Self affine tiling and Pisot numeration system *

Shigeki Akiyama

1 Introduction

Let $\beta > 1$ be a real number. Consider an expansion of positive real number x:

$$x = a_N \beta^N + a_{N-1}\beta^{N-1} + a_{N-2}\beta^{N-2} + \cdots,$$

with $a_i \in \mathbf{Z} \cap [0, \beta)$. A *greedy* expansion of x in base β is such expansion with

$$|x - \sum_{M}^{N} a_i \beta^i| < \beta^M \tag{1}$$

for any M. By using *greedy algorithm*, such an expansion always exists for any x. This is a natural extension of decimal or binary expansion to a real base. An expansion of x is *admissible*, if (1) holds for all M. Hereafter, we use a notation

$$x = a_N a_{N-1} a_{N-2} \cdots.$$

A *Pisot number* is an algebraic integer greater than 1 whose conjugates other than itself have modulus smaller than 1. We have a particular interest in the case when $\beta > 1$ is a Pisot number. Surprisingly, one can find many similar phenomena with binary or decimal expansion. See [1]. We use a term 'Pisot numeration system' to call this method to represent real numbers in a power series in Pisot number base.

In this paper, we will prove fundamental properties of tilings generated by Pisot numeration system. Let $\mathbf{Fin}(\beta)$ be a set of all finite greedy expansion in base β. Consider two properties

$$(\mathrm{F}') \qquad \mathbf{Fin}(\beta) \supset \mathbf{Z}[\beta]_{\geq 0}$$

*This research was partially supported by the Minsitry of Education, Science, Sports and Culture, Grant-in-Aid for Encouragement of Young Scientists, 09740013, 1998.

and

$$\text{(F)} \qquad \mathbf{Fin}(\beta) = \mathbf{Z}[\beta^{-1}]_{\geq 0}.$$

In [1], the author proved a necessary and sufficient condition whether β has property (F') or not. In §2 theorem 1, it is proved that (F) is equivalent to (F'). Thus, there exist a finite algorithm to determine (F). Next we define a self affine tiling of $\mathbf{R}^{n-1}$ by Pisot unit of degree n with the property (F). When β is a cubic Pisot unit with (F) which is not totally real, we obtain a self similar tiling. Otherwise, we get a self affine, not self similar, tiling of $\mathbf{R}^{n-1}$. See the figures in §3.

We can find interesting examples of these tiling in [10], [7], [11], [8]. They treat these tilings in terms of substitution or finite state automata. In contrast, this paper treat these tilings in a context of 'Pisot numeration system'. The author believes that, this method gives us a clear understanding of universal phenomena of these tilings. For instance, it is proved in theorem 2 that the origin is an inner point of the central tile $\mathcal{T}$ by using geometry of numbers. This fact is very much fundamental. We can show that the boundary of each tile is nowhere dense in $\mathbf{R}^{n-1}$ as a corollary.

It is also shown that each tile is arcwise connected in Theorem 3, under a certain weak condition. The method of the proof is the 'encircling method' developed in [3].

2 Self affine tiling generated by Pisot numeration system

Let β be a Pisot number of degree n, and $\mathbf{Fin}(\beta)$ be a set of all elements in $\mathbf{Q}(\beta)$ which have finite greedy expansion in base β. Consider a property

$$\text{(F)} \qquad \mathbf{Fin}(\beta) = \mathbf{Z}[\beta^{-1}]_{\geq 0}.$$

In [6], it is shown that a Pisot number β whose irreducible polynomial is of a form:

$$x^n - a_{n-1}x_{n-1} - a_{n-2}x_{n-2} - \ldots - a_0,$$

with $a_i \in \mathbf{Z}_{>\mathbf{0}}$ and $a_i \geq a_{i-1}$ $(i = 1, 2, \ldots, n-1)$ has property (F).

Let $\beta = \beta^{(1)}, \beta^{(2)}, \ldots \beta^{(r_1)}$ be the real conjugates and $\beta^{(r_1+1)}, \beta^{(r_1+2)}, \ldots \beta^{(r_1+r_2)}$, together with their complex conjugates, be the complex conjugates of β. We also denote by $x^{(j)}$ $(j = 1, 2, \ldots, n)$ the corresponding conjugate of $x \in \mathbf{Q}(\beta)$. Here we put $x^{(r_1+r_2+j)} = \overline{x^{(r_1+j)}}$ for $j = 1, 2, \ldots, r_2$.

Let p be a non negative integer and define $M_j(p)$ $(j = 2, 3, \dots n)$ as an upper bound of

$$|\sum_{i=0}^{p} a_{p-i} (\beta^{(j)})^i|,$$

where $\sum_{i=0}^{p} a_i \beta^{-i}$ runs through finite greedy expansions of length at most $p+1$. Let M_j be an upper bound of $M_j(p)$ $(p = 1, 2, \dots)$. One can take $M_j = [\beta]/(1 - |\beta^{(j)}|)$. Here $[x]$ is the greatest integer not exceeding x. Let b_j $(j = 1, \dots n)$ be positive real numbers and $C = C(b_1, b_2, \dots, b_n)$ be a set of elements in $\mathbf{Z}[\beta]$ such that

$$|x^{(j)}| \leq b_j.$$

Obviously, C is a finite set. Then we can show a slight generalization of Theorem 2 in [1].

Theorem 1. Let β be a Pisot number. Then β has the property (F) if and only if every element of $C = C(1, M_2, M_3, \dots, M_n)$ has finite greedy expansion in base β.

Proof. The proof is quite similar with that of Theorem 2 in [1]. We only have to note that for each element x in $\mathbf{Z}[\beta^{-1}]$ there exist q_0, that if $q \geq q_0$ with $q \in \mathbf{Z}_{\geq \mathbf{0}}$ then $\beta^q x$ is in $\mathbf{Z}[\beta]$.

Recently, the complete list of cubic Pisot units with (F) is established in [2], by using this theorem. Define a map $\Phi : \mathbf{Q}(\beta) \to \mathbf{R}^{n-1}$ by

$$\Phi(x) = (x^{(2)}, \dots, x^{(r_1)}, \Re(x^{(r_1+1)}), \Im(x^{(r_1+1)}), \Re(x^{(r_1+2)}), \Im(x^{(r_1+2)}), \dots, \Re(x^{(r_1+r_2)}), \Im(x^{(r_1+r_2)})).$$

Proposition 1. Let β be a Pisot number of degree n. Then $\Phi(\mathbf{Z}[\beta]_{>0})$ is dense in $\mathbf{R}^{n-1}$.

Proof. First, we prove $\Phi(\mathbf{Z}[\beta])$ is dense in $\mathbf{R}^{n-1}$. Thus it suffice to show that a set consists of elements:

$$
x_1 \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}
+ x_2 \begin{pmatrix} \beta^{(2)} \\ \beta^{(3)} \\ \vdots \\ \beta^{r_1} \\ \Re(\beta^{(r_1+1)}) \\ \Im(\beta^{(r_1+1)}) \\ \Re(\beta^{(r_1+2)}) \\ \Im(\beta^{(r_1+2)}) \\ \vdots \\ \Re(\beta^{(r_1+r_2)}) \\ \Im(\beta^{(r_1+r_2)}) \end{pmatrix}
+ x_3 \begin{pmatrix} (\beta^{(2)})^2 \\ (\beta^{(3)})^2 \\ \vdots \\ (\beta^{r_1})^2 \\ \Re((\beta^{(r_1+1)})^2) \\ \Im((\beta^{(r_1+1)})^2) \\ \Re((\beta^{(r_1+2)})^2) \\ \Im((\beta^{(r_1+2)})^2) \\ \vdots \\ \Re((\beta^{(r_1+r_2)})^2) \\ \Im((\beta^{(r_1+r_2)})^2) \end{pmatrix}
+ \cdots + x_n \begin{pmatrix} (\beta^{(2)})^{n-1} \\ (\beta^{(3)})^{n-1} \\ \vdots \\ (\beta^{r_1})^{n-1} \\ \Re((\beta^{(r_1+1)})^{n-1}) \\ \Im((\beta^{(r_1+1)})^{n-1}) \\ \Re((\beta^{(r_1+2)})^{n-1}) \\ \Im((\beta^{(r_1+2)})^{n-1}) \\ \vdots \\ \Re((\beta^{(r_1+r_2)})^{n-1}) \\ \Im((\beta^{(r_1+r_2)})^{n-1}) \end{pmatrix},
$$

with integer coefficients x_i with $i = 1, 2, \ldots, n$ is dense in $\mathbf{R}^{n-1}$. Define a $(n-1) \times (n-1)$ matrix

$$
A = \begin{pmatrix}
\beta^{(2)} & (\beta^{(2)})^2 & (\beta^{(2)})^3 & \cdots & (\beta^{(2)})^{n-1} \\
\beta^{(3)} & (\beta^{(3)})^2 & (\beta^{(3)})^3 & \cdots & (\beta^{(3)})^{n-1} \\
 & \vdots & \vdots & \vdots & \\
\beta^{(r_1)} & (\beta^{(r_1)})^2 & (\beta^{(r_1)})^3 & \cdots & (\beta^{(r_1)})^{n-1} \\
\beta^{(r_1+1)} & (\beta^{(r_1+1)})^2 & (\beta^{(r_1+1)})^3 & \cdots & (\beta^{(r_1+1)})^{n-1} \\
\overline{\beta^{(r_1+1)}} & (\overline{\beta^{(r_1+1)}})^2 & (\overline{\beta^{(r_1+1)}})^3 & \cdots & (\overline{\beta^{(r_1+1)}})^{n-1} \\
 & \vdots & \vdots & \vdots & \\
\beta^{(r_1+r_2)} & (\beta^{(r_1+r_2)})^2 & (\beta^{(r_1+r_2)})^3 & \cdots & (\beta^{(r_1+r_2)})^{n-1} \\
\overline{\beta^{(r_1+r_2)}} & (\overline{\beta^{(r_1+r_2)}})^2 & (\overline{\beta^{(r_1+r_2)}})^3 & \cdots & (\overline{\beta^{(r_1+r_2)}})^{n-1}
\end{pmatrix}
$$

Let P be a $(n-1) \times (n-1)$ matrix of a form:

$$
P = I_{r_1 - 1} \oplus \overbrace{\begin{pmatrix} 1 & \sqrt{-1} \\ 1 & -\sqrt{-1} \end{pmatrix} \oplus \begin{pmatrix} 1 & \sqrt{-1} \\ 1 & -\sqrt{-1} \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 1 & \sqrt{-1} \\ 1 & -\sqrt{-1} \end{pmatrix}}^{r_2 \text{ times}}.
$$

Here $A \oplus B$ is a matrix of a form:

$$
\begin{pmatrix} A & \mathbf{O} \\ \mathbf{O} & B \end{pmatrix},
$$

and I_k is the identity matrix of size k. Then

$$P \begin{pmatrix} \beta^{(2)} & (\beta^{(2)})^2 & (\beta^{(2)})^3 & \cdots & (\beta^{(2)})^{n-1} \\ \beta^{(3)} & (\beta^{(3)})^2 & (\beta^{(3)})^3 & \cdots & (\beta^{(3)})^{n-1} \\ & \vdots & \vdots & \vdots & \\ \beta^{(r_1)} & (\beta^{(r_1)})^2 & (\beta^{(r_1)})^3 & \cdots & (\beta^{(r_1)})^{n-1} \\ \Re(\beta^{(r_1+1)}) & \Re((\beta^{(r_1+1)})^2) & \Re((\beta^{(r_1+1)})^3) & \cdots & \Re((\beta^{(r_1+1)})^{n-1}) \\ \Im(\beta^{(r_1+1)}) & \Im((\beta^{(r_1+1)})^2) & \Im((\beta^{(r_1+1)})^3) & \cdots & \Im((\beta^{(r_1+1)})^{n-1}) \\ & \vdots & \vdots & \vdots & \\ \Re(\beta^{(r_1+r_2)}) & \Re((\beta^{(r_1+r_2)})^2) & \Re((\beta^{(r_1+r_2)})^3) & \cdots & \Re((\beta^{(r_1+r_2)})^{n-1}) \\ \Im(\beta^{(r_1+r_2)}) & \Im((\beta^{(r_1+r_2)})^2) & \Im((\beta^{(r_1+r_2)})^3) & \cdots & \Im((\beta^{(r_1+r_2)})^{n-1}) \end{pmatrix} = A.$$

Thus it suffice to show that

$$A^{-1}P \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix} \mathbf{Z} \pmod{\mathbf{Z}^{n-1}} = A^{-1} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix} \mathbf{Z} \pmod{\mathbf{Z}^{n-1}}$$

is dense in $\mathbf{R}^{n-1}/\mathbf{Z}^{n-1}$. By Cramer's rule,

$$A^{-1} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = (\det A)^{-1} \begin{pmatrix} \det A_1 \\ \det A_2 \\ \vdots \\ \det A_{n-1} \end{pmatrix},$$

where

$$A_i = \begin{array}{c} \hphantom{\beta^{(r_1+r_2)} \quad \cdots \quad} i\text{-th column} \hphantom{\quad \cdots \quad (\beta^{(r_1+r_2)})^{n-1}} \\ \begin{pmatrix} \beta^{(2)} & \cdots & \check{1} & \cdots & (\beta^{(2)})^{n-1} \\ \beta^{(3)} & \cdots & 1 & \cdots & (\beta^{(3)})^{n-1} \\ & \vdots & \vdots & \vdots & \\ \beta^{(r_1)} & \cdots & 1 & \cdots & (\beta^{(r_1)})^{n-1} \\ \beta^{(r_1+1)} & \cdots & 1 & \cdots & (\beta^{(r_1+1)})^{n-1} \\ \overline{\beta^{(r_1+1)}} & \cdots & 1 & \cdots & (\overline{\beta^{(r_1+1)}})^{n-1} \\ & \vdots & \vdots & \vdots & \\ \beta^{(r_1+r_2)} & \cdots & 1 & \cdots & (\beta^{(r_1+r_2)})^{n-1} \\ \overline{\beta^{(r_1+r_2)}} & \cdots & 1 & \cdots & (\overline{\beta^{(r_1+r_2)}})^{n-1} \end{pmatrix} \end{array}.$$

Note that

$$\det A = \pm \prod_{i=2}^{n} \beta^{(i)} \prod_{i<j} (\beta^{(i)} - \beta^{(j)}) = c\beta^{-1} \prod_{i<j} (\beta^{(i)} - \beta^{(j)}),$$

with a non zero integer c. We see $\pm c\beta^{-1} \det A_i / \det A$ is nothing but the elementary symmetric polynomial of degree $n-1-i$ with respect to $\beta^{(2)}, \beta^{(3)}, \dots, \beta^{(n)}$. Thus n-elements

$$1, \frac{\det A_1}{\det A}, \frac{\det A_2}{\det A}, \dots, \frac{\det A_{n-1}}{\det A}$$

lie in $\mathbf{Q}(\beta)$. Moreover these n-elements are linearly independent over $\mathbf{Q}$. To show this, note that each element x of $\mathbf{Q}(\beta)$ can be uniquely expressed in a form $x = \sum_{i=0}^{n-1} \eta_i \beta^i$ with $\eta_i \in \mathbf{Q}$. Define a map $\Xi : \mathbf{Q}(\beta) \to \mathbf{Z} \cap [0, n-1]$ by the maximal index i that $\eta_i \neq 0$. By induction, $\Xi(1) = 0$ and $\Xi(\det A_i / \det A) = n-i$. This proves the linear independence. Now, one can apply Kronecker's approximation theorem to show that $\Phi(\mathbf{Z}[\beta])$ is dense in $\mathbf{R}^{n-1}$.

Let us take an arbitrarily point $x \in \mathbf{R}^{n-1}$. For any $\varepsilon > 0$, there exists an element $x' \in \mathbf{Z}[\beta]$ that $|x - \Phi(x')| < \varepsilon/2$. As β is a Pisot number, there exist a positive integer M that $|\Phi(\beta^M)| < \varepsilon/2$ and $x + \beta^M > 0$. Put $y = x' + \beta^M$, then $|x - y| < \varepsilon$ and $y \in \mathbf{Z}[\beta]_{>0}$. This proves the assertion.

Let $A = a_L a_{L-1} \cdots a_M$ be a greedy expansion in base β. Put $\deg_\beta(A) = \deg(A) = L$ and $\mathrm{ord}_\beta(A) = \mathrm{ord}(A) = M$. Define $S_A = S_{a_L a_{L-1} \cdots a_M}$ be a set of all elements in $\mathrm{Fin}(\beta)$ whose greedy expansion has the tail part A. In other words, each element of S_A has a form:

$$b_K b_{K-1} \cdots b_{L+1} a_L a_{L-1} \cdots a_M.$$

Let $T_A = \overline{\Phi(S_A)}$. For the empty word λ, we also define

$$S_\lambda = S_{.} = \{x \in \mathrm{Fin}[\beta] \mid \mathrm{ord}_\beta(x) \geq 0\}.$$

and $\deg(\lambda) = -1$. We designate by $\mathcal{S} = S_{.}$ to avoid confusion. A *tile* is a set T_A with $\deg(A) = -1$. A *subtile* is a set T_A with $\deg(A) \geq -1$. Similarly, let $\mathcal{T} = T_{.} = \overline{\Phi(\mathcal{S})}$ which is called the *central tile.* Against standard notations, we do not assume that a tile should be a set coincides with the closure of its interior. This fact is shown in a special case in corollary 1 of theorem 2.

Corollary. Let β be a Pisot number of degree n with property (F). Then

$$\mathbf{R}^{n-1} = \bigcup_{\deg(A)=-1} T_A.$$

Proof. By the assumption, it is clear that

$$\mathbf{Z}[\beta]_{>0} \subset \mathbf{Z}[\beta^{-1}]_{\geq 0} = \bigcup_{.a_{-1}a_{-2}\cdots a_{-M}} S_{.a_{-1}a_{-2}\cdots a_{-M}}.$$

Applying the map Φ to both hand sides and taking the closure, we get the result.

The next lemma is most essential in the proof Theorem 1 in [1].

Lemma 1.
Let $\beta > 1$ be an algebraic unit of degree n, and M be a positive number. Put

$$X(p) = \{x \in \mathrm{Fin}(\beta) \mid |x| \leq M, \mathrm{ord}_\beta(x) = -p\}.$$

Then

$$\lim_{p\to\infty} \min_{x \in X(p)} \max_{j=2,3,\ldots,n} \left| x^{(j)} \right| = \infty.$$

Proof. This was already proved in [3]. However, we restate it, for the convenience of the reader. Assume that there exist a constant B and an infinite sequence $x_i \quad (i = 1, 2, \ldots)$ so that both

$$|x_i^{(j)}| \leq B \quad \text{for } j = 2, 3, \ldots n \quad \text{and} \quad \lim_{i\to\infty} \mathrm{ord}_\beta(x_i) = -\infty$$

holds. Noting β is a unit and $|x_i| \leq M$, these x_i's are finite. On the other hand, by the definition of the greedy expansion, $\{x_i \mid i = 1, 2, \ldots\}$ is an infinite set. This is absurd, which proves the lemma.

Let $\mathrm{Inn}(X)$ be a set of inner points of X.

Theorem 2. Let β be a Pisot unit with property (F). Then for each element $x \in \mathcal{S}$, we have $\Phi(x) \in \mathrm{Inn}(\mathcal{T})$. Especially, the origin is an inner point of the central tile $\mathcal{T}$.

Proof. First we show that the origin $\Phi(0)$ is an inner point of $\mathcal{T}$. Let $x = x_1 + x_2$ with $\mathrm{ord}_\beta(x_1) > 0, \deg_\beta(x_2) \leq 0$. As β is a Pisot number, there exist an absolute constant C that $|x_1^{(j)}| < C$ for all $j = 2, 3, \cdots, n$. Clearly x_2 is bounded by some absolute constant M. By using $\mathrm{ord}_\beta(x) = \mathrm{ord}_\beta(x_2) = -N$ and Lemma 1, for any positive B, there exist N such that $|x_2^{(j)}| > B + C$ for a certain $j \in \mathbf{Z} \cap [2, n]$, which shows $|x^{(j)}| > B$. Thus we have proved that, for any positive B, there exist N such that if $\mathrm{ord}_\beta(x) \leq -N$ then there exist a conjugate $|x^{(j)}| > B$ with $j = 2, 3, \ldots n$. Let $x = \beta^{-N+1} y$, then

$$\mathrm{ord}_\beta(y) < 0 \Longrightarrow \exists j \in \mathbf{Z} \bigcap [2, n] \quad |y^{(j)}| > |\beta^{(j)}|^{N-1} B.$$

Thus there are no elements $\Phi(y)$ with $y \in \mathrm{Fin}(\beta)$ of negative order in the disk

$$U = \{\xi \in \mathbf{R}^{n-1} \quad | \quad |\xi| \leq (\min_{2 \leq j \leq n} |\beta^{(j)}|)^{N-1} B\}.$$

Here $|\xi|$ is the Euclidian norm of ξ in $\mathbf{R}^{n-1}$. By using Proposition 1, $U \subset \mathcal{T}$. Thus 0 is an inner point of $\mathcal{T}$. Let x be an element of $\mathrm{Fin}(\beta)$ whose expansion is

$$x = x_k x_{k-1} \cdots x_0.$$

Let m be the minimal length that $[\beta] 0^{m-1} 1 = [\beta] 00 \cdots 01$ is admissible. Subdivide $\mathcal{T}$ in a form:

$$\mathcal{T} = \bigcup T_{x_{k+m} x_{k+m-1} \cdots x_0 \cdot},$$

where the index runs over all greedy expansion of length $k + m + 1$. We see that 0 is also an inner point of $T_{0^{k+m+1}}$. By adding x, $\Phi(x_k x_{k-1} \cdots x_0)$ is an inner point of $T_{0^m x_k x_{k-1} \cdots x_0}$. Thus the theorem is proved.

Corollary 1. Let β be a Pisot unit with property (F). For each $x \in S_A$, we have $\Phi(x) \in \mathrm{Inn}(T_A)$. Moreover, $\overline{\mathrm{Inn}(T_A)} = T_A$.

Proof. Multiplying a suitable power of β, we may assume that $\deg_\beta(A) = -1$. In theorem 2, we already proved the first statement in the case when A is an empty word. One may apply the same subdividing argument to prove that $\Phi(x) \in \mathrm{Inn}(T_A)$ if $x \in S_A$. As T_A is closed, $T_A \supset \overline{\mathrm{Inn}(T_A)}$. Let $x \in T_A$. Then for any $\varepsilon > 0$, there exists $y \in S_A$ that $|x - \Phi(y)| < \varepsilon$. As $\Phi(y) \in \mathrm{Inn}(T_A)$, we have $T_A \subset \overline{\mathrm{Inn}(T_A)}$. This shows the assertion.

Corollary 2. Let β be a Pisot unit with property (F) and $\partial(T_A)$ be a set of boundary elements of T_A. Then $\partial(T_A)$ is closed and nowhere dense in $\mathbf{R}^{n-1}$.

Proof. Let $x \in \partial(T_A)$. For any $\varepsilon > 0$, there exist infinite elements $y_i \in \mathbf{Fin}(\beta)$ $(i = 1, 2, \cdots)$ that $y_i \notin S_A$ and $|x - \Phi(y_i)| < \varepsilon$. One may assume that all y_i are in a single S_B with a greedy expansion $B \neq A$, as there exist only finite number of tiles which have common points with T_A. Thus

$$\partial(T_A) = \bigcup_{B}^{\text{finite}} (T_A \cap T_B),$$

which shows $\partial(T_A)$ is closed. For any $x \in \partial(T_A)$ and $\varepsilon > 0$, there exist $y \in S_A$ with $|x - \Phi(y)| \leq \varepsilon$. As $\Phi(y) \in \mathrm{Inn}(T_A)$, there exist no elements of $\partial(T_A)$ in a sufficiently small neighborhood of $\Phi(y)$. This shows that $\partial(T_A)$ is nowhere dense in $\mathbf{R}^{n-1}$.

Let β be a Pisot unit with (F) and $1 = a_{-1}\beta^{-1} + a_{-2}\beta^{-2} + \ldots + a_{-M}\beta^{-M}$ be an expansion of 1 defined by an algorithm

$$c_{-i} = \beta c_{-i+1} - [\beta c_{-i+1}] \quad a_{-i} = [\beta c_{-i+1}]$$

with $c_0 = 1$. It is well known that the word $b_k b_{k-1} \cdots b_1$ with alphabet $b_i \in [0, \beta) \cap \mathbf{Z}$ is admissible if and only if $b_k b_{k-1} \cdots b_1$ is lexicographically less than $a_{-1} a_{-2} \cdots a_{-M}$ at any starting point. The equation

$$x^M = a_{-1}x^{M-1} + a_{-2}x^{M-2} + \cdots + a_{-M}$$

is the 'characteristic equation' defined in [9]. It is likely that $a_{-M} = 1$ holds for all Pisot unit with (F). Now we prove

Theorem 3. Let β be a Pisot unit with (F) with $a_{-M} = 1$. Then each tile is arcwise connected.

Proof. First, note that there exist only finite tiles up to parallel translation. Let $T_{A_1}, T_{A_2}, \cdots, T_{A_h}$ be a representative system of different tiles. Define an open disk

$$D(x,r) = \{z \in \mathbf{R}^{\mathbf{n-1}} \ : \ |z-x| < r\}.$$

Let $R > 0$ be a radius that $T_{A_q} \subset D(\Phi(A_q), R)$ for $q = 1, 2, \cdots, h$. Consider a fixed tile $T = T_A = T_{.a_1 a_2 \cdots a_k} = \overline{\Phi(S_A)}$. Now we have

$$S_A = \bigcup_C S_C,$$

where the sum is taken over all greedy expansion of the form $C = b_1 b_2 \cdots b_j.a_1 a_2 \cdots a_k$. This relation give rise to a subdivision of the tile T into subtiles T_C. Hence $T = \overline{\Phi(S_A)} \subset \bigcup_C D(\Phi(C), \delta^j R)$ with $\delta = \max_{i=2,3,\cdots,n} |\beta^{(i)}|$. Put $K(A,j) = \bigcup_C D(\Phi(C), \delta^j R)$ then $T \subset \cap_{j=0}^{\infty} K(A,j)$. Noting $\delta < 1$, we also have $\cap_{j=0}^{\infty} K(A,j) \subset T$, which shows $T = \bigcap_{j=0}^{\infty} K(A,j)$. Take arbitrary two points $x_1, x_2 \in T$. Our purpose is to construct a curve in T which join x_1, x_2. Note that, by the definition of R, if there exist a common point ξ of two tiles T_{B_1} and T_{B_2}, then $\xi \in D(\Phi(B_1), R) \cap D(\Phi(B_2), R)$. Using characteristic equation of β, we can construct infinitely many elements of $T_{0.} \cap T_{1.}$. Moreover we have

$$T_{f_1 e_1 e_2 \cdots e_M} \cap T_{f_2 e_1 e_2 \cdots e_M} \neq \emptyset$$

for $|f_1 - f_2| = 1$. Using these facts, one can show by induction on j, that $K(A,j)$ is arcwise connected. In fact, we can take $y(j,k)$ $k = 1, 2, \cdots M_j$ which satisfies $y(j,1) = x_1, y(j, M_j) = x_2$ and

$$L(j) = \bigcup_{k=1}^{M_j - 1} \mathcal{L}[y(j,k), y(j,k+1)] \subset K(A,j)$$

with $\max_{k=1}^{M_j} |y(j,k) - y(j,k+1)| = O(\delta^j)$ and $y(j,k)$ for $k = 2, 3, \cdots, M_j - 1$ is the center of the disk appeared in the definition of $K(A,j)$. Here $\mathcal{L}[x,y]$ is the line segment which join $x, y \in \mathbf{R}^{n-1}$. Moreover we can choose distinct $y(j,k)$ $k = 1, 2, \cdots M_j$ such that

$$\{y(j,k) \mid k = 1, 2, \cdots M_j\} \subset \{y(j+1,k) \mid k = 1, 2, \cdots M_{j+1}\}.$$

Define $L_j(k)$ by $y(j,k) = y(j+1, L_j(k))$. We may also assume that

$$y(j+1, \xi) \in D(y(j,k), \delta^j R) \cup D(y(j,k+1), \delta^j R)$$

for $\xi \in [L_j(k), L_j(k+1)] \cap \mathbf{Z}$.

Now let us define $g(j,k) \in [0,1]$ for $j = 1, 2, \cdots$ and $k = 1, 2, \cdots, M_j$ inductively. First let $g(1,k) = k/M_1$ for $j = 1, 2, \cdots, M_1$. If $j \geq 2$, then define

$$g(j,k) = \frac{g(j-1,s)(L_{j-1}(s+1)-k) + g(j-1,s+1)(k - L_{j-1}(s))}{L_{j-1}(s+1) - L_{j-1}(s)},$$

for $L_{j-1}(s) \leq k \leq L_{j-1}(s+1)$. Define the value f at a dense subset of $[0,1]$ by $f(g(j,k)) = y(j,k)$. We can extend f continuously to $[0,1]$ by

$$f(t) = \lim_{j \to \infty} y(j, k_j(t)),$$

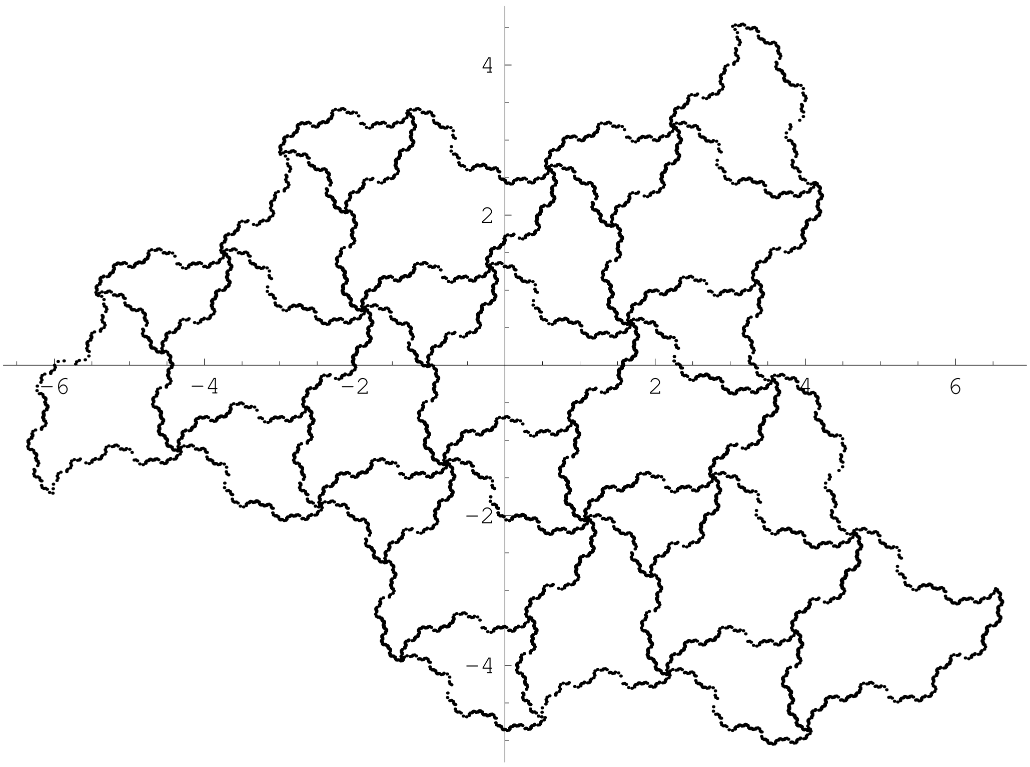
where $k_j(t)$ is defined by $t \in [g(j, k_j(t)), g(j, k_j(t)+1)]$. As $y(j,k) \in \mathrm{Inn}(T)$ for $j = 2, 3, \cdots, M_j - 1$, f is a continuous map from $[0,1]$ to T with $f(0) = x_1$ and $f(1) = x_2$. This proves the assertion.

3 Examples of the tilings

Example 1. Let $\theta = 1.3247\ldots$ be a positive root of $x^3 - x - 1 = 0$. It is known that θ is a minimal Pisot number. See [5]. The corresponding word $a_0a_1a_2\cdots$ consists of two alphabets $\{0, 1\}$ with admissiblity condition:

$$a_n = 1 \to a_{n+1} = a_{n+2} = a_{n+3} = a_{n+4} = 0.$$

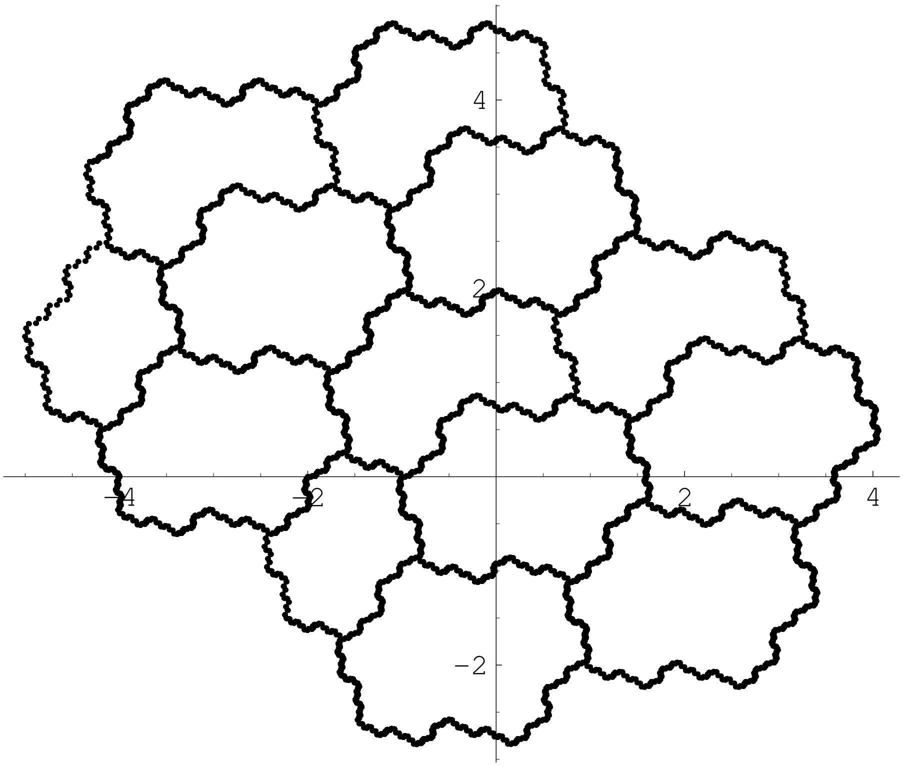
The tiling of $\mathbf{C} \simeq \mathbf{R}^2$ attached to this Pisot numeration system in base θ can be found in [11]. See also [3], for a precise study on the boundary of this tiling.



Example 2. Let $\beta = 1.839\ldots$ be a positive root of $x^3 - x^2 - x - 1$. The corresponding word consists of two alphabets $\{0, 1\}$ with admissiblity condition:

$$a_n = a_{n+1} = 1 \to a_{n+2} = 0.$$

The tiling of $\mathbf{C} \simeq \mathbf{R}^2$ attached to β is called the Rauzy Fractal. There exist many results on this tiling studied by means of substitution and finite state automata. For example, see [10], [7], [8].



Acknowledgement

The author would like to express his deep gratitude to the referee for careful reading of the manuscript.

References

[1] S.Akiyama, Pisot numbers and greedy algorithm, Number Theory, Diophantine, Computational and Algebraic Aspects, Edited by K. Györy, A. Pethö and V.T.Sós, 9–21, de Gruyter 1998.

[2] S.Akiyama, Cubic Pisot units with finite beta expansion, preprint.

[3] S.Akiyama and T.Sadahiro, A self-similar tiling generated by the minimal Pisot number, to appear in Proceedings volume of 13th Czech and Slovak Conference on Number Theory, Acta Mathematica et Informatica Universitatis Ostraviensis.

[4] S.Akiyama, Pisot numeration system and Tamura's infinite product (In Japanese), Proceedings of the conference on number theory held at Waseda Univ. (1997) 78-86

[5] M.J.Bertin, A. Decomp-Guilloux, M. Grandet-Hugot, M. Pathiaux-Delefosse and J.P. Schreiber, Pisot and Salem numbers, Birkhäuser Verlag, Basel-Boston-Berlin 1992

[6] C.Frougny and B.Solomyak, Finite beta-expansions, Ergod. Th. and Dynam. Sys. 12 (1992) 713–723

[7] S.Ito and M.Kimura, On Rauzy Fractal, Japan J. Indust. Appl. Math. 8 (1991) 461-486

[8] A.Messaoudi, Frontière du fractal de Rauzy et système de numeration complexe, preprint.

[9] W.Parry, On the β-expansions of real numbers, Acta Math. Acad. Sci. Hungar. 11 (1960) 269-278

[10] G.Rauzy, Nombres Algébriques et substitutions, Bull. Soc. France 110 (1982) 147-178.

[11] W.P.Thurston, Groups, Tilings and Finite state automata, AMS Colloquium lectures, 1989.

Shigeki Akiyama

Dept. of Math., Fac. of Science
Niigata University,
Ikarashi-2 8050, Niigata
950-2181, JAPAN
E-mail: akiyama@math.sc.niigata-u.ac.jp