarly, in the fourth case, we can apply Lemma 3. □

Example 1. Let $\{\alpha_n\}$ and $\{t_n\}$ be real sequences with Condition (i) of Theorem 2. Let γ be a positive real number. Let E be the two dimensional real Hilbert space and put $C = \{x \in E : \|x\| \leq 1\}$. For $t \geq 0$, define a 2×2 matrix $T(t)$ by

$$T(t) = \begin{bmatrix} \cos(\gamma\, t) & -\sin(\gamma\, t) \\ \sin(\gamma\, t) & \cos(\gamma\, t) \end{bmatrix}.$$

We can consider that $\{T(t) : t \geq 0\}$ is a linear nonexpansive semigroup on C. Let P be the metric projection from C onto $\bigcap_{t\geq 0} F\big(T(t)\big)$, that is, $Px = 0$ for all $x \in C$. Put $u = (1, 0)$ and define a sequence $\{u_n\}$ by (1). Assume that the conjunction of Condition (ii) and Condition (iii) of Theorem 2 does not hold. Then there exists γ such that $\{u_n\}$ does not converge strongly to Pu.

Proof. By Lemma 4, there exist a positive real number τ and a subsequence $\{f(n)\}$ of $\{n\}$ such that either (3) or (4) holds. We note that both (3) and (4) do not hold simultaneously. We put

$$\gamma = 4\,\pi/\tau.$$

We also put

$$\eta := \begin{cases} \lim_n (t_{f(n)} \bmod \tau)/\alpha_{f(n)} \in [0, \infty) & \text{if (3) holds} \\ \lim_n \big((t_{f(n)} \bmod \tau) - \tau\big)/\alpha_{f(n)} \in (-\infty, 0] & \text{if (4) holds.} \end{cases}$$

In the case where (3) holds, since

$$\sin(\gamma\, t_{f(n)}) = \sin(\gamma\, t_{f(n)} \bmod 4\,\pi) = \sin\big(\gamma\,(t_{f(n)} \bmod \tau)\big),$$

we have

$$\lim_{n\to\infty} \frac{\sin(\gamma\, t_{f(n)})}{\alpha_{f(n)}} = \lim_{n\to\infty} \frac{\gamma\,(t_{f(n)} \bmod \tau)}{\alpha_{f(n)}} = \gamma\,\eta.$$

In the case where (4) holds, since

$$\sin(\gamma\, t_{f(n)}) = \sin\big((\gamma\, t_{f(n)} \bmod 4\,\pi) - 4\,\pi\big) = \sin\big(\gamma\,\big((t_{f(n)} \bmod \tau) - \tau\big)\big),$$

we have

$$\lim_{n\to\infty} \frac{\sin(\gamma\, t_{f(n)})}{\alpha_{f(n)}} = \lim_{n\to\infty} \frac{\gamma\,\big((t_{f(n)} \bmod \tau) - \tau\big)}{\alpha_{f(n)}} = \gamma\,\eta.$$

Similarly, $\lim_n \sin(\gamma\, t_{f(n)}/2)/\alpha_{f(n)} = \gamma\,\eta/2$ holds in both cases. For $n \in \mathbb{N}$, we put a 2×2 matrix P_n by

$$P_n = \frac{\alpha_n}{4\,(1 - \alpha_n)\,\sin^2(\gamma\, t_n/2) + {\alpha_n}^2} \begin{bmatrix} a_n & -b_n \\ b_n & a_n \end{bmatrix},$$

where $a_n = \alpha_n + 2\,(1-\alpha_n)\,\sin^2(\gamma\,t_n/2)$ and $b_n = (1-\alpha_n)\,\sin(\gamma\,t_n)$. It is easy to verify that $u_n = P_n u$ for $n \in \mathbb{N}$ (cf. [14]). We obtain

$$\lim_{n\to\infty} P_{f(n)} = \frac{1}{\gamma^2\,\eta^2+1}\begin{bmatrix} 1 & -\gamma\,\eta \\ \gamma\,\eta & 1 \end{bmatrix} = \frac{1}{\sqrt{\gamma^2\,\eta^2+1}}\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix},$$

where $\theta := \arctan(\gamma\,\eta) \in (-\pi/2, \pi/2)$. Therefore

$$\lim_{n\to\infty} u_{f(n)} = \frac{1}{\sqrt{\gamma^2\,\eta^2+1}}\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} u \neq 0 = Pu$$

holds. □

From Corollary 1 and Example 1, we obtain the following.

Theorem 3. *Let E be a Hilbert space whose dimension is more than* 1*. Let $\{\alpha_n\}$ and $\{t_n\}$ be real sequences satisfying Condition* (i) *of Theorem 2. Then the following are equivalent*:

- *Conditions* (ii) *and* (iii) *of Theorem 2 hold.*
- *If $\{T(t) : t \geq 0\}$ is a one-parameter nonexpansive semigroup on a bounded closed convex subset C of E, $u \in C$, $\{u_n\}$ is a sequence defined by* (1) *and P is the metric projection from C onto $\bigcap_{t\geq 0} F\big(T(t)\big)$, then $\{u_n\}$ converges strongly to Pu.*

4. Additional Results

In [13], we have improved Theorem 1 as follows. In this section, we first compare Theorem 2 with Theorem 4.

Theorem 4 ([13])**.** *Let $\{\alpha_n\}$ and $\{t_n\}$ be real sequences satisfying Conditions* (i) *and* (ii) *of Theorem 2 and*

(iii) $s_n := \liminf_m |t_m - t_n| > 0$ *for* $n \in \mathbb{N}$ *and* $\lim_n \alpha_n/s_n = 0$.

Then the same conclusion of Theorem 2 holds.

(iii) of Theorem 4 is stronger than Condition (iii) of Theorem 2 because Condition (iii) of Theorem 2 is a sufficient and necessary condition. It is a natural question of whether (iii) of Theorem 4 is strictly stronger.

Example 2. Define functions f and g from $\mathbb{N}$ into $\mathbb{N} \cup \{0\}$ and real sequences $\{\alpha_n\}$ and $\{t_n\}$ by

- $f(n) = \max\big\{k \in \mathbb{N} \cup \{0\} : k\,(k+1)/2 < n\big\}$
- $g(n) = n - f(n)\,\big(f(n)+1\big)/2$
- $t_n = 2^{-g(n)}$ if $n = g(n)\,\big(g(n)+1\big)/2$, and $t_n = 2^{-g(n)} + 4^{-n}$ otherwise.
- $\alpha_n = 4^{-2\,n}$.

Then $\{\alpha_n\}$ and $\{t_n\}$ satisfy Conditions (i)–(iii) of Theorem 2, however, do not satisfy (iii) of Theorem 4.

Remark. The sequence $\{t_n\}$ is

$$\frac{1}{2},\ \frac{1}{2}+\frac{1}{4^2},\ \frac{1}{2^2},\ \frac{1}{2}+\frac{1}{4^4},\ \frac{1}{2^2}+\frac{1}{4^5},\ \frac{1}{2^3},\ \frac{1}{2}+\frac{1}{4^7},\ \frac{1}{2^2}+\frac{1}{4^8},\ \frac{1}{2^3}+\frac{1}{4^9},\ \frac{1}{2^4},\ \frac{1}{2}+\frac{1}{4^{11}},\ \cdots.$$

Proof. We note that if $n = m\,(m+1)/2$ for some $m \in \mathbb{N}$, then $g(n) = m$. It is obvious that Conditions (i) and (ii) of Theorem 2 hold. Since $2^{-\nu}$ is a cluster point of $\{t_n\}$ for every $\nu \in \mathbb{N}$, we have

$$s_{m\,(m+1)/2} := \liminf_{j\to\infty} |t_j - t_{m\,(m+1)/2}| = \liminf_{j\to\infty} |t_j - 2^{-m}| = 0$$

for all $m \in \mathbb{N}$. Hence (iii) of Theorem 4 does not hold. Let us prove Condition (iii) of Theorem 2. Fix $\tau \in [0,\infty)$. We consider the following three cases:

- $\tau = 0$
- $\tau = 2^{-\nu}$ for some $\nu \in \mathbb{N}$
- otherwise

In the first case, we have

$$\lim_{n\to\infty} \frac{\alpha_n}{t_n - \tau} = \lim_{n\to\infty} \frac{4^{-2\,n}}{t_n} \le \lim_{n\to\infty} \frac{4^{-2\,n}}{2^{-g(n)}} \le \lim_{n\to\infty} \frac{4^{-2\,n}}{2^{-n}} = 0.$$

In the second case, considering the two cases of $g(n) \le \nu$ and $g(n) > \nu$, we have

$$|t_n - \tau| \ge \min\left\{4^{-n},\ 2^{-\nu-1} - 4^{-n}\right\}$$

for $n \in \mathbb{N}$ with $n > \nu\,(\nu+1)/2$. Hence

$$\lim_{n\to\infty} \frac{\alpha_n}{|t_n - \tau|} \le \lim_{n\to\infty} \frac{4^{-2\,n}}{\min\left\{4^{-n},\ 2^{-\nu-1} - 4^{-n}\right\}} = 0.$$

In the third case, we have

$$\lim_{n\to\infty} \frac{\alpha_n}{|t_n - \tau|} \le \frac{\lim_n 4^{-2\,n}}{\liminf_n |t_n - \tau|} = \frac{0}{\liminf_n |t_n - \tau|} = 0.$$

Therefore Condition (iii) of Theorem 2 holds. □

Finally we study Condition (iii) of Theorem 2 more deeply.

For an arbitrary set A, we denote by $\sharp A$ the cardinal number of A. For a subset A of a topological space, we also denote by A^d the derived set of A. That is, $x \in A^d$ if and only if x belongs to the closure of $A \setminus \{x\}$. We recall that A is *dense in itself* if $A \subset A^d$. We define A^p by

$$A^p = \bigcup\{B \subset A : \ B \text{ is dense in itself}\}.$$

A^p is called the *perfect kernel* of A. A is called *scattered* if $A^p = \varnothing$. We know that A^p is perfect under the relative topology for A. We also know that $A \setminus A^p$ is scattered, that is, A can be written as the union of two disjoint sets, one perfect, the other scattered. See [8, 16].

Let α be an ordinal number. We denote by α^+ and α^- the successor and the predecessor of α, respectively. We recall that α is *isolated* if α^- exists. α is *limit* if α^- does not exist.

Proposition 1. *Let A be a subset of a topological space. Let γ be an ordinal number with $\sharp\gamma > \sharp A$ and $\sharp\gamma \ge \sharp\mathbb{N}$. Put $D = \{\alpha : \alpha \le \gamma\}$. Define a net $\{A_\alpha\}_{\alpha\in D}$ of subsets of A by*

$$A_\alpha = \begin{cases} A & \text{if } \alpha = 0 \\ A_{\alpha^-} \cap (A_{\alpha^-}{}^d) & \text{if } \alpha \text{ is isolated} \\ \bigcap\{A_\beta : \beta < \alpha\} & \text{if } \alpha \text{ is limit.} \end{cases}$$

Then $A_\gamma = A^p$ holds.

Proof. It is obvious that $\alpha \le \beta$ implies $A_\beta \subset A_\alpha$. We can easily show by transfinite induction $A^p \subset A_\alpha$ because $A^p \subset B$ implies $A^p \subset B \cap B^d$. Arguing by contradiction, we assume $A^p \subsetneqq A_\gamma$. Since $A^p \subsetneqq B \subset A$ implies $B \cap B^d \subsetneqq B$, we have $A_{\alpha^+} \subsetneqq A_\alpha$. Thus

$$\begin{aligned}\sharp\gamma &= \sharp\{\alpha :\ \alpha < \gamma\} = \sharp\{\alpha :\ \alpha \le \gamma\}\\ &= \sharp\{\alpha :\ \alpha \le \gamma,\ \alpha \text{ is isolated}\}\\ &\le \sharp\bigsqcup\{A_{\alpha^-} \setminus A_\alpha :\ \alpha \le \gamma,\ \alpha \text{ is isolated}\}\\ &= \sharp(A \setminus A_\gamma) \le \sharp A,\end{aligned}$$

which contradicts $\sharp A < \sharp\gamma$. Therefore we obtain $A_\gamma = A^p$. □

Proposition 2. *Let $\{t_n\}$ be a real sequence and put $A = \{t_n : n \in \mathbb{N}\}$. Then the following are equivalent*:

(i) *There exists a sequence $\{\alpha_n\}$ of positive real numbers satisfying $\lim_n \alpha_n/(t_n - \tau) = 0$ for all $\tau \in \mathbb{R}$.*
(ii) *A is scattered, and $\sharp\{n : t_n = \tau\} < \infty$ for all $\tau \in \mathbb{R}$.*

Remark. If $\{t_n\}$ satisfies the assumption of Theorem 4, then A is obviously scattered.

Proof. In order to show (i) implies (ii), we assume that (ii) does not hold and let $\{\alpha_n\}$ be a sequence of positive real numbers. In the case where $\sharp\{n : t_n = \tau\} = \infty$ for some $\tau \in \mathbb{R}$, it is obvious $\limsup_n \alpha_n/(t_n - \tau) = \infty$. So we consider the other case, where $A^p \ne \varnothing$. We first choose $f(1) \in \mathbb{N}$ such that $t_{f(1)} \in A^p$, and put $B_1 = (t_{f(1)} - \alpha_{f(1)}, t_{f(1)} + \alpha_{f(1)})$. Then from $t_{f(1)} \in (A^p)^d$, we have $\sharp(A^p \cap B_1) = \infty$. So we can choose $f(2) \in \mathbb{N}$ such that $f(2) > f(1)$ and $t_{f(2)} \in A^p \cap B_1$. We put

$$B_2 = B_1 \cap (t_{f(2)} - \alpha_{f(2)}, t_{f(2)} + \alpha_{f(2)}).$$

Then since $t_{f(2)} \in (A^p)^d$, we have $\sharp(A^p \cap B_2) = \infty$. So we can choose $f(3) \in \mathbb{N}$ such that $f(3) > f(2)$ and $t_{f(3)} \in A^p \cap B_2$. Continuing this argument, we have a subsequence $\{f(n)\}$ of $\{n\}$ and a sequence $\{B_n\}_{n=1}^\infty$ of nonempty open intervals satisfying

- $B_1 \supset B_2 \supset B_3 \supset \cdots$;
- $B_n \subset [t_{f(n)} - \alpha_{f(n)}, t_{f(n)} + \alpha_{f(n)}]$ for all $n \in \mathbb{N}$.

So $\big\{[t_{f(n)} - \alpha_{f(n)}, t_{f(n)} + \alpha_{f(n)}]\big\}$ has the finite intersection property. Hence there exists $\tau \in \mathbb{R}$ such that $\tau \in \bigcap_{n=1}^\infty [t_{f(n)} - \alpha_{f(n)}, t_{f(n)} + \alpha_{f(n)}]$. Then we have

$$\limsup_{n\to\infty} \frac{\alpha_n}{|t_n - \tau|} \ge \limsup_{n\to\infty} \frac{\alpha_{f(n)}}{|t_{f(n)} - \tau|} \ge 1.$$

Therefore (i) does not hold in both cases. We have shown (i) implies (ii). Let us prove (ii) implies (i). We assume (ii). Let γ be an ordinal number with $\sharp\gamma = \sharp\mathbb{R}$ and put $D = \{\alpha : \alpha \le \gamma\}$. Define a net $\{A_\alpha\}_{\alpha \in D}$ of subsets of A as in Proposition 1. By Proposition 1, $A^p = \varnothing$ holds. So we can define a function κ from $\mathbb{N}$ into D such that

$$t_n \in A_{\kappa(n)} \quad \text{and} \quad t_n \notin A_{\kappa(n)^+}.$$

Define a function δ from $\mathbb{N}$ into $(0, \infty]$ by

$$\delta(n) = \inf\big\{|t_n - s| : s \in A_{\kappa(n)} \setminus \{t_n\}\big\},$$

where $\inf\varnothing = \infty$. We note $\delta(n) > 0$ because $t_n \notin A_{\kappa(n)^+}$. We choose a real sequence $\{\alpha_n\}$ satisfying

$$0 < \alpha_n < \delta(n)/n \quad \text{and} \quad \alpha_{n+1} < \alpha_n.$$

Fix $\tau \in \mathbb{R}$ and $\varepsilon > 0$. Then there exists $\nu \in \mathbb{N}$ such that $2/\nu < \varepsilon$. It is obvious that $n \geq \nu$ implies $2\,\alpha_n/\varepsilon < \delta(n)$. We shall show

- $m > n \geq \nu$, $\alpha_n/|t_n - \tau| > \varepsilon$, $\alpha_m/|t_m - \tau| > \varepsilon$ and $t_m \neq t_n$ imply $\kappa(m) < \kappa(n)$.

Arguing by contradiction, we assume $\kappa(m) \geq \kappa(n)$. Then since $t_m \in A_{\kappa(n)} \setminus \{t_n\}$, we have

$$|t_n - t_m| \geq \delta(n) > 2\,\alpha_n/\varepsilon.$$

Since $\alpha_m < \alpha_n$, we have

$$2\,\alpha_n/\varepsilon < |t_n - t_m| \leq |t_n - \tau| + |t_m - \tau| < \alpha_n/\varepsilon + \alpha_m/\varepsilon < 2\,\alpha_n/\varepsilon,$$

which is a contradiction. Therefore we have shown $\kappa(m) < \kappa(n)$. Since there does not exist a strictly decreasing infinite sequence of ordinal numbers, we have

$$\sharp\{n \in \mathbb{N} : \alpha_n/|t_n - \tau| > \varepsilon\} < \infty.$$

Since $\varepsilon > 0$ is arbitrary, we obtain $\lim_n \alpha_n/|t_n - \tau| = 0$. $\square$

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