On the connectedness of self-affine attractors

Shigeki AKIYAMA * and Nertila GJINI †

Abstract

Let $T = T(A, D)$ be a self-affine attractor in $\mathbb{R}^n$ defined by an integral expanding matrix A and a digit set D. In the first part of this paper, in connection with canonical number systems, we study connectedness of T when D corresponds to the set of consecutive integers $\{0, 1, \ldots, |\det(A)| - 1\}$. It is shown that in $\mathbb{R}^3$ and $\mathbb{R}^4$, for any integral expanding matrix A, $T(A, D)$ is connected.

In the second part, we study connectedness of Pisot dual tiles which play an important role in the study of β-expansions, substitutions and symbolic dynamical systems. It is shown that each tile of the dual tiling generated by a Pisot unit of degree 3 is arcwise connected. This is naturally expected since the digit set consists of consecutive integers as above. However surprisingly, we found families of disconnected Pisot dual tiles of degree 4. We even give a simple necessary and sufficient condition of connectedness of the Pisot dual tiles of degree 4. Detailed proofs will be given in [4].

1 Introduction

In this paper, we shall give a brief summary of the paper [4]. Proofs given here are representative parts of detailed ones in [4]. Let $M_n(\mathbb{Z})$ be the set of $n \times n$ matrices with entries in $\mathbb{Z}$. Let A be an expanding integral matrix in $M_n(\mathbb{Z})$. The word 'expanding' means that all its eigenvalues have modulus greater than 1. Let $|\det A| = q$ and $D = \{d_1, \ldots d_q\} \subset \mathbb{R}^n$ be a set of q distinct vectors, called a $q-$*digit set*. If we put $S_j(x) = A^{-1}(x + d_j)$, $1 \le j \le q$, then they are contractive maps under a suitable norm in $\mathbb{R}^n$ (c.f. [20]). Furthermore it is well known that there is a unique compact set T satisfying $T = \bigcup_{j=1}^q S_j(T)$, which is explicitly given by

$$T := T(A, D) = \left\{ \sum_{i=1}^{\infty} A^{-i} d_{j_i} : d_{j_i} \in D \right\}.$$

T is the *attractor* of the system $\{S_j\}_{j=1}^q$, and it is called a *self-affine tile* if its Lebesgue measure $\mu(T)$ is positive. Basic questions and detailed studies on the tiling generated by T are found for example in J. C. Lagarias and Y. Wang [20], R. Kenyon [17], C. Bandt [8], Y. Wang [31], A. Vince [30] and their references.

One of the important aspects of the self-affine attractors is connectedness. Hata [15, Theorem 4.6] has shown that if $\{f_j\}_{1 \le j \le m}$ is a finite set of weak contractions of X, then the attractor $K = K(f_1, \cdots, f_m)$ is a locally connected continuum if and only if, for any $1 \le i < j \le m$, there exists a sequence $\{r_0, r_1, \cdots, r_n, r_{n+1}\} \subset \{1, 2, \cdots, m\}$ with $r_0 = i$ and $r_{n+1} = j$ such that $f_{r_k}(K) \cap f_{r_{k+1}}(K) \neq \emptyset$ for

[0]2000 Mathematics Subject Classification: 05B45, 52C22, 54D05, 11R06, 28A80.

*Supported by the Japanese Ministry of Education, Culture, Sports, Science and Technology, Grand-in Aid for fundamental research, 14540015.

†Supported by the Japan Society for the Promotion of Science, 12000267.

$k \in \{0, 1, \cdots, n\}$. Note that if a tile is connected then it must be arcwise connected. This is seen in the proof of Hata [15, Theorem 4.6]. Thus after all *arcwise connectedness and connectedness are equivalent* in our framework. Hacon-Saldanha-Veerman [14] showed that, if $|\det A| = 2$ and $D = \{0, v\} \subset \mathbb{Z}^n$ is a set of complete representatives of the quotient group $\mathbb{Z}^n / A\mathbb{Z}^n$, then $T(A, D)$ is a connected tile. Gröchenig-Haas [13] have proved the existence of connected self-similar lattice tilings for parabolic and elliptic dilations in dimension two. Kirat-Lau [18], using a graph theoretical argument on D, rediscovered Hata's above criterion of connectedness. Also Kirat and Lau showed the following sufficient condition, which will be used in the proof of Theorems 2.1 and 2.2. Afterwards we call it Kirat-Lau Condition.

Let $A \in M_n(\mathbb{Z})$ be an expanding matrix with $|\det A| = q$ having a characteristic polynomial $p(x)$. Let $D = \{0, v, \cdots, (q-1)v\}$ with $v \in \mathbb{R}^n \setminus \{0\}$. Suppose that there exists a polynomial $g(x) \in \mathbb{Z}[x]$ (which will be called a multiplying factor) such that

$$h(x) = g(x)p(x) = x^k + a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \cdots + a_1 x \pm q \tag{1}$$

with $|a_i| \leq q - 1$, for $1 \leq i \leq k - 1$. Then $T(A, D)$ is connected.

The idea of this condition is to give a concrete point on consecutive two tiles $T + kv$ and $T + (k+1)v$ and to apply Hata's criterion mentioned above. Using this, Kirat and Lau succeeded in proving the connectedness of a tile for a suitable digit set in dimension 2.

In the first part of this paper, we are interested in generalizing their results to higher dimensional cases using digit sets corresponding to consecutive integers $\{0, 1, \ldots, |\det(A)| - 1\}$. We need to recall the classical theory to count the number of roots of a polynomial which are inside and outside the unit circle by using quadratic forms and Sturm sequences for the separation of roots (briefly reviewed in [4, Section 2]). All expanding polynomials are classified by this theory. We will show

Theorem 1.1. *Let $d = 3, 4$ and $A \in M_d(\mathbb{Z})$ be an expanding matrix with $|\det A| = q$ and $D = \{0, v, \cdots, (q-1)v\}$ with $v \in \mathbb{R}^d \setminus \{0\}$. Then $T(A, D)$ is connected.*

Proofs are given separately in Theorem 2.1 and 2.2. Especially for Theorem 2.2, we have a lot of subcases. This result gives an evidence of a widely believed speculation that all such 'consecutive integer digit tiles' may be connected. This gives a good contrast with the second part of this paper.

We do not intend to consider general digit sets but only use digits which correspond to consecutive integers. One reason of this restriction is that this case is essential and widely studied in relation to canonical number systems. For canonical number systems and associated tilings, see Kátai-Kőrnyei [16], Kovács-Pethő [19], Gilbert [11]. Recent progress on topological studies of this tiling can be seen in Akiyama-Thuswaldner [5, 6].

Now we shall explain the second part of this paper. Let $\beta > 1$ be a real number which is not an integer. A *greedy expansion* of a positive real x in base β is an expansion of the form:

$$x = \sum_{i=N_0}^{\infty} a_{-i}\beta^{-i} = a_{-N_0}, a_{-N_0-1}, \cdots$$

with $a_i \in [0, \beta) \cap \mathbb{Z}$ and a *greedy condition*

$$0 \leq x - \sum_{N_0}^{N} a_{-i}\beta^{-i} < \beta^{-N} \qquad \forall N \geq N_0.$$

Let $1 = d_{-1}\beta^{-1} + d_{-2}\beta^{-2} + \cdots$ be an expansion of 1 defined by the algorithm

$$c_{-i} = \beta c_{-i+1} - \lfloor \beta c_{-i+1} \rfloor, \quad d_{-i} = \lfloor \beta c_{-i+1} \rfloor \tag{2}$$

with $c_0 = 1$, where $\lfloor x \rfloor$ denotes the maximal integer not exceeding x.
$d_\beta(1) = .d_{-1}, d_{-2}, \cdots$ is called $\beta-$*expansion of* 1. (Here we have that $d_{-1} = \lfloor \beta \rfloor$.)

Parry [23] has shown that a sequence $x = x_1, x_2, \cdots$ of nonnegative integers is realized as a $\beta-$expansion of some positive real number if and only if it satisfies the following lexicographical condition:

$$\forall p \geq 0, \quad \sigma^p(x) <_{\text{lex}} d^*(1)$$

with $d^*(1) = \begin{cases} d_\beta(1), & \text{if } d_\beta(1) \text{ is infinite;} \\ (d_{-1}, d_{-2}, \cdots, d_{-n+1}, d_{-n} - 1),^\omega & \text{if } d_\beta(1) = d_{-1}, \cdots, d_{-n}, \end{cases}$
where σ is a shift map which acts on right infinite words defined by $\sigma(x_1, x_2, \dots) = x_2, x_3 \dots$. Here a^ω is a periodic right infinite word generated by a. In this case this sequence $x = x_1, x_2, \cdots$ is called *admissible.* A *tail* of the word x is a word $\sigma^n(x)$ for some positive integer n.

From now on, let β be a *Pisot number* which is a real algebraic integer greater than 1 whose Galois conjugates other than itself have modulus smaller than 1. Let $\mathbb{Q}(\beta)_{\geq 0}$ be the nonnegative elements of the minimum field containing the rational numbers $\mathbb{Q}$ and β. Bertrand [9] and Schmidt [26] showed that any greedy expansion of $x \in \mathbb{Q}(\beta)_{\geq 0}$ is eventually periodic. Here we call a *Pisot unit* a Pisot number which is also a unit of the integer ring of $\mathbb{Q}(\beta)$.

The symbolic dynamical system associated to $\beta-$expansion is sofic if and only if the $\beta-$expansion of 1 is eventually periodic. Especially when β is a Pisot number it gives a sofic system. Thurston [29] introduced an idea to construct a self-affine tiling generated by a Pisot unit β in connection to this sofic system. Akiyama [1] and Praggastis [24] studied in detail such self-affine tilings. G. Rauzy [25] already constructed this kind of tiling in a different approach closely related to substitutions. This tiling has a strong connection to an explicit construction of Markov partitions of toral automorphisms. See also P. Arnoux and Sh. Ito [7].

Let us recall the definition of this tiling according to [1]. Let

$$\beta = \beta^{(1)}, \beta^{(2)}, \cdots, \beta^{(r_1)} \text{ and } \beta^{(r_1+1)}, \overline{\beta^{(r_1+1)}}, \cdots, \beta^{(r_1+r_2)}, \overline{\beta^{(r_1+r_2)}}$$

be the real and the complex conjugates of β, respectively. We also denote by $x^{(j)}$ $(j = 1, 2, \cdots, r_1 + 2r_2)$ the corresponding conjugates of $x \in \mathbb{Q}(\beta)$. Define a map $\Phi\colon \mathbb{Q}(\beta) \to \mathbb{R}^{\,r_1+2\,r_2-1}$ by:

$$\Phi(x) = \left(x^{(2)}, \cdots, x^{(r_1)}, \Re(x^{(r_1+1)}), \Im(x^{(r_1+1)}), \cdots, \Re(x^{(r_1+r_2)}), \Im(x^{(r_1+r_2)}) \right).$$

Let $A = .\,a_{-1}, a_{-2}, \cdots$ be a greedy expansion in base β. Define S_A to be the set of elements of $\mathbb{Z}[\beta]_{\geq 0}$ whose greedy expansion has its fractional part A. In other words we just classify all elements of $\mathbb{Z}[\beta]_{\geq 0}$ by their fractional parts and map via Φ to have a protile $T_A = \overline{\Phi(S_A)}$. Akiyama [3] has shown that the Euclidean space is covered by these tiles, there are only finitely many tiles up to translation and the number of tiles coincides with the number of different tails of the $\beta-$expansion of 1. So, unlike in the first part of the paper, the dual tiling of the space has several tile types (see [3]), and they are obtained by a graph-directed iterated function system, rather than the standard iterated function system (see [4]). The finiteness condition in [10] or its weaker version in [3] implies that these T_A will give a non overlapping tiling of the space $\mathbb{R}^{r_1+2r_2-1}$ (see also [1]).

The second aim of this paper is to explore connectedness problem of resulting tiles of Pisot dual tiling of low degree. We use again the classical theory on the separation of roots of polynomials, and a sufficient condition for connectedness of Pisot dual tiles which is established in Akiyama-Gjini [4, Theorem 4.1]. We prove that

Theorem 1.2. *Each tile corresponding to a Pisot unit β is arcwise connected if $d_\beta(1)$ is finite and terminates with* 1.

Our main result is:

Theorem 1.3. *Let β be a Pisot unit of degree 3 or 4 defined by the monic polynomial $p(x) \in \mathbb{Z}[x]$. If $\deg \beta = 3$ or $p(0) = 1$ then each tile is connected. If $\deg \beta = 4$ and $p(0) = -1$ then each tile is connected if and only if*

$$a + c - 2[\beta] \neq 1$$

for $p(x) = x^4 - ax^3 - bx^2 - cx - 1$.

These statements are a combination of Theorem 3.1, 3.2, 3.3 and 3.4. In spite of a quite simple nature of the statements, the proof is pretty involved having a lot of subcases.

In fact, if $\deg \beta = 4$, $p(0) = -1$ and $a + c - 2[\beta] = 1$, there exists a disconnected tile. As far as we know, no example of disconnected Pisot dual tiles was known before. As these tiles are generated by consecutive integers, it was even expected that Pisot dual tiles are always connected. Thus this result gives an unfortunate surprise that there exists a concrete family of Pisot units one of whose dual tiles is disconnected. The idea of the proof of disconnectedness is found in Lemma 3.1.

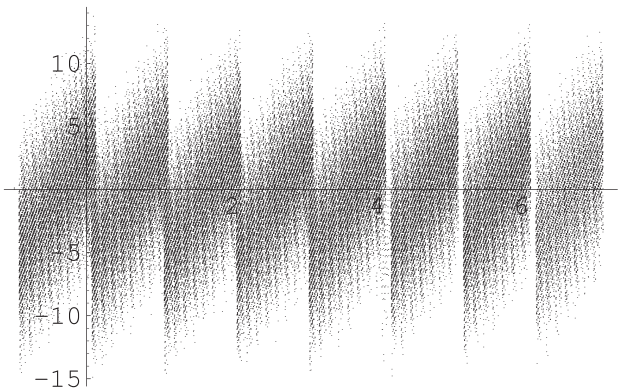

Figure 1: The projection of a disconnected tile generated by the Pisot unit β with the minimal equation $x^4 - 3x^3 - 7x^2 - 6x - 1 = 0$

2 Connectedness of self-affine tiles generated by an expanding matrix

First suppose that tiles are generated by an expanding matrix A of degree 3. Our proof of connectedness is based on the following lemma (c.f. [27], [28]) which gives a characterization of the expanding polynomials of degree 3.

Lemma 2.1. *A polynomial $p(x) = x^3 + ax^2 + bx + c$ with integer coefficients is expanding if and only if*

$$\begin{cases} |\, b - ac \,| < c^2 - 1 \\ |\, b + 1 \,| < |\, a + c \,|. \end{cases} \tag{3}$$

Theorem 2.1. *Let $A \in M_3(\mathbb{Z})$ be an expanding matrix with $|\det A| = q$ and $D = \{0, v, \cdots, (q-1)v\}$ with $v \in \mathbb{R}^3 \backslash \{0\}$. Then $T(A, D)$ is connected.*

Proof. Let $p(x) = x^3 + ax^2 + bx + c$ with $a, b, c \in \mathbb{Z}$ be the characteristic polynomial of A, which is expanding. We use the Kirat-Lau Condition to show the connectedness. Among a lot of subcases, let us only show the case $c \geq 2$, $c \leq b \leq 2c - 1$ and $a = 1 + c$ for example. Then we have $b \geq c + 2$, $-c + 1 < b - 2c < 0$, $-2c + 2 \leq 2c - 2b + 1 \leq 0$.

♢ If $-c + 1 \leq 2c - 2b + 1 \leq 0$ then the required polynomial $h(x)$ is

$$x^5 + (c-1)x^4 + (b-2c-1)x^3 + (2c-2b+1)x^2 + (b-2c)x + c = (x-1)^2 p(x)$$

♢ If $-2c + 2 \leq 2c - 2b + 1 \leq -c$ then $-c + 1 < 3c - 2b + 1 \leq 0$ and $-c \leq 2b - 4c - 1 < -1$.

⋄ If $2b - 4c - 1 > -c$ then the required polynomial $h(x)$ is

$$x^7 + (c-1)x^6 + (b-2c)x^5 + (3c-2b)x^4 + (2b-4c-1)x^3 + (3c-2b+1)x^2 + (b-2c)x + c = (x^2+1)(x-1)^2 p(x)$$

⋄ If $2b - 4c - 1 = -c$ then the required polynomial $h(x)$ is

$$x^6 + (c-1)x^5 + (b-2c)x^4 + (2c-b)x - c = (x^3 - 2x^2 + 2x - 1)p(x)$$

□

In the following part of this Section, we shall use a necessary and sufficient condition (c.f. [27], [28]) on coefficients of polynomial $p(x)$ of degree 4 for $p(x)$ to be an expanding polynomial. Also we claim that the attractor generated by an expanding integral matrix of degree 4 is connected.

Lemma 2.2. *The polynomial $p(x) = x^4 + ax^3 + bx^2 + cx + d$ with integer coefficients is expanding if and only if*

$$\begin{aligned} &\begin{cases} d \geq 2 \\ |c - ad| \leq d^2 - 2 \\ |a + c| < 1 + b + d \\ -1 + b - ac + c^2 + d + a^2 d - 2bd - acd + d^2 + bd^2 - d^3 < 0 \end{cases} \\ &\text{or} \\ &\begin{cases} d \leq -2 \\ |c - ad| \leq d^2 - 2 \\ |a + c| < -1 - b - d \\ -1 + b - ac + c^2 + d + a^2 d - 2bd - acd + d^2 + bd^2 - d^3 > 0. \end{cases} \end{aligned} \tag{4}$$

Theorem 2.2. *Let $A \in M_4(\mathbb{Z})$ be an expanding matrix with $|\det A| = q$ and $D = \{0, v, \cdots, (q-1)v\}$ with $v \in \mathbb{R}^4 \backslash \{0\}$. Then $T(A, D)$ is connected.*

Remark 1. The characteristic polynomial of the matrix A is not necessary irreducible.

3 Connectedness of self-affine tiles of the tiling generated by a Pisot unit of low degree

Let $\beta > 1$ be a Pisot unit. Let us recall the definition of graph directed attractors and graph directed iterated function systems. Let $G = G(V, E)$ be a strongly connected graph where $V = \{1, \ldots, q\}$ is the set of vertices and E is the set of directed edges. Let $E_{i,j}$ be the set of edges from i to j. Now for each $e \in E$ define a uniformly contractive map $F_e : \mathbb{R}^d \to \mathbb{R}^d$. Then by [22, Theorem 1] there exists a unique family $K_1, \ldots, K_q$ of compact non-empty sets satisfying

$$K_i = \bigcup_{j=1}^{q} \bigcup_{e \in E_{i,j}} F_e(K_j). \tag{5}$$

The set of contractions $\{F_e \,|\, e \in E\}$ is called a *graph directed iterated function system* and the sets K_i are called *graph directed attractors*. Connectedness and arcwise connectedness of these graph directed attractors are studied in [21] as well.

For words a, b, we denote by $a \oplus b$ the concatenation of words. Let G_{-1} be the natural map defined by the following commutative diagram:

$$\begin{array}{ccc} \mathbb{Q}(\beta) & \xrightarrow{\times\beta} & \mathbb{Q}(\beta) \\ {\scriptstyle \Phi}\big\downarrow & & \big\downarrow{\scriptstyle \Phi} \\ \mathbb{R}^{d-1} & \xrightarrow[G_{-1}]{} & \mathbb{R}^{d-1}. \end{array} \tag{6}$$

Then G_{-1} is contractive since β is a Pisot number. The set equations are given in this form:

$$T_{.A} = \bigcup_{.i \oplus A} G_{-1}(T_{.i \oplus A}), \tag{7}$$

where the summation is taken over all possible $i \in [0, \beta) \cap \mathbb{Z}$ such that $i \oplus A$ is admissible (see [3]). Note that we identify $.i \oplus A$ with the corresponding $\beta-$expansion to realize it as a non negative real number. Since there are finitely many tiles up to translation, it is easy to show that they form graph directed self-affine attractors by using Parry's result mentioned in Section 1.

Define η by

$$\eta := \max\{\mu \ : \ \mu \text{ is a tail of the } \beta - \text{expansion of } 1\}$$

The maximum exists because, since β is a Pisot unit, $\beta-$expansion of 1 has only a finite number of different tails. Let us call $T_{.\eta}$ the smallest tile (the name is justified because η gives the strongest constraint on its integer parts). Akiyama-Gjini [4] proved that every tile of the dual tiling generated by a Pisot unit β is arcwise connected if

$$T_{.\eta} \cap (T_{.\eta} - \Phi(\beta^{-1})) \neq \emptyset \tag{8}$$

where $T_{.\eta}$ is the smallest tile. Especially if there exist $a_i \in \mathbb{Z}$ $(i = 1, 2, \cdots)$ such that $|a_i| < \lfloor \beta \rfloor$ and $1 + \sum_{i=1}^{\infty} a_i \Phi(\beta^i) = 0$ then

$$\sum_{i=1}^{\infty} a_i^+ \Phi(\beta^{i-1}) + \Phi(\eta) = \sum_{i=1}^{\infty} a_i^- \Phi(\beta^{i-1}) + \Phi(\eta) - \Phi(1/\beta)$$

where $a_i^+ = \max\{a_i, 0\}$ and $a_i^- = -\min\{a_i, 0\}$. Since $|a_i| < \lfloor \beta \rfloor$, the words $\cdots, a_3^+ a_2^+ a_1^+ .\eta$ and $\cdots, a_3^- a_2^- a_1^- .\eta$ are admissible. So the above condition (8) is satisfied. Usually this trick works but there are some cases to be treated separately.

In course of proofs of Theorem 3.1, Theorem 3.2 and Theorem 3.3, we are required to check that each formal expansion is admissible.

Theorem 3.1. *Let β be a Pisot unit of degree 3. Then each tile is arcwise connected.*

Proof. To prove this theorem we use the characterization of Pisot units of degree 3 given by Akiyama [2]. Also we use the $\beta-$ expansion of 1 (see Gjini [12]) to find the common point of tiles which appear in (8). $\square$

To study the connectedness of each tile of the dual tiling of degree 4 first we give the characterization of Pisot units of degree 4:

Proposition 3.1. *Let $\beta > 1$ be an algebraic unit and let*

$$p(x) = x^4 - ax^3 - bx^2 - cx - d$$

with $d = \pm 1$ be its minimal polynomial. Then β is a Pisot unit if and only if

$$\begin{cases} |b-2| < a+c \\ a-c > 0 \end{cases} \quad \textit{for} \ \ d=-1; \qquad \begin{cases} |b| < a+c \\ a^2+4b-c^2 > 0 \end{cases} \quad \textit{for} \ \ d=1$$

Theorem 3.2. *Let β be a Pisot unit of degree 4 having its minimal polynomial $p(x) = x^4 - ax^3 - bx^2 - cx + 1$. Then each tile is arcwise connected.*

Proof. We write β_i instead of $\Phi(\beta^i)$ for simplicity. Let us prove only the case $a \geq 1$, $a+1 \leq b \leq 2a$ and $b-a-1 \leq c \leq a-1$.

* If $c-b+a \geq 0$ then $\lfloor\beta\rfloor = a+1$ and $d_\beta(1) = .a+1, (b-a-1, c+a-b, b-c-1, c, a)^\omega$. Since every conjugate of β is also a root of $p(x)(x-1)(x^4-1)\sum_{i=0}^{\infty} x^{8i} = 0$ then

  $$1-(c+1)\beta_1+((c-b)\beta_2+(b-a)\beta_3+a\beta_4+c\beta_5)\,(1-x^4)\sum_{i=0}^{\infty}\beta_{8i} = 0$$

  and all the coefficients have absolute value less than $\lfloor\beta\rfloor$.

* If $c-b+a = -1$ and $b \geq a+2$ then $d_\beta(1) = .a+1, (b-a-2, a+1, b-a-2, 0, a-1, b-a, a-1, b-a-1, a)^\omega$. Since every conjugate of β is also a root of $p(x)(x^2-x+1)\sum_{i=0}^{\infty} x^{5i} = 0$ then

  $$1-(c+1)\beta_1-(a\beta_2-\beta_3+c\beta_4+a\beta_5+c\beta_6)\sum_{i=0}^{\infty}\beta_{5i} = 0$$

  and all the coefficients have absolute value less than $\lfloor\beta\rfloor = a+1$.

* If $c-b+a = -1$ and $b = a+1$ then $c = 0$, $\lfloor\beta\rfloor = a$ and $d_\beta(1) = .\,a, a, (a, a-1)^\omega$. Since every conjugate of β is also a root of the $p(x)(x^2+1)\sum_{i=0}^{\infty}\beta_{4i} = 0$ then

  $$.\eta = {}^\omega(a-1, a), a, 0.\eta \quad -0.1$$

  is a common point of the smallest tile $T_{.\eta}$ and $T_{.\eta} - \Phi(\beta^{-1})$.

$\square$

When β is a Pisot unit of degree 4 with minimal polynomial $p(x) = x^4 - ax^3 - bx^2 - cx - 1$, we have the following Lemma:

Lemma 3.1. *If the negative root γ of the polynomial $x^2 - \lfloor\beta\rfloor x - 1$ has the property*

$$p(\gamma) > 0$$

then at least one of the tiles is not connected.

The rough geometric idea of this lemma is to project the tiles along the negative conjugate direction as in Figure 1 and to show that the restriction on digits which comes from the sofic system, gives an obstacle to connect subdivided pieces. Using this Lemma we get that

Theorem 3.3. *Let β be a Pisot unit of degree 4 with its minimal polynomial $p(x) = x^4 - ax^3 - bx^2 - cx - 1$. Then each tile is arcwise connected except for the following cases*

$$\begin{cases} a \geq 5 \\ c = a - 3 \\ \frac{5-3a}{2} \leq b \leq -a \end{cases} \quad \begin{cases} a \geq 3 \\ c = a - 1 \\ \frac{1-a}{2} \leq b \leq -1 \end{cases} \quad \begin{cases} a \geq 3 \\ c = a + 1 \\ \frac{1+a}{2} \leq b \leq a-1 \end{cases} \quad \begin{cases} a \geq 1 \\ c = a + 3 \\ \frac{5+3a}{2} \leq b \leq 2a+2 \end{cases}$$

Proof. Let us prove only the case: $c = a + 1$ then $\frac{a+1}{2} \leq b \leq 2a$ and

$$d_\beta(1) = \begin{cases} .a\ , b+1, (0\ , a-b,\ b\ ,\ b\ , a-b+1,\ 0\ ,\ b)^\omega & \text{if } b \leq a-1 \\ .a+1,\ 0\ ,\ 0\ , (0\ ,\ a\ ,\ 0\ ,\ 0\ ,\ a\ ,\ a\ ,\ 1)^\omega & \text{if } b = a \\ .a+1, b-a-1, 2a-b+1, b-a,\ a\ ,\ 1 & \text{if } b \geq a+1 \end{cases}$$

* For $b \leq a - 1$, to show that one of the tiles is not connected, according to Lemma 3.1, it is enough to prove that $p(\gamma) > 0$. Since $\gamma^2 - a\gamma - 1 = 0$ we have that

$$p(\gamma) \geq \gamma^4 - a\gamma^3 - (a-1)\gamma^2 - (a+1)\gamma - 1 = \gamma^2(1-\gamma) > 0.$$

* For $b = a$, since every conjugate of β is also a root of $p(x)(x-1)\sum_{i=0}^{\infty} x^{3i} = 0$, then

$$1 + a\beta_1 - \beta_2 + (\beta_3 - \beta_4) \textstyle\sum_{i=0}^{\infty} \beta_{3i} = 0$$

  and all the coefficients have absolute value less than $\lfloor\beta\rfloor = a + 1$.

* For $b \geq a + 1$, we use Theorem 1.2.

$\square$

From the proof of this Theorem we can easily see that $a + c - 2\lfloor\beta\rfloor = 1$ for the cases when at least one of the tiles is disconnected and $a + c - 2\lfloor\beta\rfloor \leq 0$ for the cases when each tile is connected. So, the above theorem can be written in the following equivalent way:

Theorem 3.4. *Let β be a Pisot unit of degree 4 with its minimal polynomial $p(x) = x^4 - ax^3 - bx^2 - cx - 1$. Then*

- $a + c - 2\lfloor\beta\rfloor \leq 1$,
- *each tile is arcwise connected if and only if* $a + c - 2\lfloor\beta\rfloor \leq 0$.

References

[1] S. Akiyama, *Self affine tiling and Pisot numeration system*, Number Theory and its Applications, ed. by K. Győry and S. Kanemitsu, 7–17 Kluwer 1999.

[2] S. Akiyama, *Cubic Pisot Units with finite beta expansions*, Algebraic Number Theory and Diophantine Analysis, ed. by F. Halter-Koch and R.F. Tichy, de Gruyter (2000), 11–26.

[3] S. Akiyama, *On the boundary of self affine tilings generated by Pisot numbers*, Journal of Math. Soc. Japan, vol. 54, no. 2 (2002), 283–308.

[4] S. Akiyama and N. Gjini, *Connectedness of number theoretic tilings*, submitted.

[5] S. Akiyama and J. Thuswaldner, *Topological properties of two-dimensional number systems*, J. Théor. des Nombres de Bordeaux **12** (2000), 69–79.

[6] S. Akiyama and J. Thuswaldner, *On the topological structure of fractal tilings generated by quadratic number systems*, submitted.

[7] P. Arnoux and Sh. Ito, *Pisot substitutions and Rauzy fractals*, Journés Montoises d'Informatique Theorique (Marne-la-Vallé, 2000). Bull. Belg. Math. Soc. Simon Stevin **8** (2001), no. 2, 181–207.

[8] C. Bandt, *Self-similar tilings and patterns described by mappings*, The Mathematics of Long-Range Aperiodic Order, Kluwer, (1997), 45–83.

[9] A. Bertrand, *Développements en base de Pisot et répartition modulo* 1. C. R. Acad. Sci. Paris, **285** (1977), 419–421.

[10] C. Frougny and B. Solomyak, *Finite beta-expansions*, Ergod. Th. and Dynam. Sys. **12** (1992), 713–723.

[11] W. J. Gilbert, *Radix representations of quadratic number fields*, J. Math. Anal. Appl. **83** (1981), 263–274.

[12] N. Gjini *$\beta-$expansion of* 1 *for quartic Pisot units*, to appear in Periodica Mathematica Hungarica.

[13] K. Gröchenig, A. Haas, *Self-similar lattice tilings*, J. Fourier. Anal. Appl. **1** (1994), 131-170.

[14] D. Hacon, N. C. Saldanha and J. J. P. Veerman, *Remarks on self-affine tilings*, Experiment. Math. **3** (1994), 317-327

[15] M. Hata, *On the Structure of Self-Similar Sets*, Japan J. Appl. Math. **2** (1985), 381–414.

[16] I. Kátai and I. Kőrnyei, *On Number Systems in Algebraic Number Fields*, Publ. Math. Debrecen **41** no. 3–4 (1992), 289–294.

[17] R. Kenyon, *Self-replicating tilings*, Symbolic Dynamics and Its Applications (P. Walters Ed.), Contemporary Math. **135** 239–263, Amer. Math. Providence RI, 1992.

[18] I. Kirat and K. S. Lau, *On the Connectedness of Self-Affine Tiles*, J. London. Math. Soc. **2** 62 (2000), 291–304.

[19] B. Kovács and A. Pethő, *Number systems in integral domains, especially in orders of algebraic number fields*, Acta Sci. Math. Szeged, **55** (1991), 287–299.

[20] J. C. Lagarias and Y. Wang, *Self-affine tiles in* $\mathbb{R}^n$, Adv. Math. **121** (1996), 21–49.

[21] J. Luo, S. Akiyama and J. Thuswaldner, *On boundary connectedness of connected tiles*, to appear in Math. Proc. Camb. Phil. Soc.

[22] R. D. Mauldin and S. C. Williams, *Hausdorff dimension in graph directed constructions*, Trans. Amer. Math. Soc. **309** (1988), 811–829.

[23] W. Parry, *On the $\beta-$expansions of real numbers*, Acta Math. Acad. Sci. Hungar. **11** (1960), 401–416.

[24] B. Praggastis, *Markov partition for hyperbolic toral automorphism*, Ph. D. Thesis, Univ. of Washington, 1992.

[25] G. Rauzy, *Nombres Algébriques et substitutions*, Bull. Soc. France **110** (1982),147–178.

[26] K. Schmidt, *On periodic expansions of Pisot numbers and Salem numbers*, Bull. London Math. Soc., **12** (1980), 269–278.

[27] I. Schur, *Über Potenzreihen, die im Inneren des Einheitskreises bescränkt sind*, J. Reine angew. Math. **147**, 205–232 (1917).

[28] T. Takagi, *Introduction to Algebra*, 1965, in Japanese.

[29] W. P. Thurston, *Groups, Tilings and Finite state automata*, AMS Colloquium lectures, 1989.

[30] A. Vince, *Digit Tiling of Euclidean Space*, in Directions in mathematical quasicristals, Amer. Math. Soc., Providence, RI, 2000, 329–370.

[31] Y. Wang, *Self-affine tiles. Advances in wavelets* (Hong Kong, 1997), Springer, Singapore, 1999, pp. 261–282.

Shigeki Akiyama
Department of Mathematics, Faculty of Science, Niigata University, Ikarashi 2-8050, NIIGATA 950-2181, JAPAN
e-mail address: akiyama@math.sc.niigata-u.ac.jp

Nertila Gjini
Department of Mathematics, Faculty of Science, Tirana University, TIRANA, ALBANIA
e-mail address: nkokona@hotmail.com