Multiple zeta values at non-positive integers

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Abstract. Values of Euler-Zagier's multiple zeta function at non-positive integers are studied, especially at $(0, 0, \ldots, -n)$ and $(-n, 0, \ldots, 0)$. Further we prove a symmetric formula among values at non-positive integers.

Key words: Multiple zeta function, Bernoulli numbers, Stirling numbers

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1 Introduction.

One of remarkable properties of the Riemann zeta-function $\zeta(s)$ is that

$$\zeta(2n) = (-1)^{n+1} \frac{2^{2n-1} B_{2n}}{(2n)!} \times \pi^{2n}$$

for positive integers n, due to L. Euler. Here B_m denote the Bernoulli numbers defined by $\dfrac{x}{e^x - 1} = \sum_{m=0}^{\infty} \dfrac{B_m}{m!} x^m$. Further he observed, by a farsighted argument,

$$\zeta(0) = -\frac{1}{2}, \quad \zeta(-2n) = 0 \quad \text{and} \quad \zeta(1-2n) = -\frac{B_{2n}}{2n}. \tag{1}$$

for positive integers n, giving a prototype of the functional equation of the Riemann zeta-function. From this discovery to this date, many profound researches have been done for values at non-positive integers of various zeta functions and L-functions. For example, values of Dirichlet's L-function at non-positive integers are essentially the generalized Bernoulli numbers. It is found that they play a fundamental role in the theory of p-adic zeta function and Iwasawa theory. One may say that such studies are one of the main themes in number theory.

In this paper, we shall study values at non-positive integers of Euler-Zagier's multiple zeta function defined by

$$\zeta_k(s_1, s_2, \ldots, s_k) = \sum_{0<n_1<n_2<\cdots<n_k} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}} \tag{2}$$

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with complex variables s_i $(i = 1, 2, \ldots, k)$ [22]. When $\Re s_i \geq 1$ for $i = 1, 2, \ldots, k-1$ and $\Re s_k > 1$, the summation is absolutely convergent and defines a holomorphic function in k variables there.

Values at positive integers of this function have been attracting much attention in various fields of mathematics and physics, e.g. in knot theory [9, 16, 21], perturbative quantum field theory [8], and the analysis of algorithms [10]. Many relations among these values are known. The simplest such relation is

$$\sum_{n_1 < n_2} \frac{1}{n_1 n_2^2} = \zeta(3)$$

which is also due to Euler. Hoffman [12] systematically studied some relations among them and presented two conjectures, so called, sum and duality conjectures. They are first proved by Zagier [22] and extensively generalized by Ohno [19]. The articles [7] and [11] are recommended for further study of multiple zeta function and other related functions.

On the other hand, analytic properties of the general multiple zeta function have not been studied so much. In the case $k = 2$, Atkinson [5] gave an analytic continuation of $\zeta_2(s_1, s_2)$ by the Poisson summation formula and applied it to the study of the mean value formula of the Riemann zeta-function [13, 14, 18]. Arakawa and Kaneko [4] showed that (2) can be meromorphically continued as a function of s_k to the whole complex s_1-plane. Zhao [23] gave an analytic continuation as a function of s_i $(i = 1, 2, \ldots, k)$ to $\mathbb{C}^k$ using Gelfand and Shilov's generalized functions. In our previous paper [1], we developed a simple way of an analytic continuation using the Euler-Maclaurin summation formula and discussed some relations among values at non-positive integers. [1]

We will continue this study of multiple zeta values at non-positive integers in detail. As is shown in [1], $\zeta_k(s_1, \ldots, s_k)$ is a meromorphic function of k variables and has singularities on

$$\left\{ (s_1, s_2, \ldots, s_k) \in \mathbb{C}^k \;\middle|\; \begin{array}{c} s_k = 1,\ s_{k-1} + s_k = 2, 1, 0, -2, -4, \ldots, \\ \sum_{i=1}^{j} s_{k-i+1} \in \mathbb{Z}_{\leq j}\ (j = 3, 4, \ldots, k) \end{array} \right\}.$$

Therefore the point $(-r_1, -r_2, \ldots, -r_k)$ where r_i are non-negative integers lies on the set of singularities, in particular it is a point of indeterminacy. This fact might be a little discouraging to number theorists who wish to find an unknown 'functional equation' in this new zeta function. The aim of the present paper can be grasped as an experimental attempt to overcome this difficulty. We define regular values of multiple zeta function at non-positive integers by

$$\zeta_k(-r_1, -r_2, \ldots, -r_k) = \lim_{s_1 \to -r_1} \lim_{s_2 \to -r_2} \cdots \lim_{s_k \to -r_k} \zeta_k(s_1, s_2, \ldots, s_k), \tag{3}$$

[1]Recently, meromorphic continuation of more general multiple zeta functions has been done by Matsumoto [17] applying the Mellin-Barnes integral formula. See also [2].

and we are concerned about values at $(0,\dots,0,-n)$, $(-n,0,\dots,0)$ and the symmetric representation of them, which we call the *tangent symmetry*. The authors believe that our results suggest us the existence of some hidden 'duality' in multiple zeta values.

To state our results, we need the Stirling numbers. Let k be a positive integer. The Stirling numbers of the first kind $s(k,j)$ and the second kind $S(k,j)$ are defined by

$$x(x-1)\cdots(x-k+1)=\sum_{j=1}^{k}s(k,j)x^{j} \tag{4}$$

and

$$x^{k}=\sum_{j=1}^{k}S(k,j)x(x-1)\cdots(x-j+1), \tag{5}$$

respectively.

Now we will state our results.

Theorem 1. *Let n be a non-negative integer. Then we have*

$$\zeta_k(0,\dots,0,-n)=-\frac{1}{(k-1)!\,(n+1)}\sum_{j=1}^{k}s(k,j)B_{n+j}+(-1)^{k}\delta_n \tag{6}$$

where δ_n is the function defined by

$$\delta_n=\begin{cases}1 & \textit{if} \quad n=0,\\ 0 & \textit{if} \quad n\neq 0.\end{cases} \tag{7}$$

This theorem can be regarded as a generalization of (1).

Corollary 1. *It holds that*

$$\zeta_k(0,\dots,0)=\frac{(-1)^{k}}{k+1}.$$

This corollary is announced in [1], to be proved in this paper.

Corollary 2. *Let n be a non-negative integer. Then we have*

$$\begin{aligned}(8)\qquad &(n+2)\zeta_k(0,\dots,0,-n-1)\\ &= k(n+1)\Big\{\zeta_{k+1}(0,\dots,0,-n)+\zeta_k(0,\dots,0,-n)\Big\}.\end{aligned}$$

These two corollaries provide us with an easy way to calculate values $\zeta_k(0,\dots,0,-n)$ like the Pascal triangle (see (31)).

Values of $\zeta(s)$ at negative integers, which are essentially the Bernoulli numbers, can be written as a linear combination of Stirling numbers of the second kind (see (30) below). As a generalization of this type of expressions, we have

Corollary 3. *Let n be a non-negative integer. Then we have*

$$\zeta_k(0,\dots,0,-n) = \frac{(-1)^{n+1}}{n+1}\sum_{j=1}^{n+1}\frac{(-1)^{k+j}j\,!\,S(n+1,j)}{k+j}. \tag{9}$$

This Corollary 3 also implies Corollary 1 by taking $n=0$. Next theorem concerns values at $(-n,0,\dots,0)$.

Theorem 2. *Let n be a non-negative integer. Then we have*

$$\zeta_k(-n,0,\dots,0) = \frac{(-1)^n}{k\,!}\sum_{j=1}^{k}s(k,j)\frac{jB_{n+j}}{n+j}. \tag{10}$$

The above theorems and corollaries are proved in §4 and §5. Also in this paper, we will give an affirmative answer on Conjecture 1 in [1]. To explain it, we prepare some notation from [1]. Let S be the ordered index set $\{1,2,\dots,k\}$ of k elements and let $\mathcal{D}_l^k$ be the set of all ways of dividing S into l parts preserving the order of $1,2,\dots,k$. The element J in $\mathcal{D}_l^k$ can be expressed as

$$J = (1,\dots,i_1\big|\, i_1+1,\dots,i_2\big|\, i_2+1,\dots,i_{l-1}\big|\, i_{l-1}+1,\dots,k).$$

Let $A=(-r_1,-r_2,\dots,-r_k)$ be a sequence of k non-positive integers. For $J\in\mathcal{D}_l^k$ as above, we set

$$A^J = (-r_1-r_2-\cdots-r_{i_1},\ -r_{i_1+1}-\cdots-r_{i_2},\dots,-r_{i_{l-1}+1}-\cdots-r_k)$$

and

$$\zeta_l(A^J) = \zeta_l\big(-r_1-r_2-\cdots-r_{i_1},\ -r_{i_1+1}-\cdots-r_{i_2},\dots,-r_{i_{l-1}+1}-\cdots-r_k\big).$$

Then Conjecture 1 in [1], which is now a theorem, can be stated as

Theorem 3 (Tangent Symmetry). *Let r_i be non-negative integers, $r_1>0$ and $\sum_{i=1}^k r_i \not\equiv k \pmod 2$. Let $A=(-r_1,-r_2,\dots,-r_k)$. Then we have*

$$\zeta_k(A) = \sum_{j=1}^{k-1}c(j)\Big(\sum_{J\in\mathcal{D}_{k-j}^k}\zeta_{k-j}(A^J)\Big)$$

where $c(j)$ is the number defined by

$$c(j) = 2(1-2^{j+1})\frac{B_{j+1}}{j+1}$$

for $j \geq 1$.

The proof will be given in §6. Note that this theorem implies Theorem 4 of [1]. The value of $c(j)$ is zero when j is an even positive integer. The first several values of $c(j)$ for odd j are $c(1) = -1/2, c(3) = 1/4, c(5) = -1/2, c(7) = 17/8, c(9) = -31/2$ and $c(11) = 691/4$. They appear in the coefficients of Taylor expansion of tangent function:

$$\tan x = \sum_{n=1}^{\infty}(-1)^n 2^{2n-1} c(2n-1)\frac{x^{2n-1}}{(2n-1)!}. \tag{11}$$

We show some examples. Let r_i be non-negative integers which satisfy the assumption of Theorem 3.

1. The case $k = 2$;

$$\zeta_2(-r_1, -r_2) = -\frac{1}{2}\zeta(-r_1 - r_2).$$

2. The case $k = 3$;

$$\zeta_3(-r_1, -r_2, -r_3) = -\frac{1}{2}\Big\{\zeta_2(-r_1 - r_2, -r_3) + \zeta_2(-r_1, -r_2 - r_3)\Big\}.$$

3. The case $k = 4$;

$$\begin{aligned}
&\zeta_4(-r_1, -r_2, -r_3, -r_4) \\
&= \ -\frac{1}{2}\Big\{\zeta_3(-r_1,\ -r_2,\ -r_3 - r_4) + \zeta_3(-r_1,\ -r_2 - r_3,\ -r_4) \\
&\qquad +\zeta_3(-r_1 - r_2,\ -r_3,\ -r_4)\Big\} + \frac{1}{4}\zeta(-r_1 - r_2 - r_3 - r_4).
\end{aligned}$$

These three cases were proved in [1]. Apparent analogy suggests that there would exist some bridges between tangent symmetry and known relations in positive values, such as Hoffman's or Le-Murakami's.

Corollary 4. *Let n be a positive integer with $n \not\equiv k \pmod 2$. Then we have*

$$\zeta_k(-n, 0, \dots, 0) = \sum_{j=1}^{k-1} c(j)\binom{k-1}{j}\zeta_{k-j}(-n, 0, \dots, 0). \tag{12}$$

Each side of the above corollary is a linear combination of $\dfrac{B_{n+j}}{n+j}$. We can expect that their coefficients coincide with each other. In fact, we have

Corollary 5. *Let k and l be positive integers such that $1 \leq l \leq k-1$ and $l \not\equiv k \pmod 2$. Then*

$$\frac{s(k,l)}{(k-1)\,!} = 2\sum_{j=1}^{k-l}(1-2^{j+1})\binom{k}{j+1}\frac{s(k-j,l)B_{j+1}}{(k-j)\,!}.$$

The proofs of these corollaries will be given in §6.

As already stated, non-positive integer points are the points of indeterminacy. Hence we can employ another definitions of multiple zeta values at such points, in particular, central values and reverse values (see (39) and (40)). In the last section, we shall show the corresponding theorems for reverse values.

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2 Analytic continuation and recurrence relations.

For positive integers l and m, we put

$$(13)\quad \phi_l(m,s) = \sum_{n=1}^{m}\frac{1}{n^s} - \left\{\frac{m^{1-s}-1}{1-s} + \frac{1}{2m^s} - \sum_{q=1}^{l}\frac{(s)_q a_q}{m^{s+q}} + \zeta(s) - \frac{1}{s-1}\right\}$$

with $(s)_n = s(s+1)\cdots(s+n-1)$ and $a_q = \dfrac{B_{q+1}}{(q+1)!}$. By using the Euler-Maclaurin summation formula, we have $\phi_l(m,s) = O\big(|(s)_{l+1}|m^{-\Re(s)-l-1}\big)$. In [1], we applied (13) for the sum with respect to n_k and got

$$\begin{aligned}(14)\quad \zeta_k(s_1,s_2,\ldots,s_k) = {}& \frac{\zeta_{k-1}(s_1,s_2,\ldots,s_{k-2},s_{k-1}+s_k-1)}{s_k-1} \\ & - \frac{\zeta_{k-1}(s_1,s_2,\ldots,s_{k-2},s_{k-1}+s_k)}{2} \\ & + \sum_{q=1}^{l}(s_k)_q a_q \zeta_{k-1}(s_1,s_2,\ldots,s_{k-2},s_{k-1}+s_k+q) \\ & - \sum_{0<n_1<n_2<\ldots<n_{k-1}}\frac{\phi_l(n_{k-1},s_k)}{n_1^{s_1}n_2^{s_2}\ldots n_{k-1}^{s_{k-1}}}.\end{aligned}$$

This gave us the analytic continuation of the multiple zeta function. Furthermore, we have the recurrence relation of the multiple zeta values at non-positive integers:

$$
\begin{aligned}
(15)\qquad & \zeta_k(-r_1,-r_2,\dots,-r_k) \\
&= -\frac{\zeta_{k-1}(-r_1,-r_2,\dots,-r_{k-2},-r_{k-1}-r_k-1)}{1+r_k} \\
&\quad -\frac{\zeta_{k-1}(-r_1,-r_2,\dots,-r_{k-2},-r_{k-1}-r_k)}{2} \\
&\quad +\sum_{q=1}^{r_k}(-r_k)_q a_q \zeta_{k-1}(-r_1,-r_2,\dots,-r_{k-2},-r_{k-1}-r_k+q)
\end{aligned}
$$

where r_i are non-negative integers. Note that values at non-positive integers are defined by (3).

Now we show an another method of analytic continuation of $\zeta_k(s_1,s_2,\dots,s_k)$. Let s_i be complex numbers with $\Re s_i > 1$. Then applying (13) for the sum with respect to n_1 in (2), we have

$$
\begin{aligned}
\zeta_k(s_1,\dots,s_k) &= \sum_{n_k=1}^{\infty}\frac{1}{n_k^{s_k}}\sum_{n_{k-1}=1}^{n_k-1}\frac{1}{n_{k-1}^{s_{k-1}}}\cdots\sum_{n_2=1}^{n_3-1}\frac{1}{n_2^{s_2}}\left\{\frac{n_2^{1-s_1}}{1-s_1}-\frac{1}{2n_2^{s_1}}\right. \\
&\qquad \left. -\sum_{q=1}^{l}(s_1)_q a_q\frac{1}{n_2^{s_1+q}}+\zeta(s_1)+\phi_l(n_2,s_1)\right\} \\
&= \frac{\zeta_{k-1}(s_1+s_2-1,s_3,\dots,s_k)}{1-s_1}-\frac{\zeta_{k-1}(s_1+s_2,s_3,\dots,s_k)}{2} \\
&\qquad -\sum_{q=1}^{l}(s_1)_q a_q\zeta_{k-1}(s_1+s_2+q,s_3,\dots,s_k) \\
&\qquad +\zeta(s_1)\zeta_{k-1}(s_2,\dots,s_k)+\sum_{n_2<n_3<\cdots<n_k}\frac{\phi_l(n_2,s_1)}{n_2^{s_2}n_3^{s_3}\cdots n_k^{s_k}}.
\end{aligned}
$$

The last summation can be transformed as

$$
\begin{aligned}
&\sum_{n_3<n_4<\cdots<n_k}\frac{1}{n_3^{s_3}\cdots n_k^{s_k}}\left\{\sum_{n_2=1}^{\infty}\frac{\phi_l(n_2,s_1)}{n_2^{s_2}}-\sum_{n_3\le n_2}\frac{\phi_l(n_2,s_1)}{n_2^{s_2}}\right\} \\
&= \zeta_{k-2}(s_3,\dots,s_k)\sum_{n_2=1}^{\infty}\frac{\phi_l(n_2,s_1)}{n_2^{s_2}} \\
&\quad -\sum_{n_4<n_5<\cdots<n_k}\frac{1}{n_4^{s_4}\cdots n_k^{s_k}}\left\{\sum_{n_3\le n_2}\frac{\phi_l(n_2,s_1)}{n_2^{s_2}n_3^{s_3}}-\sum_{n_4\le n_3\le n_2}\frac{\phi_l(n_2,s_1)}{n_2^{s_2}n_3^{s_3}}\right\}.
\end{aligned}
$$

Repeating this procedure, we finally obtain

(16)

$$
\begin{aligned}
\zeta_k(s_1, \dots, s_k) &= \frac{\zeta_{k-1}(s_1+s_2-1, s_3, \dots, s_k)}{1-s_1} - \frac{\zeta_{k-1}(s_1+s_2, s_3, \dots, s_k)}{2} \\
&- \sum_{q=1}^{l} (s_1)_q a_q \zeta_{k-1}(s_1+s_2+q, s_3, \dots, s_k) + \zeta(s_1)\zeta_{k-1}(s_2, \dots, s_k) \\
&+ \zeta_{k-2}(s_3, \dots, s_k) \sum_{n_2=1}^{\infty} \frac{\phi_l(n_2, s_1)}{n_2^{s_2}} - \zeta_{k-3}(s_4, \dots, s_k) \sum_{n_3 \le n_2} \frac{\phi_l(n_2, s_1)}{n_2^{s_2} n_3^{s_3}} \\
&+ \cdots + (-1)^{k-1} \zeta(s_k) \sum_{n_{k-1} \le n_{k-2} \le \cdots \le n_2} \frac{\phi_l(n_2, s_1)}{n_2^{s_2} n_3^{s_3} \cdots n_{k-1}^{s_{k-1}}} \\
&+ (-1)^k \sum_{n_k \le n_{k-1} \le \cdots \le n_2} \frac{\phi_l(n_2, s_1)}{n_2^{s_2} n_3^{s_3} \cdots n_k^{s_k}}.
\end{aligned}
$$

We can see that the summations involving $\phi_l(n_2, s_1)$ are absolutely convergent if

$$
l > -\Re s_1 - \Re s_2 + k - 2 - \sum_{\substack{3 \le i \le k \\ \Re s_i < 0}} \Re s_i.
$$

Since l can be chosen arbitrarily large, it follows that the multiple zeta function can be continued to the whole $\mathbb{C}^k$ as a meromorphic functions of s_i. Values at non-positive integers are given by

(17)
$$
\begin{aligned}
&\zeta_k(-r_1, \dots, -r_k) \\
&= \frac{\zeta_{k-1}(-r_1-r_2-1, -r_3, \dots, -r_k)}{1+r_1} - \frac{\zeta_{k-1}(-r_1-r_2, -r_3, \dots, -r_k)}{2} \\
&\quad + \zeta(-r_1)\zeta_{k-1}(-r_2, \dots, -r_k) - R_k(-r_1, \dots, -r_k)
\end{aligned}
$$

where

$$
\begin{aligned}
&R_k(-r_1, \dots, -r_k) \\
&= \lim_{s_1 \to -r_1} \lim_{s_2 \to -r_2} \cdots \lim_{s_k \to -r_k} \sum_{q=1}^{l} (s_1)_q a_q \zeta_{k-1}(s_1+s_2+q, s_3, \dots, s_k)
\end{aligned}
$$

for sufficiently large l.

3 Some lemmas on Stirling numbers.

In this section, we prove several lemmas on Stirling numbers after recalling some well-known properties.

Let $s(k,j)$ and $S(k,j)$ be the Stirling numbers of the first and second kind respectively. The generating function of $S(k,j)$ is

$$\sum_{k\geq r}\frac{S(k,r)}{k\,!}x^k=\frac{1}{r\,!}(e^x-1)^r. \tag{18}$$

Throughout the paper, we put $s(k,j)=S(k,j)=0$ for $j<0$ and $j>k$, and $s(k,0)=S(k,0)=\delta_k$ to simplify our notation. Stirling numbers are known to have the recurrence relations:

$$s(k+1,j)=-ks(k,j)+s(k,j-1) \tag{19}$$

and

$$S(k+1,j)=jS(k,j)+S(k,j-1). \tag{20}$$

Furthermore, it is well known that

$$a_k=\sum_{j=1}^{k}s(k,j)b_j \Longleftrightarrow b_k=\sum_{j=1}^{k}S(k,j)a_j \tag{21}$$

for any sequences $\{a_n\}$ and $\{b_n\}$.

First we prove

Lemma 1. *Let a and c be positive integers, and $a\geq c$. Then*

$$\sum_{b=c}^{a}\frac{1}{b}S(a,b)s(b,c)=\frac{B_{a-c}+\delta_{a-c-1}}{a}\binom{a}{c}.$$

Proof. By using (18), we have

$$\frac{x}{e^x-1}\sum_{k=b}^{\infty}\frac{S(k,b)}{k\,!}x^k=\frac{x}{b}\sum_{k=b-1}^{\infty}\frac{S(k,b-1)}{k\,!}x^k.$$

Thus when $b\geq 1$,

$$\left(\sum_{l=0}^{\infty}\frac{B_l}{l\,!}x^l\right)\left(\sum_{k=0}^{\infty}\frac{S(k,b)}{k\,!}x^k\right)=\frac{1}{b}\sum_{k=0}^{\infty}\frac{S(k-1,b-1)}{(k-1)\,!}x^k.$$

Comparing the coefficients of x^a on both sides we have

$$\sum_{l=0}^{a}\frac{B_{a-l}}{(a-l)\,!}\frac{S(l,b)}{l\,!}=\frac{S(a-1,b-1)}{(a-1)\,!\,b}. \tag{22}$$

Now we use (20) to obtain

$$\sum_{l=0}^{a}\binom{a}{l}B_{a-l}S(l,b)=\frac{a}{b}\Big\{S(a,b)-bS(a-1,b)\Big\}.$$

Multiplying $s(b,c)$ and taking summation on $b=c,\dots,a$, we have

$$\sum_{b=c}^{a}\sum_{l=0}^{a}\binom{a}{l}B_{a-l}S(l,b)s(b,c)=a\sum_{b=c}^{a}\frac{1}{b}S(a,b)s(b,c)-a\sum_{b=c}^{a-1}S(a-1,b)s(b,c).$$

Thus we have

$$\sum_{l=0}^{a}\binom{a}{l}B_{a-l}\sum_{b=c}^{l}S(l,b)s(b,c)=a\sum_{b=c}^{a}\frac{1}{b}S(a,b)s(b,c)-a\delta_{a-1-c}.$$

By using (21), we have proved

$$\frac{1}{a}\binom{a}{c}B_{a-c}=\sum_{b=c}^{a}\frac{1}{b}S(a,b)s(b,c)-\delta_{a-c-1}.$$

□

Lemma 2. *Let a and c be positive integers, and $a\geq c-1$. Then*

$$\sum_{b=c-1}^{a}\frac{S(a,b)s(b+1,c)}{b+1}=\frac{B_{a+1-c}}{a+1}\binom{a+1}{c}.$$

Proof. Again by (22), we have

$$\sum_{l=0}^{a}\binom{a}{l}B_{a-l}S(l,b)=\frac{a}{b}S(a-1,b-1).$$

Using (21), we get the assertion. □

We can also show

Lemma 3. *Let a and c be non-negative integers, and $a\geq c$. Then*

$$\sum_{b=c}^{a}S(a,b)s(b,c)b=(-1)^{a+c}\binom{a}{c-1}.$$

We omit the proof, since it will not appear later in this paper. Remark that one can use Lemma 3 instead of Lemma 2 in the later sections but the proof of our theorems become more involved.

4 Proof of Theorem 1 and its Corollaries.

Proof of Theorem 1. Let δ_n be the function defined by (7) and let $Z(k,n) = \zeta_k(0,0,\dots,0,-n)$. Then our task is to show

$$Z(k,n) = -\frac{1}{(k-1)\,!\,(n+1)}\sum_{m=1}^{k} s(k,m)B_{n+m} + (-1)^k\delta_n.$$

The case $k=1$ follows from (1). We shall use an induction on k. By (15), we see

$$Z(k+1,n) = -\frac{Z(k,n+1)}{n+1} - \frac{1}{2}Z(k,n) + \sum_{q=1}^{n}(-n)_q a_q Z(k,n-q). \tag{23}$$

Using the inductive assumption, the right-hand side of (23) is rewritten as

$$\frac{1}{(k-1)\,!}\sum_{m=1}^{k} s(k,m)\left\{\frac{B_{n+1+m}}{(n+1)(n+2)} + \frac{B_{n+m}}{2(n+1)} - \sum_{q=1}^{n}(-n)_q a_q \frac{B_{n-q+m}}{n-q+1}\right\}$$
$$-\frac{(-1)^k B_{n+1}}{n+1} - (-1)^k\delta_n.$$

Here we have to distinguish the case $n=0$ with the case $n \neq 0$ to derive this. By using (19), the left-hand side of (23) should be

$$\frac{1}{(k-1)\,!}\sum_{m=1}^{k} s(k,m)\left\{\frac{B_{n+m}}{n+1} - \frac{B_{n+m+1}}{k(n+1)}\right\} - (-1)^k\delta_n.$$

Adding $(-1)^k\delta_n$, multiplying $(k-1)\,!\,S(M,k)$ and taking summation on $k=1,2,\dots,M$, we have

$$\frac{B_{n+1}}{n+1}\sum_{k=1}^{M} S(M,k)(-1)^k(k-1)\,! - \sum_{m=1}^{M}\frac{B_{n+m+1}}{n+1}\sum_{k=m}^{M}\frac{1}{k}S(M,k)s(k,m)$$
$$= \frac{B_{n+1+M}}{(n+1)(n+2)} - \frac{B_{n+M}}{2(n+1)} + \sum_{q=1}^{n}\frac{(-n)_q(-1)^q}{(q+1)\,!\,(n-q+1)}B_{q+1}B_{n-q+M}.$$

Note that this summing operation on k is a reversible procedure. In fact, in the light of (21), we can recover our inductive assertion and (23) from this identity by multiplying $s(L,M)$ and taking summation on $M=1,2,\dots,L$. Now we employ an identity

$$\sum_{k=1}^{M} S(M,k)(-1)^k(k-1)\,! = -\delta_{M-1},$$

which is also easily seen by the definition (5). From this and Lemma 1, what we have to show is the following identity

$$
\begin{aligned}
(24) \qquad & -\frac{1}{M}\sum_{m=1}^{M}\binom{M}{m}B_{n+m+1}B_{M-m} \\
&= \frac{B_{n+1+M}}{n+2}+\frac{B_{n+M}}{2}+\sum_{q=1}^{n}\binom{n+1}{q}\frac{B_{q+1}B_{n-q+M}}{q+1}
\end{aligned}
$$

for $M, n \geq 1$. One should be careful enough to classify the case $M = 1$ and the case $M \neq 1$ to deduce (24).

First we assume that $M > 1$. The left-hand side of (24) can be written as

$$
\frac{1}{2}B_{n+M}-\frac{1}{M}\sum_{j=0}^{[\frac{M-1}{2}]}\binom{M}{2j}B_{n+1+M-2j}B_{2j}.
$$

By putting $q + 1 = 2r$ and noting

$$
\binom{n+1}{2r-1}\frac{1}{2r}=\frac{1}{n+2}\binom{n+2}{2r},
$$

the sum on the right-hand side of (24) is equal to

$$
\frac{1}{n+2}\sum_{r=1}^{[\frac{n+1}{2}]}\binom{n+2}{2r}B_{2r}B_{n+1+M-2r}.
$$

Therefore the identity (24) is equivalent to

(25)

$$
(n+2)\sum_{j=0}^{[\frac{M-1}{2}]}\binom{M}{2j}B_{2j}B_{n+M+1-2j}+M\sum_{r=0}^{[\frac{n+1}{2}]}\binom{n+2}{2r}B_{2r}B_{n+M+1-2r}=0.
$$

To show this identity, we recall the well-known formula of Bernoulli polynomials:

$$
\begin{aligned}
(26) \qquad B_m(x)B_n(x) &= \sum_r\left\{\binom{m}{2r}n+\binom{n}{2r}m\right\}\frac{B_{2r}B_{m+n-2r}(x)}{m+n-2r} \\
&\quad + (-1)^{m+1}\frac{m!\,n!}{(m+n)!}B_{m+n}
\end{aligned}
$$

for $m, n \geq 1$. (see Apostol[3], p.276 19 (b)). Differentiating this formula with respect to x and putting $x = 0$, we get

$$
(27) \quad mB_{m-1}B_n+nB_mB_{n-1}=\sum_r\left\{\binom{m}{2r}n+\binom{n}{2r}m\right\}B_{2r}B_{m+n-1-2r}.
$$

Now we substitute M into m and $n+2$ into n in (27). Then the above equation becomes

(28)

$$\begin{aligned} &MB_{M-1}B_{n+2}+(n+2)B_M B_{n+1} \\ &= (n+2)\sum_{r=0}^{[M/2]}\binom{M}{2r}B_{2r}B_{M+n+1-2r} + M\sum_{r=0}^{[n/2]+1}\binom{n+2}{2r}B_{2r}B_{M+n+1-2r}. \end{aligned}$$

We note that

$$\left[\frac{M-1}{2}\right]=\left[\frac{M}{2}\right],\ \left[\frac{n+1}{2}\right]=\left[\frac{n}{2}\right]+1, \text{ and } \ B_{n+2}=B_M=0.$$

when M and n are odd integers. Therefore the equation (28) is the equation (25) itself in this case. The coincidence of (28) and (25) is shown similarly for the other cases, so we have proved (24) for $M>1$.

When $M=1$, we must show that

$$\begin{aligned} -B_{n+2} &= \frac{B_{n+2}}{n+2}+\frac{B_{n+1}}{2}+\sum_{q=1}^{n}\binom{n+1}{q}\frac{B_{q+1}B_{n+1-q}}{q+1} \\ &= \frac{B_{n+1}}{2}+\frac{1}{n+2}\sum_{r=0}^{[\frac{n+1}{2}]}\binom{n+2}{2r}B_{2r}B_{n+2-2r}. \end{aligned} \tag{29}$$

But (29) is also immediately obtained from (28). (Note that (28) is valid even for $M=1$.) This completes the proof of Theorem 1. □

Proof of Corollary 1. As a special case of Lemma 2, we have

$$B_n=\sum_{j=0}^{n}\frac{(-1)^j j\,!}{j+1}S(n,j) \tag{30}$$

for $n>0$. (In fact, we put $c=1$, and use $s(j+1,0)=(-1)^j j!$.) This gives, by the inversion formula (21),

$$\sum_{r=0}^{k}s(k,r)B_r=\frac{(-1)^k k\,!}{k+1}.$$

Therefore we have

$$\begin{aligned} \zeta_k(0,\dots,0) &= -\frac{1}{(k-1)\,!}\sum_{j=1}^{k}s(k,j)B_j+(-1)^k \\ &= -\frac{1}{(k-1)\,!}\frac{(-1)^k k\,!}{k+1}+(-1)^k \\ &= \frac{(-1)^k}{k+1}. \end{aligned}$$

$\square$

Proof of Corollary 2. Let $Z(k,n) = \zeta_k(0,\ldots,0,-n)$. We use the recurrence relation (19) of Stirling numbers in the right-hand side of (6). Thus we have

$$\begin{aligned}
(n+2)Z(k,n+1) &= -\frac{1}{(k-1)\,!}\sum_{r=0}^{k}\Big\{s(k+1,r+1)+ks(k,r+1)\Big\}B_{n+1+r} \\
&= -\frac{1}{(k-1)\,!}\sum_{r'=1}^{k+1}s(k+1,r')B_{n+r'} - \frac{k}{(k-1)\,!}\sum_{r'=1}^{k}s(k,r')B_{n+r'} \\
&= k(n+1)\Big\{Z(k+1,n)+(-1)^{k+1}\delta_n + Z(k,n)+(-1)^k\delta_n\Big\} \\
&= k(n+1)\Big\{Z(k+1,n)+Z(k,n)\Big\}.
\end{aligned}$$

This is the assertion which we wanted to prove. $\square$

The values of $Z(k,n)$ can be reproduced by above two corollaries. In fact, let $P(k,n) = (n+1)Z(k,n)$. Then the recurrence relation $P(k,n+1) = k\big\{P(k+1,n)+P(k,n)\big\}$ is quite convenient to calculate values $Z(k,n)$, like the Pascal triangle:

$$\begin{array}{c|ccccccccccc}
k\backslash n & 0 & & 1 & & 2 & & 3 & & 4 & & 5 \\
\hline
1 & -\frac{1}{2} & \rightarrow & -\frac{1}{6} & \rightarrow & 0 & \rightarrow & \frac{1}{30} & \rightarrow & 0 & \rightarrow & -\frac{1}{42} \\
 & & \nearrow & & \nearrow & & \nearrow & & \nearrow & & \nearrow & \\
2 & \frac{1}{3} & \rightarrow & \frac{1}{6} & \rightarrow & \frac{1}{30} & \rightarrow & -\frac{1}{30} & \rightarrow & -\frac{1}{42} & & \\
 & & \nearrow & & \nearrow & & \nearrow & & \nearrow & & & \\
3 & -\frac{1}{4} & \rightarrow & -\frac{3}{20} & \rightarrow & -\frac{1}{20} & \rightarrow & \frac{3}{140} & & & & \\
 & & \nearrow & & \nearrow & & \nearrow & & & & & \\
4 & \frac{1}{5} & \rightarrow & \frac{2}{15} & \rightarrow & \frac{2}{35} & & & & & & \\
 & & \nearrow & & \nearrow & & & & & & & \\
5 & -\frac{1}{6} & \rightarrow & -\frac{5}{42} & & & & & & & & \\
 & & \nearrow & & & & & & & & & \\
6 & \frac{1}{7} & & & & & & & & & &
\end{array} \tag{31}$$

$P(k,n)$

Since $P(1,n) = (n+1)Z(1,n) = -B_{n+1}$ for $n \geq 1$, we get an alternative way to calculate Bernoulli numbers. This algorithm resembles the one in [15] using tangent numbers, but it is essentially different. This might be the simplest algorithm for a computer language equipped with rational arithmetic.

Remark 1. If we restrict our attention to the numerical calculation of Bernoulli numbers, the above algorithm is essentially the same as (30). Professor M. Kaneko kindly pointed out this fact.

Corollary 2 can also be expressed in the following way. Let $G_n(x)$ be a polynomial of x defined recursively by $G_1(x) = 1$ and

$$G_{n+1}(x) = x(x+n+2)G_n(x) - (x+1)^2 G_n(x+1).$$

Then (8) gives

$$Z(k,n) = \frac{(-1)^k k G_n(k)}{(n+1)\prod_{j=1}^{n+1}(k+j)}.$$

The examples of $G_n(k)$ are

$$\begin{aligned} G_2(k) &= k-1, \\ G_3(k) &= k^2 - 5k, \\ G_4(k) &= k^3 - 16k^2 + 11k + 4 \end{aligned}$$

and so on. We also have $B_{n+1} = \dfrac{G_n(1)}{(n+2)\,!}$ for $n \geq 1$.

Proof of Corollary 3. Let

$$Q(k,n) = (-1)^{n+1} \sum_{j=1}^{n+1} \frac{(-1)^{k+j} j\,!\, S(n+1,j)}{k+j}.$$

It is trivially seen that $Q(k,0) = P(k,0) = \frac{(-1)^k}{k+1}$. By using the recurrence relation (20), we have

$$\begin{aligned} Q(k,n+1) &= (-1)^{k+n+2} \sum_{j=1}^{n+2} j\,! \Big\{ jS(n+1,j) + S(n+1,j-1) \Big\} \frac{(-1)^j}{k+j} \\ &= (-1)^{k+n+2} \sum_{j=1}^{n+1} j\,! \, S(n+1,j)(-1)^j \left\{ \frac{j}{k+j} - \frac{j+1}{k+j+1} \right\} \\ &= (-1)^{k+n+2} k \sum_{j=1}^{n+1} j\,! \, S(n+1,j)(-1)^j \left\{ -\frac{1}{k+j} + \frac{1}{k+j+1} \right\} \\ &= k \Big\{ Q(k,n) + Q(k+1,n) \Big\}. \end{aligned}$$

Hence the function $Q(k,n)$ satisfies the same recurrence relation as $P(k,n)$. Since the value $Q(k,n)$ is uniquely determined by the initial values $Q(k,0)$ and the above recurrence relation, we can conclude that $Q(k,n) = P(k,n)$ for any k and n. This completes the proof of Corollary 3. $\square$

Remark 2. From the recurrence relation, the value $P(k,n)$ can be written in the form: $P(k,n) = \sum_{j=0}^{n} f_k(n,j) P(k+j,0)$ with the coefficient $f_k(n,j)$. Hence we have $\zeta_k(0,\ldots,0,-n) = \frac{1}{n+1}\sum_{j=0}^{n} f_k(n,j)\frac{(-1)^{k+j}}{k+j+1}$ trivially. But we remark that the assertion of Corollary 3 is somewhat different from above because the coefficients of $\frac{(-1)^{k+j}}{k+j+1}$ are independent on k in (9).

5 Proof of Theorem 2.

To prove Theorem 2, we have to utilize an analytic continuation in 'reverse order' developed in §2. Our task is to determine $R_k(-r_1, 0, 0, \dots, 0)$ in a concrete form.

Let $\zeta_k(s, 0, \dots, 0) = \lim_{s_2 \to 0} \cdots \lim_{s_k \to 0} \zeta_k(s, s_2, \dots, s_k)$. Then, as a function of s, $\zeta_k(s, 0, \dots, 0)$ has simple pole at n $(0 < n \leq k)$. First we shall determine the residues at these points.

Lemma 4. *Let n be a non-negative integer and k a positive integer with $k \geq n$. Then we have*

$$\lim_{s \to 0} s\zeta_k(s+n, 0, 0, \dots, 0) = \frac{n(-1)^{n-1}s(k,n)}{k\,!}.$$

Proof. The formula is trivial when $k = 1$. We will prove this by induction on k. Let $\mathcal{R}(k,n) = \lim_{s \to 0} s\zeta_k(s+n, 0, 0, \dots, 0)$. Using (16) with $l = k-1$, we see

$$\mathcal{R}(k,n) = \frac{\mathcal{R}(k-1, n-1)}{1-n} - \frac{\mathcal{R}(k-1,n)}{2} - \sum_{q=1}^{k-1} (n)_q a_q \mathcal{R}(k-1, n+q)$$

for $n \geq 2$. Thus, by inductive assumption, it suffices to show

$$\begin{aligned} ns(k,n) &= ks(k-1, n-1) - \frac{kn}{2}s(k-1,n) \\ &\quad -k\sum_{q=1}^{k-1} (-1)^q (n)_q (n+q) a_q s(k-1, n+q). \end{aligned}$$

One can show that this formula is also valid when $n = 1$, by considering the cancellation of poles. Thus we have

$$n\frac{s(k+1,n)}{k+1} = s(k, n-1) - \frac{n}{2}s(k,n) + \sum_{q=1}^{k} \binom{n+q}{q+1} B_{q+1} s(k, n+q).$$

Multiplying $S(M,k)$ and summing through $k = 1, 2, \dots, M$, we get

$$n\frac{B_{M-n+1}}{M+1}\binom{M+1}{n} = \delta_{M-n+1} - \frac{n}{2}\delta_{M-n} + \sum_{q=1}^{M} \binom{n+q}{q+1} B_{q+1} \delta_{M-n-q},$$

by Lemma 2. Similarly as in the proof of Theorem 1, this summing operation can be reversed. Thus we only have to show the last identity, which is quite trivial. $\square$

Proof of Theorem 2. Let $\mathcal{Z}(k,n)=\zeta_k(-n,0,0,\dots,0)$. Putting $q=n+l$ with $l>0$, we have

$$\begin{aligned}&\lim_{s_1\to -n}(s_1)_q a_q \zeta_{k-1}(s_1+q,0,\dots,0)\\ &\quad=(-n)(-n+1)\cdots(-1)1\cdot 2\cdot\cdots\cdot(l-1)a_{n+l}\mathcal{R}(k-1,l)\\ &\quad=-\frac{n\,!\,l\,!}{(n+l+1)\,!}\frac{B_{n+l+1}}{(k-1)\,!}s(k-1,l).\end{aligned}$$

Therefore

$$R_k(-n,0,0,\dots,0)=-\sum_{q=1}^{n}(n)_q a_q\mathcal{Z}(k-1,n-q)-\sum_{l=1}^{k-1}\frac{n\,!\,l\,!\,B_{n+l+1}s(k-1,l)}{(n+l+1)\,!\,(k-1)\,!}.$$

Thus by using (17), we have derived a recurrence relation for $\mathcal{Z}(k,n)$:

$$\begin{aligned}\mathcal{Z}(k,n)=&\frac{\mathcal{Z}(k-1,n+1)}{1+n}-\frac{\mathcal{Z}(k-1,n)}{2}\\ &-\sum_{q=1}^{n}(n)_q a_q\mathcal{Z}(k-1,n-q)+\zeta(-n)\mathcal{Z}(k-1,0)\\ &-\sum_{l=1}^{k-1}\frac{n\,!\,l\,!\,B_{n+l+1}s(k-1,l)}{(n+l+1)\,!\,(k-1)\,!}.\end{aligned}$$

Theorem 2 can be proved quite similarly as Theorem 1 by this recurrence relation. Namely, replacing both sides by inductive assumption, multiplying $(k-1)!S(M,k-1)$ and summing through $k=1,...,M$, we arrive at showing

$$\begin{aligned}&\sum_{j=1}^{M+1}\frac{j(-1)^j}{M+1}\binom{M+1}{j}\frac{B_{M+1-j}B_{n+j}}{n+j}+M(-1)^M\sum_{q=1}^{n-1}(-n)_q a_q\frac{B_{n-q+M}}{n-q+M}\\ &=\left(\frac{M(-1)^M}{n+1}-\binom{n+M}{n}^{-1}\right)\frac{B_{n+M+1}}{n+M+1}-M(-1)^M\frac{B_{n+M}}{2(n+M)}\end{aligned}$$

for any positive integer M and n. We used Lemma 2 on the way. This equality is shown by using the equation obtained by putting $x=0$ in (26).

Here we will give another simple proof of Theorem 2. From (14) and the definition of $\zeta_k(s,0,\dots,0)$, we see easily that there exist constants $A_k(j)$ such that

$$\zeta_k(s,0,\dots,0)=\sum_{j=1}^{k}A_k(j)\zeta(s-j+1).$$

For example,

$$\zeta_2(s,0)=-\zeta(s-1)-\frac{1}{2}\zeta(s)$$

and

$$\begin{aligned}\zeta_3(s,0,0) &= -\zeta_2(s,-1) - \frac{1}{2}\zeta_2(s,0)\\ &= \frac{1}{2}\zeta(s-2) + \zeta(s-1) + \frac{1}{3}\zeta(s).\end{aligned}$$

The coefficients $A_k(j)$ of $\zeta(s-j+1)$ is determined by Lemma 3. In fact, we have

$$\begin{aligned}A_k(j) &= \lim_{s\to j}(s-j)\zeta_k(s,0,\dots,0)\\ &= \frac{s(k,j)(-1)^{j-1}j}{k\,!}.\end{aligned}$$

Hence

$$\zeta_k(s,0,\dots,0) = \sum_{j=1}^{k}\frac{s(k,j)(-1)^{j-1}j}{k\,!}\zeta(s-j+1).$$

Now we evaluate the both sides at $s=-n$. Then we obtain

$$\begin{aligned}\zeta_k(-n,0,\dots,0) &= \sum_{j=1}^{k}\frac{s(k,j)(-1)^{j-1}j}{k\,!}\zeta(1-n-j)\\ &= \sum_{j=1}^{k}\frac{s(k,j)(-1)^{j}j}{k\,!}\frac{B_{n+j}}{n+j} - \frac{s(k,1)}{k\,!}\delta_n,\end{aligned}$$

which is easily proved to be equal to the right-hand side of (10).

6 Proof of Theorem 3 and its corollaries.

The assertion of Theorem 3 is known to be true for $k \le 4$ ([1]). Let V_j be vectors with k components. In this section, in order to simplify notations, we introduce an abuse of terminology, that is, a formal sum of vectors $V_1 + \cdots + V_l$. Note that the sum $V_1 + \cdots + V_l$ does not mean the componentwise addition of vectors and the value of ζ_k at such a formal sum is defined by

$$\zeta_k(V_1+\cdots+V_l) = \zeta_k(V_1)+\cdots+\zeta_k(V_l).$$

Definition. Let R be a vector with k components. Then we define

$$\partial_k(R) = \partial_k^1(R) = \sum_{J\in\mathcal{D}_{k-1}^k} R^J$$

and

$$\partial_k^i(R) = \partial_{k-i+1}(\partial_k^{i-1}(R))$$

recursively. Here $\mathcal{D}_l^k$ is the set defined in the Introduction.

It is immediately seen that

$$\partial_k^i(R) = i\,! \sum_{J\in\mathcal{D}_{k-i}^k} R^J \tag{32}$$

and

$$\partial_{k-j}^i \partial_k^j = \partial_k^{i+j}.$$

Let r_i be non-negative integers, $r_1 > 0$ and $R = (-r_1, \dots, -r_k)$. We put

$$\begin{aligned}
c_j &= c(j)/j\,!,\\
U &= (-r_1, \dots, -r_{k-1}),\\
R_q &= (-r_1, \dots, -r_{k-2}, -r_{k-1} - r_k + q) \quad (-1 \le q \le r_k)
\end{aligned}$$

and

$$(s)_{-1} = \frac{1}{s-1}.$$

to simplify our notation. Then in our new notation the recurrence relation (15) can be written simply as

$$\zeta_k(R) = -\frac{1}{2}\zeta_{k-1}(R_0) + \sum_{\substack{q=-1\\ q:odd}}^{r_k} (-r_k)_q a_q \zeta_{k-1}(R_q). \tag{33}$$

We want to show that

$$\mathbf{P}(k) : \zeta_k(R) = \sum_{j=1}^{k-1} c_j \zeta_{k-j}\big(\partial_k^j(R)\big)$$

under the condition $\sum_{i=1}^k r_i \not\equiv k \pmod 2$.

As already stated, $\mathbf{P}(1), \dots, \mathbf{P}(4)$ are true. We suppose that $\mathbf{P}(m)$ is true for $m < k$ and consider $\mathbf{P}(k)$. In (33), the summation on the right-hand side is taken over odd integers q, hence we can apply the inductive assumption to $\zeta_{k-1}(R_q)$. Interchanging the summation on j and q, we get

$$\zeta_k(R) = -\frac{1}{2}\zeta_{k-1}(R_0) + \sum_{\substack{j=1\\ j:odd}}^{k-2} c_j \Big\{ \sum_{\substack{q=-1\\ q:odd}}^{r_k} (-r_k)_q a_q \zeta_{k-j-1}\big(\partial_{k-1}^j(R_q)\big) \Big\}.$$

From (33) and the inductive assumption, we get

$$\begin{aligned}
&\sum_{\substack{q=-1\\ q:odd}}^{r_k}(-r_k)_q a_q \zeta_{k-j-1}(\partial_{k-1}^j(R_q))\\
&\qquad = \zeta_{k-j}\big(\partial_{k-1}^j(U),-r_k\big)+\frac{1}{2}\zeta_{k-j-1}\big(\partial_{k-1}^j(R_0)\big)\\
&\qquad = \zeta_{k-j}\big(\partial_{k-1}^j(U),-r_k\big)+\frac{1}{2}\sum_{\substack{i=1\\ i:odd}}^{k-j-2} c_i\zeta_{k-i-j-1}\big(\partial_{k-1}^{i+j}(R_0)\big).
\end{aligned}$$

Here we must note that

$$\zeta_{k-j}\big(\partial_{k-1}^j(U),-r_k\big)=j\,!\sum_{J\in\mathcal{D}_{k-1-j}^{k-1}}\zeta_{k-j}\big(U^J,-r_k\big).$$

Therefore

(34)

$$\begin{aligned}
&\zeta_k(R)=-\frac{1}{2}\zeta_{k-1}(R_0)+c_1\zeta_{k-1}\big(\partial_{k-1}(U),-r_k\big)\\
&+\sum_{\substack{m=3\\ m:odd}}^{k-2}\Big\{c_m\zeta_{k-m}\big(\partial_{k-1}^m(U),-r_k\big)+\frac{1}{2}\Big(\sum_{\substack{i=1\\ i:odd}}^{m-2}c_ic_{m-i-1}\Big)\zeta_{k-m}\big(\partial_{k-1}^{m-1}(R_0)\big)\Big\}\\
&+\frac{1}{2}\Big(\sum_{i=1}^{k-3}c_ic_{k-2-i}\Big)\zeta\big(\partial_{k-1}^{k-2}(R_0)\big).
\end{aligned}$$

The last term appears only when k is even. The first line on the right-hand side of (34) is equal to

$$-\frac{1}{2}\Big\{\zeta_{k-1}(R_0)+\zeta_{k-1}\big(\partial_{k-1}(U),-r_k\big)\Big\}=-\frac{1}{2}\zeta_{k-1}\big(\partial_k(R)\big).$$

To deal with general terms, we first note that

$$mc_m=\frac{1}{2}\sum_{i=1}^{m-2}c_ic_{m-i-1}$$

for an odd integer m. This is obtained by combing (11) and

$$(\tan x)'=1+\tan^2 x.$$

Hence we have

$$\begin{aligned}
&c_m\zeta_{k-m}(\partial_{k-1}^{m}(U),-r_k)+\frac{1}{2}\Big(\sum_{\substack{i=1\\ i:odd}}^{m-2}c_ic_{m-i-1}\Big)\zeta_{k-m}\big(\partial_{k-1}^{m-1}(R_0)\big)\\
&=c_m m!\Big\{\sum_{J\in\mathcal{D}_{k-m-1}^{k-1}}\zeta_{k-m}(U^J,-r_k)+\sum_{J\in\mathcal{D}_{k-m}^{k-1}}\zeta_{k-m}\big(R_0^J\big)\Big\}\\
&=c_m\, m!\sum_{J\in\mathcal{D}_{k-m}^{k}}\zeta_{k-m}\big(R^J\big)\\
&=c_m\zeta_{k-m}\big(\partial_k^m(R)\big).
\end{aligned}$$

When k is even, the last term of (34) is equal to

$$(k-1)c_{k-1}(k-2)!\,\zeta_1(-r_1-r_2-\cdots-r_k)=c_{k-1}\zeta_1\big(\partial_k^{k-1}(R)\big).$$

We have thereby proved Theorem 3. □

Proof of Corollary 4. We apply Theorem 3 to $A=(-n,0,\ldots,0)$. Since the set $\mathcal{D}_l^k$ consists of $\binom{k-1}{l-1}$ elements, Corollary 4 follows immediately. □

Proof of Corollary 5. Let n and l be positive integers with $n\not\equiv k \pmod 2$, and let

$$a_l=\frac{1}{k!}s(k,l)-2\sum_{j=1}^{k-1}(1-2^{j+1})\frac{B_{j+1}}{j+1}\binom{k-1}{j}\frac{s(k-j,l)}{(k-j)!}.$$

Then from Theorem 2 and Corollary 4, we have

$$\sum_{\substack{l=1\\ l\not\equiv k \pmod 2}}^{k-1}a_l\,\frac{lB_{n+l}}{n+l}=0$$

for all n with $n\not\equiv k \pmod 2$. Therefore it is enough to show that linear relations

$$\sum_{\substack{l=1\\ l\not\equiv k \pmod 2}}^{k-1}\frac{lB_{n+l}}{n+l}x_l=0\quad \text{for all } n \text{ with } n\not\equiv k \pmod 2$$

hold simultaneously only when $x_l=0$ ($l\not\equiv k \pmod 2$). We wish to pick out integers $n_1<n_2<\cdots$ so that the above linear equations restricted to n_i have only a trivial solution.

First we consider the case that n is odd and k is even. Let $B^*_n = B_n/n$ and

$$D(n_1, n_2, \dots, n_i) = \det \begin{pmatrix} B^*_{n_1+1} & B^*_{n_1+3} & \cdots & B^*_{n_1+2i-1} \\ B^*_{n_2+1} & B^*_{n_2+3} & \cdots & B^*_{n_2+2i-1} \\ \vdots & \vdots & \ddots & \vdots \\ B^*_{n_i+1} & B^*_{n_i+3} & \cdots & B^*_{n_i+2i-1} \end{pmatrix}.$$

We show that there are integers $n_1 < n_2 < \cdots < n_{\frac{k}{2}}$ such that

$$D(n_1, n_2, \dots, n_{\frac{k}{2}}) \neq 0, \tag{35}$$

from which the assertion $x_l = 0$ ($l \not\equiv k \pmod 2$) follows immediately. Let n_1 be an arbitrary positive odd integer. We have, of course, $D(n_1) = B^*_{n_1+1} \neq 0$. Assume that $D(n_1, n_2, \dots, n_i) \neq 0$ for $i < k/2$. Let p_{i+1} be a prime number such that

$$v_{p_{i+1}}\bigl(D(n_1, n_2, \dots, n_i)\bigr) = 0, \quad \text{and} \quad v_{p_{i+1}}\bigl(B^*_{n_a+2b-1}\bigr) \geq 0$$

for $1 \leq a, b \leq i$, where v_p is a p-adic valuation. We put $n_{i+1} = p_{i+1} - 2i - 2$. We may assume $n_{i+1} > n_i$, since we can take a larger p_{i+1} if the condition $n_{i+1} > n_i$ is not satisfied. Then from the von Staudt and Clausen theorem, we have

$$v_{p_{i+1}}\bigl(B^*_{n_{i+1}+2i+1}\bigr) = -1, \quad v_{p_{i+1}}\bigl(B^*_{n_j+2i+1}\bigr) \geq 0, \quad \text{and} \quad v_{p_{i+1}}\bigl(B^*_{n_{i+1}+2j-1}\bigr) \geq 0$$

for $j = 1, 2, \dots, i$. Hence we have

$$D(n_1, n_2, \dots, n_{i+1}) \neq 0.$$

Repeating this process, we can find integers $n_1, n_2, \dots, n_{\frac{k}{2}}$ which satisfy (35), therefore Corollary 5 is proved in this case.

The case that n is even and k is odd is proved similarly. □

We can also prove Corollary 5 without using multiple zeta function. For the sake of completeness, we shall give our proof. It seems to be interesting in itself.

Another proof of Corollary 5. Let us put $c(0) = -1$ for convenience. We have to show that

$$\sum_{j=0}^{k-l} c(j) \binom{k-1}{j} \frac{s(k-j, l)}{(k-j)\,!} = 0 \tag{36}$$

for $k \not\equiv l \pmod 2$. This equation is equivalent to

$$\sum_{j=0}^{k-l} c(j) \binom{k-1}{j} \frac{s(k-j, l)}{(k-j)\,!} = (-1)^{k-l} \sum_{j=0}^{k-l} c(j) \binom{k-1}{j} \frac{s(k-j, l)}{(k-j)\,!} \tag{37}$$

for any k and l. Multiplying $S(l,h)$ on both sides of (37) and summing through $l=0,\ldots,k$, we have

$$c(k-h)\binom{k-1}{k-h}\frac{1}{h\,!} = (-1)^k \sum_{j=0}^{k-h} c(j)\binom{k-1}{j}\frac{1}{(k-j)\,!} \times \sum_{l\le k-j} (-1)^l s(k-j,l)S(l,h). \tag{38}$$

The above procedure is reversible, so we have only to show (38). The last sum is called a Lah's number and it is known that $\sum_l (-1)^l s(k,l)S(l,h) = (-1)^k \frac{k\,!}{h\,!}\binom{k-1}{h-1}$. (See [20], p.44.) Hence the right-hand side of (38) is equal to

$$\begin{aligned} &\frac{1}{h\,!}\sum_{j=0}^{k-h} c(j)(-1)^j \binom{k-1}{j}\binom{k-j-1}{h-1} \\ =\ &\frac{1}{h\,!}\binom{k-1}{h-1}\sum_{j=0}^{k-h} c(j)(-1)^j\binom{k-h}{j}. \end{aligned}$$

Therefore it is enough to show that (by replacing k for $k-h$)

$$c(k) = \sum_{j=1}^{k} c(j)(-1)^j \binom{k}{j}.$$

This follows immediately by comparing the coefficients of x^k of

$$\Bigl(1-\tanh\frac{x}{2}\Bigr)e^x = 1+\tanh\frac{x}{2}.$$

(Note that $\tanh\frac{x}{2} = \frac{1}{i}\tan(\frac{ix}{2}) = -\sum_{n=1}^{\infty}\frac{c(n)}{n!}x^n$.) □

7 Another definitions of multiple zeta values

Let r_i $(i=1,2,\ldots,k)$ be non-negative integers. So far, we used the definition (3) for the multiple zeta values at non-positive integers. They are calculated by the recurrence relation (15) or (17). As mentioned in the introduction, the values depend on the choice of the limiting process. In this section, we wish to reconsider this problem. In [1], we remarked that it seems appropriate to define by

$$\zeta_k^C(-r_1,-r_2,\ldots,-r_k) = \lim_{\varepsilon\to 0}\zeta_k(-r_1+\varepsilon,-r_2+\varepsilon,\ldots,-r_k+\varepsilon) \tag{39}$$

at least in $k=2,3$, which is called 'central values'. The central values seems difficult to calculate by a recurrence relation. So we do not even know whether they have finite values. This is the main reason why we do not employ the definition by 'central values' in this paper.

There is another way to define values at non-positive integers:

$$\lim_{s_k\to -r_k}\lim_{s_{k-1}\to -r_{k-1}}\cdots\lim_{s_1\to -r_1}\zeta_k(s_1,s_2,\ldots,s_k), \tag{40}$$

which we call them 'reverse values' and denote by $\zeta_k^R(-r_1,\ldots,-r_k)$. In the scope of (16), we have a recurrence relation

$$\begin{aligned}
&\zeta_k^R(-r_1,\ldots,-r_k)\\
&\quad=\frac{\zeta_{k-1}^R(-r_1-r_2-1,-r_3,\ldots,-r_k)}{1+r_1}-\frac{\zeta_{k-1}^R(-r_1-r_2,-r_3,\ldots,-r_k)}{2}\\
&\qquad+\zeta(-r_1)\zeta_{k-1}^R(-r_2,\ldots,-r_k)\\
&\qquad-\sum_{q=1}^{r_1}(-r_1)_q a_q\zeta_{k-1}^R(-r_1-r_2+q,-r_3,\ldots,-r_k).
\end{aligned}$$

Especially when $r_1>0$, we have

$$\begin{aligned}
&\quad=\frac{\zeta_{k-1}^R(-r_1-r_2-1,-r_3,\ldots,-r_k)}{1+r_1}-\frac{\zeta_{k-1}^R(-r_1-r_2,-r_3,\ldots,-r_k)}{2}\\
&\qquad-\sum_{q=1}^{r_1-1}(-r_1)_q a_q\zeta_{k-1}^R(-r_1-r_2+q,-r_3,\ldots,-r_k).
\end{aligned}$$

Thus it is an interesting task to extend our results to $\zeta_k^R(-r_1,\ldots,-r_k)$. First we have

Theorem 4. *Let n be a non-negative integer. Then we have*

$$\zeta_k^R(0,\ldots,0,-n)=-\frac{1}{(k-1)\,!}\sum_{j=1}^{k}s(k,j)\frac{B_{n+j}}{n+j}+(-1)^k\delta_n.$$

This theorem is obtained by

$$\lim_{s\to 0}s\zeta_k^R(0,\ldots,0,s+n)=\frac{1}{(k-1)!}s(k,n),$$

which is proved quite similarly as Lemma 4.

Theorem 5. *Let n and k be integers with $n\geq 0$ and $k\geq 2$. Then we have*

$$\begin{aligned}
&\zeta_k^R(-n,0,\ldots,0)-\zeta(-n)\zeta_{k-1}^R(0,0,\ldots,0)\\
&\qquad=\frac{(-1)^n}{(k-2)\,!\,(n+1)}\sum_{j=1}^{k-1}s(k-1,j)\frac{B_{n+1+j}}{j}.
\end{aligned}\tag{41}$$

Proof. This theorem is proved analogously as Theorem 1, so we will only show the outline of proof. Let

$$c(n,j) = B_j' B_{n+1} + (-1)^{j+1} B_{n+j+1}$$

for $j \geq 1$ and $n \geq 1$, where we put $B_n' = B_n + \delta_{n-1}$. From Theorem 4, we can reduce (41) to showing

$$\begin{aligned}(42)\qquad &\sum_{j=2}^{M+1} B_{M-j+1}' \binom{M}{j-1} \frac{c(n,j)}{j} \\ &\quad = \frac{c(n+1,M)}{n+2} + \frac{c(n,M)}{2} - (n+1) \sum_{q=1}^{n-1} (-n)_q a_q \frac{c(n-q,M)}{n-q+1}.\end{aligned}$$

By repeated application of (27), we can show that the both sides of (42) are equal to

$$\frac{1}{n+2} \sum_r \binom{n+2}{2r} B_{2r} B_{M+n+2-2r} - B_{M+1} B_{n+1} + \frac{1}{2} B_{M+n+1}$$

when M is odd, and

$$\frac{1}{M+1} \sum_r \binom{M+1}{2r} B_{2r} B_{M+n+2-2r} - B_M B_{n+2} + \frac{1}{2} B_M B_{n+1} - \frac{1}{2} B_{M+n+1}$$

when M is even. This completes the proof of Theorem 5. $\square$

We can show the tangent symmetry for reverse values as well, by a similar argument:

Theorem 6. *Under the same condition of Theorem 3, we have*

$$\zeta_k^R(A) = \sum_{j=1}^{k-1} c(j) \Big(\sum_{J \in \mathcal{D}_{k-j}^k} \zeta_{k-j}^R (A^J) \Big).$$

However, the same statement does not hold for central values. For example,

$$\zeta_3^C(-2,-4,-6) = \frac{5003}{411840}, \qquad -\frac{\zeta_2^C(-2,-10) + \zeta_2^C(-6,-6)}{2} = \frac{13141}{1081080}.$$

One can find strong resemblance between our Theorems on reverse values and those on regular values. This fact suggests us that there would exist some principle between multiple zeta values which does not depend on the definition of values at the points of indeterminacy.

Finally, we prove the fact remarked in [1].

Proposition 1. *Let $r_1, r_2, \ldots, r_{k-1}$ be non-negative integers and not equal to zero simultaneously. Then $(-r_1, -r_2, \ldots, -r_{k-1}, 1)$ is a point of indeterminacy, at which the multiple zeta value $\zeta_k^R(-r_1, -r_2, \ldots, -r_{k-1}, 1)$ is rational.*

Proof. By using (16), we have similarly

$$\begin{aligned}
&\zeta_k^R(-r_1, \ldots, -r_{k-1}, 1) \\
&\quad = \frac{\zeta_{k-1}^R(-r_1 - r_2 - 1, -r_3, \ldots, -r_{k-1}, 1)}{1 + r_1} - \frac{\zeta_{k-1}^R(-r_1 - r_2, -r_3, \ldots, -r_{k-1}, 1)}{2} \\
&\qquad + \sum_{q=1}^{r_1 - 1} (-r_1)_q a_q \zeta_{k-1}^R(-r_1 - r_2 - q, -r_3, \ldots, -r_{k-1}, 1),
\end{aligned}$$

for $r_1 > 0$. Note that the pole at $s_k = 1$ of $\zeta(-r_1)\zeta_{k-1}^R(-r_2, \ldots, -r_{k-1}, s_k)$ cancels out with the one in the former sum. Thus we see the assertion by induction. □

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