MAHLER’S Z-NUMBER AND 3/2 NUMBER SYSTEMS

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ABSTRACT. We improve the results in [1] on the characterization of multiple points in rational based number system, in connection with Mahler’s Z-number problem. As a by-product, we show that when $p > q^2$, there exists a positive $x$ such that the fractional part of $x(p/q)^n$ ($n = 0, 1, \ldots$) stays in a Cantor set (Theorem 2.5). Hausdorff dimension of the set is positive but tends to zero as $p \to \infty$ when $q$ is fixed.

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1. Representations in a rational base

Let us review the result in [1]. Let $p, q$ be coprime integers with $p > q > 1$ and consider a digit set $\mathcal{A} = \{0, 1, \ldots, p-1\}$. Every positive integer $u$ has a unique representation:

$$u = u_0 \frac{1}{q} + u_1 \frac{p}{q^2} + u_2 \frac{p^2}{q^3} + \cdots + u_\ell \frac{p^\ell}{q^{\ell+1}}$$

with $u_i \in \mathcal{A}$. The digits $u_i$ are successively determined by taking module $p$ of both sides in the ring $\mathbb{Z}_q = \{z/q^n \mid z \in \mathbb{Z}, n \geq 0\}$, the localization of $\mathbb{Z}$ by $q$. Following the convention of decimal expression, we write $u = u_\ell u_{\ell-1} \ldots u_1 u_0$ and identify with the word in $\mathcal{A}^*$. The set of words which represent positive integers is denoted by $L_{p/q} \subset \mathcal{A}^*$. Then the set $L_{p/q}$ is not even context free since no infinite repetition is allowed but $0^\infty$. However the odometer is given by an automaton. A positive real number $x$ not greater than a given constant $\theta = \theta(p/q) > 1$ has a representation in a form:

$$x = x_{-1} \frac{1}{p} + x_{-2} \frac{q}{p^2} + \cdots + x_{-\ell} \frac{q^\ell}{p^{\ell+1}} + \cdots = x_{-1} x_{-2} \ldots$$

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with the property that \( x_{-1} x_{-2} \ldots x_{-m} \in L_{p/q} \) for all positive integers \( m \). This \( \theta(p/q) \) corresponds to the maximal word \( W(0) \) by the notation of [1] and is explicitly written as:

\[
\theta \left( \frac{p}{q} \right) = \sum_{i=0}^{\infty} \left( \frac{qG_{i+1}}{p} - G_i \right) \left( \frac{q}{p} \right)^i
\]

with \( G_0 = 0 \) and \( G_{i+1} = \lfloor (pG_i + p - 1)/q \rfloor \). For any real number \( x \), there exists \( M > 0 \) such that \( x(q/p)^M \leq \theta(p/q) \). This means that we can expand any \( x > 0 \) into

\[
x = x_M x_{M-1} \ldots x_0 \cdot x_{-1} x_{-2} \ldots
\]

using the decimal point ‘.’ in a usual manner. From the property of \( L_{p/q} \), there are no eventually periodic expansions. The \( p/q \)-integer part (resp. \( p/q \)-fractional part) of \( x \) is defined to be \( x_M x_{M-1} \ldots x_0 \) (resp. \( x_{-1} x_{-2} \ldots \)). We put the decimal point to distinguish them with other representations. This representation is unique but countably many exceptions.

We only review an easy part, the necessity of the condition (1) for the purpose of this note. It is shown in [1] that the digit-wise difference of eventually \( p/q \)-representations if and only if there exists \( n_0 \) so that

\[
\left\langle \frac{x}{q} \left( \frac{p}{q} \right)^n \right\rangle \in \bigcup_{0 \leq c \leq q-1} \left[ \frac{k_c}{p}, \frac{k_c + 1}{p} \right]
\]

holds for all \( n \geq n_0 \). The number \( k_c \in A \) is defined by \( qk_c \equiv c \pmod{p} \).

**Theorem 1.1** (Akiyama-Frougny-Sakarovitch [1]). Let \( p \geq 2q - 1 \). Then a positive real number \( x \) has two \( p/q \)-representations if and only if there exists \( n_0 \) so that

\[
.f\quad \text{ } x_{c_{-1} c_{-2} \ldots} < \frac{q - 1}{p - q} \leq 1.
\]

Since \( p/q \)-integer parts have integer values, this inequality means that the \( p/q \)-fractional part (resp. \( p/q \)-integer part) of \( x \) coincides with the usual fractional part (resp. integer part) of \( x \). Let us consider a function \( f(x) = q([xp/q] - (p/q)[x]) \). By the above claim, if \( x \) admits double expressions, then we have

\[
f((p/q)^n x) = q(c_M c_{M-1} \ldots c_0 c_{-1}).c_M c_{M-1} \ldots c_0 0) = q \times c_{-1} = c_{-1}
\]
Thus for large \( n \), \( f((p/q)^n x) \) takes values only in \( \{0, 1, \ldots, q - 1\} \). Note that \( f \) is a periodic function of period \( q \) and the value of \( f((p/q)^n x) \) is determined by \( x \pmod{q} \in \mathbb{R}/q\mathbb{Z} \). Now using \( p \geq 2q - 1 \), it is easy to show
\[
\left( \frac{p}{q} \right) \left( \frac{p}{q} \right)^{-1} \left( \frac{p}{q} \right)^n = \left( \frac{p}{q} \right)^{n-1} \left( \frac{p}{q} \right) \leq \left( \frac{p-2}{p-q} \right) \leq 1.
\]
which shows the necessity. \( \square \)

The same idea allows us to show

**Theorem 1.2.** For an integer \( k \) with \( p - 1 \leq k(p-q) \), if a real number \( x \) has \( k \) different \( p/q \)-representations then there exists \( n_0 \) so that
\[
\left\langle x \left( \frac{p}{q} \right)^n \right\rangle \in \bigcup_{0 \leq c \leq (k-1)q-(k-2)p-1} \left[ \frac{k_c}{p}, \frac{k_c + 1}{p} \right]
\]
holds for all \( n \geq n_0 \).

**Proof.** We proceed in a similar manner as the above proof of Theorem 1.1. Only thing to note is that \( x \) has \( k \) different representations if and only if \( x \) has a suffix in \( \{0, \ldots, (k-1)q-(k-2)p-1\}^* \), since \( p - 1 - (k-1)(p-q) = (k-1)q-(k-2)p-1 \) and \( (p/q)^n x = c_M c_{M-1} \ldots c_0 . c_{-1} c_{-2} \ldots \) for large \( n \) satisfies
\[
.c_{-1} c_{-2} \ldots < \frac{(k-1)q-(k-2)p-1}{p-q} \leq 1.
\]
\( \square \)

**Corollary 1.3.** A real number has at most \( 1 + \left\lceil \frac{p-2}{p-q} \right\rceil \) different \( p/q \)-representations.

**Proof.** An inequality \( 1 \leq (k-1)q-(k-2)p-1 \) is necessary to have an aperiodic expansion of \( x > 0 \). \( \square \)

As far as we computed, there seems no triple points for any \( p/q \)-representations. Perhaps it is reasonable to pose a

**Conjecture 1.4.** There are no positive real \( x \) so that
\[
\left\langle x \left( \frac{p}{q} \right)^n \right\rangle \in \bigcup_{0 \leq c \leq 2q-p-1} \left[ \frac{k_c}{p}, \frac{k_c + 1}{p} \right]
\]
holds for all \( n \),
which implies that there are no \( x \) with triple expressions when \( p - 1 \leq 3(p - q) \). For e.g., if \( p = 4 \) and \( q = 3 \) then the conjecture asserts that there are no positive \( x \) such that
\[
\left \langle x \left( \frac{4}{3} \right) \right \rangle \in [0, 1/4) \cup [3/4, 1)
\]
holds for all \( n \geq 0 \). This is also equivalent to the statement that there are no real \( x \) such that \( ||x(4/3)^n|| < 1/4 \) for all \( n \), where \( ||y|| \) is the distance of \( y \) from the nearest integer. Here the left endpoint of \([3/4, 1)\) can be neglected. In fact, \( \langle x(4/3)^n \rangle = 3/4 \) occurs only when \( x \) is rational and at most once for such a \( x \) by seeing the denominator of \( x \). However we may substitute \( x \) by \( x(4/3)^{n+1} \) in such a case. The end points usually do no harm by this trick.

2. A generalization of Mahler’s \( Z \)-number

One can show stronger results than the ones in the previous section. Before stating the result, we begin with some terminologies. Let \( F \) be a finite union of half open subintervals \([a, b)\) of \([0, 1)\) and \( \mu(F) \) be the 1-dimensional Lebesgue measure of \( F \). We study two sets \( Z_{p/q}^+(F) = \{ 0 < x \in \mathbb{R} \mid \langle x(p/q)^n \rangle \in F \} \) and \( Z_{p/q}(F) = \{ x \in \mathbb{R} \mid \langle x(p/q)^n \rangle \in F \} \). In fact, our framework is much suitable for the study of \( Z_{p/q}^+(F) \) but occasionally we can deduce results on \( Z_{p/q}(F) \) as well. The notorious problem in this context is due to Mahler [4] whether \( Z_{3/2}^+(\{0, 1/2\}) \) is empty or not. For developments on the distribution of limit points of \( \langle x(p/q)^n \rangle \), the reader should consult series of papers by Dubickas for e.g. [2, 3]. He also derived a large \( \mu(F) \) with \( Z_{p/q}(F) = \emptyset \).

Theorem 1.1 implies that if \( p \geq 2q - 1 \) then there exists some \( F \) with \( \mu(F) = q/p \) that \( Z_{p/q}^+(F) \) is countably infinite. For e.g., using Theorem 1.1 with \( p = 3 \) and \( q = 2 \), we see \( Z_{3/2}^+(\{0, 1/3\} \cup [2/3, 1)) \) is countably infinite. Thus there exists a real \( x \)’s such that \( ||x(3/2)^n|| < 1/3 \) for all \( n \). As a refinement of Theorem 1.1, we have

**Theorem 2.1.** Let \( p > q > 1 \) with \( p \geq 2q - 1 \). Then a positive real number \( x \) has two \( p/q \)-representations if and only if there exists \( n_0 \) so that
\[
\left \langle \frac{x}{q} \left( \frac{p}{q} \right) \right \rangle \in \bigcup_{0 \leq c \leq q-1} k_c \frac{k_c}{p} - \frac{q - 1}{p(p - q)} [\frac{1}{2}]
\]
holds for all \( n \geq n_0 \). The number \( k_c \in A \) is defined by \( qk_c \equiv c \pmod{p} \).
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This Theorem implies that for $p \geq 2q - 1$, two conditions (1) and (3) are equivalent, in fact. Further, this implies that there exists a finite union of intervals $F$ with $\mu(F) = \frac{q(q-1)}{p(p-q)}$ such that $Z^{+}_{p/q}(F)$ is countably infinite.

**Proof.** As the right hand side of (3) is narrower than (1), the sufficiency of the condition (3) is obvious. We only prove the necessity. Firstly we shall show a weaker statement. The open intervals of (3) are substituted by closed ones. Here the idea is to generalize the function $f(x)$ to

$$f_m(x) = q^m \left\lfloor \frac{x p^m}{q^m} \right\rfloor - p^m [x]$$

with a large integer $m(\geq 2)$ in the proof of Theorem 1.1. Using the same idea, this function $f_m$ has period $q^m$ and if $x$ is a double point then

$$f_m((p/q)^n x) = q^m(c_{m-1} \ldots c_0 c_{-1} \ldots c_{-m} - c_{M} c_{M-1} \ldots c_0 0^m)$$

holds for a large $n$. Our task is to construct concretely the inverse image of $f_m$.

Take $k^* = k^*(c_{-1}, c_{-2}, \ldots, c_{-m}) \in \{0, 1, \ldots, p^m - 1\}$ which satisfies

$$\sum_{j=1}^{m} p^{m-j} q^{-1} c_{-j} \equiv q^m k^* \pmod{p^m}. \quad (4)$$

By using the same proof of Theorem 1.1, we have

$$\left\langle \frac{x}{q^m} \left( \frac{p^m}{q^m} \right)^n \right\rangle \in \bigcup_{(c_{-1}, \ldots, c_{-m}) \in \{0, \ldots, q-1\}^m} \left[ \frac{k^*}{p^m}, \frac{k^* + 1}{p^m} \right]$$

for $mn \geq n_0$ where $n_0$ is the same as in the proof of Theorem 1.1. Multiplying $q^{m-1}$, we have

$$\frac{x}{q} \left( \frac{p^m}{q^m} \right)^n \pmod{q^{m-1}} \in \bigcup_{(c_{-1}, \ldots, c_{-m})} \left[ \frac{q^{m-1} k^*}{p^m}, \frac{q^{m-1} (k^* + 1)}{p^m} \right]. \quad (5)$$

From (4), one see

$$p^{m-1} k_{c_{-1}} + \sum_{j=2}^{m} p^{m-j} q^{-2} c_{-j} \equiv q^{m-1} k^* \pmod{p^m}. \quad (6)$$
Without loss of generality, we may assume that \( c_{-2} \ldots c_{-m} \neq (q - 1)^{m-1} \).

Therefore we have an estimate
\[
\sum_{j=2}^{m} p^{m-j} q^{j-2} c_{-j} < p^{m-2} \frac{q - 1}{1 - q/p} \left( 1 - \left( \frac{q}{p} \right)^{m-1} \right) \leq p^{m-1} - q^{m-1}.
\]

Thus the left hand side of (6) belongs to \([0, p^m - 1] \cap \mathbb{Z}\). Taking modulo 1 of (5), we have
\[
\left< \frac{x}{q} \left( \frac{p^m}{q^m} \right)^n \right> \in \bigcup_{c_{-1} \ldots c_{-m}} \left[ \frac{k_{c_{-1}}}{p} + \sum_{j=2}^{m} c_{-j} \frac{q^{j-2}}{p^j}, \frac{k_{c_{-1}}}{p} + \sum_{j=2}^{m} c_{-j} \frac{q^{j-2}}{p^j} + \frac{q^{m-1}}{p^m} \right].
\]

Note that
\[
\bigcup_{c_{-2} \ldots c_{-m}} \left[ \frac{k_{c}}{p} + \sum_{j=2}^{m} c_{-j} \frac{q^{j-2}}{p^j}, \frac{k_{c}}{p} + \sum_{j=2}^{m} c_{-j} \frac{q^{j-2}}{p^j} + \frac{q^{m-1}}{p^m} \right]
\]

is contained in the interval
\[
\left[ \frac{k_{c}}{p}, \frac{k_{c}}{p} + \sum_{j=2}^{m} \frac{(q - 1)q^{j-2}}{p^j} + \frac{q^{m-1}}{p^m} \right].
\]

Thus we have
\[
\left< \frac{x}{q} \left( \frac{p^m}{q^m} \right)^n \right> \in \bigcup_{c} \left[ \frac{k_{c}}{p}, \frac{k_{c}}{p} + \frac{q - 1}{p(p - q)} + \frac{q^{m-1} p - 1}{p^m (p - q)} \right].
\]

This implies that there exists \( n_1 \) so that for any positive \( \varepsilon \),
\[
\left< \frac{x}{q} \left( \frac{p^m}{q^m} \right)^n \right> \in \bigcup_{c} \left[ \frac{k_{c}}{p}, \frac{k_{c}}{p} + \frac{q - 1}{p(p - q)} + \varepsilon \right]
\]
holds for \( n \geq n_1 \). This shows the weaker statement for closed intervals. Consider end points of the intervals of (3). As we may assume that \( c_{-2} \ldots c_{-m} \neq 0^{m-1} \) or \((q - 1)^{m-1}\), we easily see that such end points can not be attained in the above proof. \( \square \)

**Remark 2.2.** In the above proof, if \( q^2 \geq p \geq 2q - 1 \), then
\[
\bigcup_{c_{-2} \ldots c_{-m}} \left[ \frac{k_{c}}{p} + \sum_{j=2}^{m} c_{-j} \frac{q^{j-2}}{p^j}, \frac{k_{c}}{p} + \sum_{j=2}^{m} c_{-j} \frac{q^{j-2}}{p^j} + \frac{q^{m-1}}{p^m} \right]
\]
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is exactly equal to
\[
\begin{bmatrix}
k_c / p, & k_c / p + \sum_{j=2}^{m} \frac{(q - 1)q^{j-2}}{p^j} + \frac{q^{m-1}}{p^m}
\end{bmatrix}.
\]

To see this, we note
\[
\frac{q^{m-k-1}}{p^{m-k+1}} \leq \sum_{j=2}^{\infty} \frac{(q - 1)q^{j-2}}{p^{j+2}} \leq \sum_{j=2}^{k} \frac{(q - 1)q^{j-2}}{p^{j+2}} + \frac{q^{m-1}}{p^m}.
\]

The left inequality follows from \(q^2 \geq p\) and the right from \(p \geq 2q - 1\).

Following the same proof, we have

**Theorem 2.3.** For an integer \(k\) with \(p - 1 \leq k(p - q)\), if a real number \(x\) has \(k\) different \(p/q\)-representations then there exists \(n_0\) so that
\[
\left\langle \frac{x}{q} \left(\frac{p}{q}\right)^n \right\rangle \in \bigcup_{0 \leq c \leq (k-1)q-(k-2)\text{p-1}} \left[ \frac{k_c}{p}, \frac{k_c}{p} + \frac{(k-1)q-(k-2)p}{p(p-q)} \right]
\]

holds for all \(n \geq n_0\).

It is remarkable that if \(p > q^2\), the result becomes better.

**Theorem 2.4.** Let \(p > q > 1\) with \(p > q^2\). Then for any positive \(\varepsilon\), there exists a finite union of intervals \(F\) with \(\mu(F) < \varepsilon\) and a positive real number \(x\) has two \(p/q\)-representations if and only if there exists \(n_0\) so that
\[
\left\langle \frac{x}{q} \left(\frac{p}{q}\right)^n \right\rangle \in F
\]

holds for all \(n \geq n_0\).

Thus for \(p > q^2\) there exists a finite union of intervals \(F\) of an arbitrary small size \(\mu(F)\) such that \(Z_{p/q}(F)\) is countably infinite. In the following, we shall prove a stronger result.

A set \(X = X(p/q)\) is given as a non empty compact set in \(\mathbb{R}\) satisfying an iterated function system:
\[
X = \bigcup_{j=0}^{q-1} qX+j/p.
\]

It is approximated by a decreasing sequence of sets defined by \(X_0 = [0, (q - 1)/(p - q)]\) and \(X_{k+1} = \bigcup_{j=0}^{p-1} (qX_k + j)/p\) for \(k = 0, 1, \ldots\). We see \(X = \cap_k X_k\) and all end points of \(X_k\) are in \(X\). As \(p > q^2\), \(\mu(X) = 0\) follows from the definition. The pieces \((qX+j)/p\) do not overlap, this system gives a
Cantor set in $[0,(q-1)/(p-q)]$ of Hausdorff dimension $\log q/\log(p/q) < 1$ which is positive but tends to zero as $p \to \infty$ and $q$ is fixed.

**Theorem 2.5.** Let $p > q > 1$ with $p > q^2$. Then a positive $x$ has two $p/q$-representations if and only if there exists $n_0$ that

$$\left\langle \frac{x}{q} \left( \frac{p}{q} \right)^n \right\rangle \in \bigcup_{c=0}^{q-1} \frac{X(p/q) + kc}{p}$$

for $n \geq n_0$.

As $X_k (k = 0, 1, \ldots)$ are finite unions of intervals, Theorem 2.4 follows immediately from Theorem 2.5.

**Proof.** Since $X(p/q) \subset [0,(q-1)/(p-q)]$, the sufficiency of (10) follows from Theorem 2.1. We show the necessity. It is easily seen that

$$X(p/q) = \left\{ \sum_{i=0}^{\infty} c_i \frac{q^i}{p^{i+1}} \middle| c_i \in [0,q-1] \cap \mathbb{Z} \right\}.$$

We proceed in the same manner as the proof of Theorem 2.1. If $x$ admits two $p/q$-representations, then there exists $n_0$ such that (7) holds for $mn \geq n_0$. Each element $u$ of

$$\left\lfloor \frac{k_c}{p} + \sum_{j=2}^{m} c_{j-1} \frac{q^{j-2}}{p^j}, \frac{k_c}{p} + \sum_{j=2}^{m} c_{j-1} \frac{q^{j-2}}{p^j} + \frac{q^{m-1}}{p^n} \right\rfloor$$

has distance at most $q^{m-1}/p^m$ from the compact set $(X(p/q) + kc)/p$. As we can choose $m$ large, the distance of the point $\langle x/q(p/q) \rangle^n$ and the compact set $\bigcup_{c=0}^{q-1} (X(p/q) + kc)/p$ is zero, which proves the theorem. □

Denominators of end points of $X_k$ are divisors of $(p(p-q))^{k+1}$ which are coprime to $q$. Thus, as $n$ increases, $\langle x/q(p/q) \rangle^n$ can visit the end points at most once only when $x$ is rational.

Note that if $x$ is a double point, then there exists $n_0$ such that $\langle x(p/q) \rangle^n = .c_{-1}c_{-2}\ldots$ with $c_{-i} \in [0,q-1] \cap \mathbb{Z}$ for $n \geq n_0$. This already implies that $\langle x(p/q) \rangle^n \in X(p/q)$. We observe that (10) is stronger than this inclusion. Indeed, (10) implies

$$x \left( \frac{p}{q} \right)^n (\mod q) \in \bigcup_{c} \left( \frac{qX}{p} + \frac{qk_c}{p} \right)$$

and taking modulo 1, we get $\langle x(p/q) \rangle^n \in X(p/q)$ again.
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At any rate, it is unexpected that when \( p > q^2 \) there exists \( x > 0 \) that the closure of \( \langle x(p/q)^n \rangle (n = 0, 1, \ldots) \) is contained in the Cantor set \( X(p/q) \), a compact set of measure zero. We do not know whether the closure of \( K = \{ \langle x(p/q)^n \rangle | n = 0, 1, \ldots \} \) could be of Hausdorff dimension 0. In the other direction, Vijayaraghavan [5] showed that the number of accumulation points of \( K \) is infinite but it is not known whether the closure of \( K \) could be countable.

REFERENCES


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