On the boundary connectedness of connected tiles

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1. Introduction

In the present paper we want to dwell upon topological properties of ordinary as well as graph directed iterated function systems. Let us first give the basic definitions.

It is well known that for a family of contractions $f_1, \dots, f_k$ on $\mathbb{R}^d$, there is a unique non-empty compact set $T = T(f_1, \dots, f_k)$ with $T = \cup_i f_i(T)$ (cf. [16]). Here T is called the attractor of the *iterated function system* (IFS for short) $f_1, \dots, f_k$.

If there exists a non-empty open set $V \subset \mathbb{R}^d$ such that $f_i(V) \cap f_j(V) = \emptyset$ for $i \neq j$ and $\cup_i f_i(V) \subset V$, then we say that $f_1, \dots, f_k$ (or T) satisfy the *open set condition* (cf. [11]).

Define a graph $\mathcal{G}_T$ as follows. The set of vertices is $\{f_1, \dots, f_k\}$, and two distinct vertices f_i, f_j are incident with each other whenever $f_i(T)$ intersects $f_j(T)$. It is easy to show that T is connected if and only if the graph $\mathcal{G}_T$ is connected (cf. [14, 22]).

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The principal aim of this paper is to study the connectedness of the boundary of the attractor T for dimensions $d \geq 2$. Connectedness of T is an obvious necessary condition to have a connected boundary. However the following simple example shows that it is not sufficient.

Example 1. For integers $i, j \in \{0, 1, \ldots, 6\}$, let $\varphi_{i,j}$ be a contraction of $\mathbb{R}^2$ defined by

$$\varphi_{i,j}(x, y) = \left(\frac{3x + i}{9}, \frac{3y + j}{9} \right).$$

Let $S = \{\varphi_{i,j} \mid i \in \{0, 6\} \text{ or } j \in \{0, 6\}\}$. Then the attractor B of this system S is

$$B = [0, 1]^2 \setminus \left(\frac{1}{3}, \frac{2}{3} \right)^2$$

(see Figure 1).

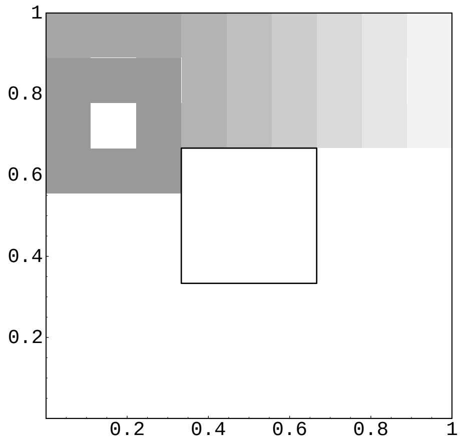

FIGURE 1. The punctured square S is an IFS attractor with overlaps

It is an interesting question to ask what kind of additional conditions would imply the boundary connectedness of connected attractors. In the case $d = 2$, it is shown (cf. [28]) under the open set condition that the boundary of a connected attractor is also connected. One of the aims of the present paper is to generalize this result to arbitrary dimensions $d \geq 2$. In particular we shall prove the following theorem.

Theorem 1.1. *Let $f_1, \ldots, f_k$ be injective contractions on $\mathbb{R}^d$ $(d \geq 2)$ satisfying the open set condition, and let T be the attractor. Then ∂T is connected whenever T is.*

We mention here that the above theorem and Theorems 1.2 to 1.4 in this section concern connected attractors with non-empty interior. Otherwise, they would be trivially true. Also, the proof of Theorem 1.1 is quite topological based on the non overlapping property inherited from the open set condition. One can imagine by Example 1 that such a non overlapping condition is indispensable.

A variant of this result can be proved if we assume that the attractor can tile the space. Let us denote by J a discrete subset of $\mathbb{R}^d$ and T be a compact set in $\mathbb{R}^d$ which coincides with the closure of its interior. If translates by J of T cover $\mathbb{R}^d$ and the interiors of two distinct translates have no intersection, then we say that T satisfies the *tiling condition*. In short, we just say that T is a *tile*.

Theorem 1.2. *Let $f_1, \dots, f_k$ be injective contractions on $\mathbb{R}^d$ $(d \geq 2)$ and assume that the attractor T satisfies the tiling condition. Then ∂T is connected whenever T is.*

We employ a graph theoretical idea in the proof. In fact, the tiling condition is strong enough in this context so that we do not need to assume that T is an attractor (see Theorem 3.1). Here is a class of attractors with the tiling condition.

Example 2 (Bandt [5], Kenyon [18]). Let A be an expanding integer matrix of size $d \times d$, i.e., each eigenvalue of A has absolute value greater than 1. Denote by D a complete representative system of $\mathbb{Z}^d/A\mathbb{Z}^d$. The attractor T of the contractions

$$v \mapsto A^{-1}(v + d) \qquad (d \in D)$$

satisfies the tiling condition.

As this example shows, IFS attractors are a fruitful source of tiles. In general, such tiles are of interest in their own right and were extensively studied, in particular in the case where the contractions are affinities (in this case we call a tile *self affine*). Many important theorems on the characterization of self affine tiles and their properties were proved for instance in Gröchenig-Haas [13] and Lagarias-Wang [24, 25, 26]. We also refer to the recent papers Vince [34] and Wang [35] where a large list of literature is provided. Also certain topological properties of self affine tiles have been studied. In Hata [14] an easy criterion when the attractor is a locally connected continuum is shown in a somewhat more general context (see also [15]). Further topological properties of plane attractors are investigated by Luo-Rao-Tan [28]. General criteria for disklikeness of lattice self affine tiles in the plane can be found for instance in Bandt-Wang [8]. The convex hull of self affine tiles as well as an algorithm for finding their neighbors was provided in Strichartz-Wang [33] and Kenyon et al. [21] (see also Scheicher-Thuswaldner [32] for a different algorithm).

As a special case, tiles occur in the study of certain generalized number systems. They have a long history, for instance, consult Kátai-Kőrnyei [20], Kovács-Pethő [19], Gilbert [12], Scheicher-Thuswaldner [31], Brunotte [7] and their references. Various topological properties of this class of tilings attached to number systems were studied in Akiyama-Thuswaldner [2, 3].

As it is well known, IFS attractors admit a generalization defined via graphs: so called *graph directed self affine attractors*. First we define these objects in full generality as it appears in Mauldin-Williams [29] and Barnsley et al. [9]. See also [11, Chapter 3] for a slightly specialized version having a uniform Hausdorff dimension for all pieces.

Let $V = \{1, \dots, q\}$ be a set of vertices and let E be a set of directed edges starting and ending in elements of V. Then $G = G(V, E)$ is a directed graph. We suppose that each vertex is the starting point of at least one edge. Let $E_{i,j}$ be the set of edges leading from i to j. Now for each $e \in E$ define a uniformly contractive map $F_e : \mathbb{R}^d \to \mathbb{R}^d$. Then by [29, Theorem 1] (or [9, Corollary 3.5]) there exists a unique family $S_1, \dots, S_q$ of compact non-empty sets fulfilling

$$S_i = \bigcup_{j=1}^{q} \bigcup_{e \in E_{i,j}} F_e(S_j). \tag{1}$$

The set of contractions $\{F_e \,|\, e \in E\}$ is called a *graph directed IFS* (GIFS for short) and the sets S_i as well as their union $S = \bigcup_i S_i$ are called graph directed attractors (see [17]).

In this more general setting we also have an analogue to the open set condition. Namely, we say that the generalized open set condition holds if there exist non-empty open sets $V_1, \dots, V_q$ such that

$$\bigcup_{j=1}^{q} \bigcup_{e \in E_{i,j}} F_e(V_j) \subset V_i.$$

Here the unions have to be disjoint for each $i \in \{1, \dots, q\}$ (cf. [6] or [30]).

The tiling condition can also be naturally generalized to a version including several pieces. Let $T_1, T_2, \dots, T_q$ be compact sets such that each of them coincides with the closure of its interior and let $J_1, \dots, J_q$ be discrete sets in $\mathbb{R}^d$. Assume that $\mathbb{R}^d = \bigcup_{i=1}^q T_i + J_i$ and for $j_s \in J_s$ and $j_t \in J_t$, $T_s^\circ + j_s \cap T_t^\circ + j_t \neq \emptyset$ implies $s = t$ and $j_s = j_t$. Then we say that $T_1, T_2, \dots, T_q$ forms a system of prototiles and each piece T_i is called a prototile. A *graph directed tile* is a GIFS attractor which is a prototile in this sense.

Graph directed tiles are much more difficult to construct than ordinary tiles (see [17]) but important in relation to symbolic dynamics and, in particular, Markov partitions of toral automorphisms. One known way of construction of graph directed self affine tiles leads via atomic surfaces related to substitutions on finite alphabets. These again are strongly related to digital expansions w.r.t. Pisot numbers (cf. [1, 4]).

Seeing this, it is expected to generalize Theorem 1.1 to graph directed self affine attractors. The third aim of the present paper is to show

Theorem 1.3. *Let $S_1, \dots, S_q$ be GIFS attractors in $\mathbb{R}^d$ satisfying the generalized open set condition. Then ∂S_i is connected for at least one $i \in \{1, \dots, q\}$ whenever S_i is connected for each $i \in \{1, \dots, q\}$.*

Remark 1. This result seems weaker than readers might expect. However the statement of Theorem 1.3 is close to best possible. If S is a connected GIFS attractor then ∂S can be disconnected. Furthermore, even if all S_i $(1 \le i \le q)$ are connected, there may well exist a $j \in \{1, \dots, q\}$ such that ∂S_j is disconnected. This may happen even if the generalized open set condition is satisfied. This is illustrated by the examples given below.

Example 3. In this example we construct a connected GIFS attractor S with disconnected boundary.

It is clear that the punctured square in Figure 1 can be represented as a union of eight congruent squares which we denote $S_0, \cdots, S_7$ as seen in Figure 2.

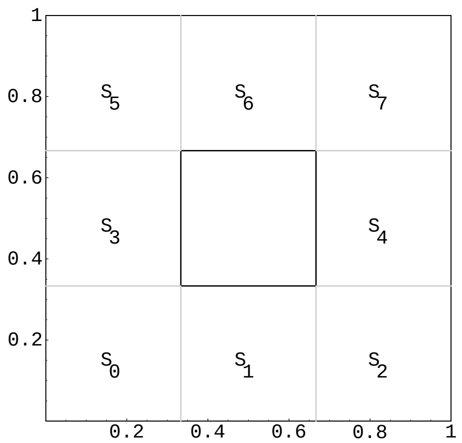

FIGURE 2. The punctured square S as GIFS attractor

Since each S_i can be divided into four squares of half the side length, we may find similarity maps (not unique) $F_k^1, \cdots, F_k^4$ with factor $\frac{1}{2}$ such that

$$S_k = F_k^1(S_k) \cup F_k^2(S_k) \cup F_k^3(S_k) \cup F_k^4(S_{k+1 \bmod 8})$$

Then we know that S is a GIFS attractor directed by a primitive graph with generalized open set condition.

Example 4. In this example we will construct a GIFS $S = S_1 \cup S_2$ with connected sets S_1, S_2, such that ∂S_1 is not connected. To this matter let

$$\begin{aligned} S_1 &:= [0,1]^2 \setminus \left(\frac{1}{3}, \frac{2}{3}\right)^2, \\ S_2 &:= \left[\frac{1}{3}, \frac{2}{3}\right]^2. \end{aligned}$$

Here, S_1 is the punctured square in Figure 1, S_2 is the central small square which is deleted, and $S = S_1 \cup S_2$ is a square. Now we show that S is graph directed. Set $M := \{(i,j) \mid 0 \le i,j \le 2,\ (i,j) \ne (1,1)\}$ and define the contractions

$$H_{i,j}(x,y) := \frac{1}{3}(x+i, y+j).$$

Then we easily see that

$$\begin{aligned} S_1 &= \bigcup_{(i,j)\in M} \left(H_{i,j}(S_1) \cup H_{i,j}(S_2)\right), \\ S_2 &= H_{1,1}(S_1) \cup H_{1,1}(S_2). \end{aligned}$$

Thus S is a GIFS directed set with the generalized open set condition by a primitive graph.

Related to Theorem 1.3, one can also show a generalization of Theorem 1.2.

Theorem 1.4. *Let $T_1, \dots, T_q$ be a system of tiles of $\mathbb{R}^d$ such that each prototile T_i is arcwise connected. Then ∂T_i is connected for at least one $i \in \{1, \dots, q\}$.*

Note that here we no longer assume that each T_i is an attractor.

Furthermore, for graph directed attractors there holds a simple criterion for their arcwise as well as local connectedness. These criteria are rather easy to prove and we will show them in Section 3.

The paper is organized as follows. Section 2 contains the proof of Theorem 1.1. In Section 3, we give a proof of Theorem 1.2 and 1.4. In Section 4 we sketch the proof of Theorem 1.3. Finally, in the same section we show the mentioned criteria for arcwise and local connectedness.

2. The open set condition

To prove Theorem 1.1, we need to show that the complement of T° is connected and arcwise connected. First, let us start from a basic result of plane topology.

Lemma 2.1. *Suppose that A, B are disjoint closed sets with*

$$\begin{aligned} A &\subset [0,1) \times [0,1], \\ B &\subset (0,1] \times [0,1]. \end{aligned}$$

Then there exists a path in $([0,1] \times [0,1]) \setminus (A \cup B)$ starting from a point in $(0,1) \times \{0\}$ and leading to a point in $(0,1) \times \{1\}$.

Proof. Let $N > 0$ be a positive integer such that $2^{-N+2}\sqrt{2}$ is less than the distances $\rho(A \cup (\{0\} \times [0,1]), B \cup (\{1\} \times [0,1]))$. Divide the square $[0,1] \times [0,1]$ into 2^{-N} horizontal strips, each of which is of width 2^{-N}. For every $1 \leq n \leq 2^{N-1}$, the $(2n-1)$-th strip is divided into a row of 2^{N-1} rectangles of length 2^{-N+1}, and the $(2n)$-th strip is divided into 2 squares of side 2^{-N} and $2^{N-1} - 1$ rectangles of length 2^{-N+1} such that the two small squares are at the very left and right ends of the strip. This is visualized in Figure 3.

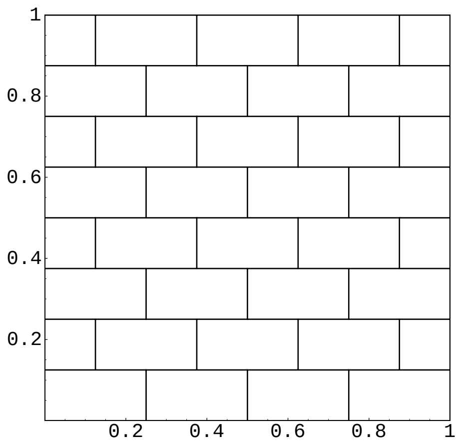

FIGURE 3. The "brick wall" corresponding to the choice $N = 3$

Denote by $\mathcal{A}$ the family of all the rectangles or small squares intersecting $A \cup (\{0\} \times [0,1])$, and $\mathcal{B}$ the family of those intersecting $B \cup (\{1\} \times [0,1])$.

Then the union U and V of elements in $\mathcal{A}$ and $\mathcal{B}$ are two disjoint compact sets each of which has finitely many components. From now on, we call a component of the complement of a set $X \subset \mathbb{R}^d$ a *complementary component of* X. Since each of the components of $U \cup V$ is a locally connected continuum which has no cut point, the boundary of the unbounded complementary component for every component of $U \cup V$ is a simple closed curve [23, §61, II, Theorem 4].

If we add to $U \cup V$ the small rectangles and cubes which lie entirely in a bounded complementary component of $U \cup V$, we will obtain finitely many pairwise disjoint topological disks $P_1, \cdots, P_m$. Note that the boundary of each disk does not intersect $A \cup B$ except at the four sides of $[0,1] \times [0,1]$.

Let P_i be the topological disk containing $\{0\}\times[0,1]$, then P_i must intersect both $(0,1)\times\{0\}$ and $(0,1)\times\{1\}$. Let $p\in(0,1)\times\{0\}$ and $q\in(0,1)\times\{1\}$ be the two points of ∂P_i such that the open line segment from p to 1×0, and that from q to 1×1 do not intersect P_i. It is now clear that the simple closed curve ∂P_i is separated by $\{p,q\}$ into two broken arcs from p to q, and the one not intersecting $\{0\}\times[0,1]$ is an arc which has no common points with $A\cup B$.

Let A be a subset of $\mathbb{R}^d$ or $\mathbb{S}^d=\{x_1\times x_2\times x_3 : x_1^2+x_2^2+x_3^2=1\}$ with $d\geq 2$. If two points x,y lie in different complementary components of A, we say that A separates x and y. Note that if A is closed then the components of its complementary set are arcwise connected regions.

Theorem 2.2. *Suppose that A,B are disjoint compact sets in $\mathbb{R}^d$ or $\mathbb{S}^d(d\geq 2)$. If x,y are separated by $A\cup B$, they must be separated either by A or by B.*

Proof. Otherwise, there would be two paths $f_1,f_2:[0,1]\to\mathbb{R}^d$ (or $\mathbb{S}^d$) from x to y with $f_1([0,1])\cap A=\emptyset$ and $f_2([0,1])\cap B=\emptyset$. Since both $\mathbb{R}^d$ and $\mathbb{S}^d$ are simply connected, there exists a homotopy $F:[0,1]\times[0,1]\to\mathbb{R}^d$ (or $\mathbb{S}^d$) between f_1,f_2. Here $F(t,0)=f_1(t)$, $F(t,1)=f_2(t)$, $F(0,s)=x$ and $F(1,s)=y$ for all $s,t\in[0,1]$. It is clear that $F^{-1}(A),F^{-1}(B)$ are disjoint compact sets in the unit square which are contained in $[0,1]\times(0,1]$ and $[0,1]\times[0,1)$, respectively. Applying Lemma 2.1 with first and second coordinate interchanged, we obtain a path P in the unit square from $p\in\{0\}\times(0,1)$ to $q\in\{1\}\times(0,1)$ which does not intersect $F^{-1}(A)\cup F^{-1}(B)$. Hence $F(P)$ is a path in $\mathbb{R}^d\setminus(A\cup B)$ joining x and y, which contradicts the condition that x,y are separated by $A\cup B$.

Corollary 2.3. *Suppose that U,V are disjoint bounded open sets in $\mathbb{R}^d$ $(d\geq 2)$. If x,y are separated by $U\cup V$, they must be separated either by U or by V.*

Proof. For all $n\geq 1$, denote by A_n the complement of the $\frac{1}{n}$-neighborhood of U's complement, and by B_n the complement of the $\frac{1}{n}$-neighborhood of V's complement. Then A_n,B_n are disjoint compact sets. Were x,y separated neither by U nor by V, then x,y would not be separated by A_n or B_n for all n. Thus, by Theorem 2.2, x,y lie in the same complementary component Q_n of $A_n\cup B_n$. Since $\overline{Q_{n+1}}$ lies in the interior of Q_n for all $n\geq 1$, the intersection $M=\bigcap_n Q_n$ equals $\bigcap_n\overline{Q_n}$. As the closures Q_n' of Q_n in $\mathbb{S}^d$ are a decreasing sequence of continua, if M is bounded then it is already a continuum containing x and y ([23, §47, II, Theorem 5]); if M is unbounded then $M\cup\{\infty\}=\bigcap_n Q_n'$ is a continuum and $M\subset\mathbb{R}^d$ is an unbounded connected set containing both x and y by the boundedness of U and V. Recall that M does not intersect $U\cup V$. Thus $\{x,y\}\subset M$ must be contained in a single complementary component of $U\cup V$, which is contradictory to the condition that x,y are separated by $U\cup V$.

Theorem 2.4. *Let $f_1, \dots, f_k$ be injective contractions on $\mathbb{R}^d$ $(d \geq 2)$ satisfying the open set condition. If the attractor T is connected, then the complement of T° is connected.*

Proof. We may assume that T has interior points. Denote by $U_n (n \geq 1)$ the components of T°, and by V_n the union of $U_1, \cdots, U_n$. We claim that the complement P_n of every U_n is connected, thus by Corollary 2.3, the complement Q_n of every V_n is connected. Regard Q_n as a subset of $\mathbb{S}^d$, then $\{\overline{Q_n} : n \geq 1\}$ is a decreasing sequence of continua, and the intersection $M = \bigcap_n \overline{Q_n}$ is connected ([23, §47, II, Theorem 5]). This indicates that $\mathbb{R}^d \setminus T^\circ = \bigcap_n Q_n = M \setminus \{\infty\}$ is connected, too.

Were there some n with P_n disconnected, P_n would have a bounded component C. For any fixed point $x \in T^\circ$, choose $\varepsilon > 0$ such that the closed ball $B := B(x, 2\varepsilon)$ lies entirely in T°. We define $f_\alpha = f_{i_1} \circ \cdots \circ f_{i_m}$ for all $m \geq 1$ and $\alpha = i_1, \dots, i_m \in \{1, \dots, k\}^m$. Then for distinct $\alpha, \beta \in \{1, \dots, k\}^m$, $f_\alpha(T^\circ)$ does not intersect $f_\beta(T^\circ)$ (cf. [28]). Choose $N > 0$ with $\mathrm{diam}(f_\alpha(T)) < \varepsilon$ for all $\alpha \in \{1, \dots, k\}^N$, then there is some $\alpha_1 \in \{1, \dots, k\}^N$ with $x \in f_{\alpha_1}(T)$, and $f_{\alpha_1}(C) \cup f_{\alpha_1}(U_n)$ is contained in the interior of the ball $B(x, \varepsilon)$. Choose $\alpha_2 \in \{1, \dots, k\}^N \setminus \{\alpha_1\}$ with $f_{\alpha_2}(T) \cap f_{\alpha_1}(C) \neq \emptyset$, then $f_{\alpha_2}(T)$ is contained in $f_{\alpha_1}(C)$ since T is connected (here the open set condition is used).

In particular, $f_{\alpha_2}(U_n)$ lies entirely in the bounded component $f_{\alpha_1}(C)$ of $\mathbb{R}^d \setminus f_{\alpha_1}(U_n)$, and $f_{\alpha_2}(C)$ is a bounded component of $\mathbb{R}^d \setminus f_{\alpha_2}(U_n)$, so $f_{\alpha_2}(C) \cap f_{\alpha_1}(U_n)$ is empty. This indicates that $f_{\alpha_2}(C) \cup f_{\alpha_2}(U_n)$ is contained in $f_{\alpha_1}(C)$.

Inductively, we can choose distinct $\alpha_1, \dots, \alpha_{k^N} \in \{1, \dots, k\}^N$ such that the union $\bigcup_{1 \leq j \leq k^N} f_{\alpha_j}(U_n)$ lies entirely in $B(x, \varepsilon)$. This indicates that $T = \cup_{\alpha \in \{1, \dots, k\}^N} f_\alpha(T)$ lies entirely in B°, which is impossible.

Corollary 2.5. *Let $f_1, \dots, f_k$ be injective contractions on $\mathbb{R}^d$ $(d \geq 2)$ satisfying the open set condition. If the attractor T is connected, then the complement of T° is arcwise connected.*

Proof. For a fixed point $x \in T^\circ$, choose $\varepsilon > 0$ such that the closed ball $B := B(x, \varepsilon)$ lies entirely in T°. Choose $N > 0$ such that $\mathrm{diam}(f_\alpha(T)) < \varepsilon$ for all $\alpha \in \{1, \dots, k\}^N$, then there is some $\alpha \in \{1, \dots, k\}^N$ with $x \in f_\alpha(T)$ and $f_\alpha(T) \subset B^\circ$. As each f_i is injective, f_α is a homeomorphism on every bounded subset of $\mathbb{R}^d$. So we just need to show that $\mathbb{R}^d \setminus f_\alpha(T^\circ)$ is arcwise connected.

Denote by M_1 the union of ∂B and its exterior, and by M_2 the union of all sets $f_\beta(T)$ with $\beta \in \{1, \dots, k\}^N \setminus \{\alpha\}$ such that $f_\beta(T)$ intersects B. Since the connectedness of $\mathbb{R}^d \setminus T^\circ$ indicates that $\mathbb{R}^d \setminus f_\alpha(T^\circ)$ is connected, the set M_2 is connected, too.

Let $\mathcal{G}$ be a graph whose vertices are those $\beta \in \{1, \ldots, k\}^N \setminus \{\alpha\}$ with $f_\beta(T) \subset M_2$, and two vertices β, γ are incident whenever $f_\beta(T)$ intersects $f_\gamma(T)$. Then $\mathcal{G}$ is a connected graph, thus M_2 is arcwise connected since every $f_\beta(T)$ is. Since $M_1 \cap M_2$ contains the circle ∂B, the union $M_1 \cup M_2 = \mathbb{R}^d \setminus f_\alpha(T^\circ)$ must be arcwise connected.

Proof of Theorem 1.1:

Let $f_1, \ldots, f_k$ be injective contractions on $\mathbb{R}^d$ ($d \geq 2$) satisfying the open set condition. If the attractor T is connected, then the complement of T° is arcwise connected by Corollary 2.5. As T is locally connected, it is arcwise connected.

Let us assume now that $\partial T = P \cup Q$ is a separation. Choose $p \in P$ and $q \in Q$, there must be two paths $g_0 : [0,1] \to T$ and $g_1 : [0,1] \to \mathbb{R}^d \setminus T^\circ$ joining p and q. By the simple connectedness of $\mathbb{R}^d$, there is a homotopy $G : [0,1] \times [0,1] \to \mathbb{R}^d$ between g_0 and g_1. It is easy to see that $G^{-1}(P), G^{-1}(Q)$ are disjoint compact sets which are contained in $[0,1) \times [0,1]$ and $(0,1] \times [0,1]$, respectively. By Lemma 2.1, there exists an arc $J \subset [0,1] \times [0,1]$ from $x_0 \in (0,1) \times \{0\}$ to $x_1 \in (0,1) \times \{1\}$ which does not intersect $G^{-1}(P) \cup G^{-1}(Q)$. Therefore, $G(J)$ is a path in $\mathbb{R}^d \setminus \partial T$ from $G(x_0) \in T^\circ$ to $G(x_1) \in (\mathbb{R}^d \setminus T)$, indicating that $G(J)$ intersects $\partial T = P \cup Q$. This leads to a contradiction and we are done.

3. The tiling condition

In this section we wish to replace the open set condition of Theorem 1.1 by the tiling condition. In fact, we will show the following stronger statement.

Theorem 3.1. *Let T be an arcwise connected compact set satisfying the tiling condition. Then ∂T is connected.*

A *cut point* x of a connected (undirected) graph G is a point for which $G \setminus \{x\}$ is not connected. A path of a graph G is a sequence of vertices denoted by $(u_1, u_2, \ldots, u_k)$ where u_i and u_{i+1} are incident. Sometimes, u_1 and u_k are called the two end points of the path. A path is called *elementary* when it consists of pairwise distinct vertices. Furthermore, an elementary path is *maximal* if it is not contained properly in any other elementary path.

For the convenience of the reader we recall an easy lemma from graph theory.

Lemma 3.2 (cf. [27]). *A finite connected graph G which contains not less than 2 vertices has at least 2 non cut points.*

Proof. Let $K = (u_1, u_2, \ldots, u_k)$ be a maximal elementary path of G. By assumption, $k \geq 2$. We shall prove that the two end points of K can not be cut points. Assume that $G \setminus \{u_1\}$ is not connected and decompose it into components $C_1, C_2, \ldots, C_m$ with $m \geq 2$. Let $K \setminus \{u_1\} \subset C_1$. As G is connected, there exists a vertex $v \in C_2$ which is incident with u_1. Now we have an elementary path $(v, u_1, u_2, \ldots, u_k)$ which is longer than K. This gives a contradiction.

Reviewing the former section, we used the open set condition only in the proof of Theorem 2.4. Thus the remaining non trivial part of the proof of Theorem 3.1 is to show

Theorem 3.3. *Assume that the compact set T satisfies the tiling condition. If T is connected, then the complement of T° is connected.*

Proof. Let J be the discrete set of $\mathbb{R}^d$ with respect to which the pair (T, J) gives a tiling of $\mathbb{R}^d$, i.e., $T + J = \mathbb{R}^d$ and for any distinct elements $i, j \in J$, we have $(T^\circ + i) \cap (T^\circ + j) = \emptyset$. Note that the slightly stronger statement

$$(T + i) \cap (T^\circ + j) = \emptyset \text{ for any distinct } i, j \in J,$$

is also valid since T coincides with the closure of its interior. Without loss of generality, we assume that $0 \in J$.

Let us introduce an infinite graph whose vertices consist of the set J and $a, b \in J$ are incident if and only if $(T+a)\cap(T+b) \neq \emptyset$. We denote this graph by G_J. It is easily seen that the graph G_J is connected as $\mathbb{R}^d$ is connected.

Suppose that the complement of T° is disconnected. Then there exists a bounded component C of the complement of T° since the unbounded component is unique by the compactness of T. Then C contains a point of $T+i$ with $i \neq 0$ since otherwise C would be contained in $\mathbb{R}^d \setminus \cup_{0 \neq j \in J}(T+j) = T^\circ$. As $T + i$ is connected and has empty intersection with T°, $T + i$ is contained entirely in C. Let us consider $T+j$ with $j \neq 0$ which has a common point with $T+i$. Then by the same reason, $T+j$ is contained entirely in C. This implies that the component Q containing i of the subgraph $G_J \setminus \{0\}$ must be finite since C is bounded. This shows that 0 is a cut point of G_J. In the same way, we see that each element of J is a cut point of G_J. Let us consider the finite connected subgraph $Q \cup 0$, then there exists a non cut point $y \in Q$ of this subgraph. Then y is a non cut point of G_J which gives a contradiction.

Corollary 3.4. *Assume that the compact set T satisfies the tiling condition. If T is arcwise connected, then the complement of T° is arcwise connected.*

Proof. We have seen the connectedness of the complement of T°. As the complement of T° is equal to $\cup_{0 \neq j \in J}(T + j)$, $G_J \setminus \{0\}$ is a connected graph. Take two points x, y of $\cup_{0 \neq j \in J}(T + j)$. Then it is easy to find a curve connecting x, y in $\cup_{0 \neq j \in J}(T + j)$ since each $T + j$ is arcwise connected.

Proof of Theorem 1.4:

The idea of the proof is almost the same as in he proof of Theorem 3.3. Let $J_1, \dots, J_q$ be the sets of the corresponding translations. Construct an infinite connected graph G_J with vertex set $J = \cup_i J_i$ and $j_s \in J_s$ such that $j_t \in J_t$ are incident whenever $T_s + j_s \cap T_t + j_t \neq \emptyset$. Assume that the complement of T_i° is not connected for each $i \in \{1, \dots, q\}$. Then similarly, each vertex must be a cut point of a graph and the same contradiction occurs. This shows that the complement of T_i° is connected for at lease one $i \in \{1, \dots, q\}$. The remaining part of the proof is likewise.

4. The graph directed case

The aim of this section is the treatment of the more general case of graph directed self affine attractors. In the first part we shall prove Theorem 1.3. Since this theorem is a result which is valid for connected attractors, we want to give a criterion for the connectedness and arcwise connectedness of GIFS in the second part of this section. To show this, an easy criterion for their local connectedness is needed.

Proof of Theorem 1.3.

We may assume that $S_j^\circ \neq \emptyset$ holds for all $j \in \{1, \dots, q\}$ because otherwise the theorem is trivially true. First we shall prove that for at least one $j \in \{1, \dots, q\}$ the complement of S_j° is connected. Similar to the proof of Theorem 2.4 we denote by $U_n^{(j)}$ $(n \geq 1)$ the components of S_j° and by $P_n^{(j)}$ the complement of $U_n^{(j)}$. By Corollary 2.3 it suffices to show that there exists some $j \in \{1, \dots, q\}$ such that $P_n^{(j)}$ is connected for each $n \in \mathbb{N}$.

Suppose on the contrary that for all $j \in \{1, \dots, q\}$, there exists some $n \in \mathbb{N}$ such that $P_n^{(j)}$ is disconnected. Then $P_n^{(j)}$ has a bounded component, and for all $j \in \{1, \dots, q\}$, there exists some $n \in \mathbb{N}$ such that $P_n^{(j)}$ has a bounded component C_j. Now fix $i \in \{1, \dots, q\}$. For some $x \in S_i^\circ$ choose $\varepsilon > 0$ such that $B(x, 2\varepsilon) \subset S_i^\circ$.

Let $W_m(G, i, j)$ be the set of paths in G having length m, starting in i and ending in j. $W_m(G, i)$ denotes the set of paths in G starting at i. For each path w in G defined by the edges $(e_1, \dots, e_m)$ set $F_w = F_{e_1} \circ \cdots \circ F_{e_m}$. For convenience, we will write $U^{(j)}$ instead of $U_{n_j}^{(j)}$.

For each $N > 0$ there is some $i_1 \in \{1, \dots, q\}$ and some $w_1 \in W_N(G, i, i_1)$ with $x \in F_{w_1}(S_{i_1})$. Now choose N large enough such that

(i) $\operatorname{diam}(F_w(S_j)) < \varepsilon$ for all $w \in W_N(G, i, j)$ and all $j \in \{1, \dots, q\}$.
(ii) $F_{w_1}(C_{i_1}) \cup F_{w_1}(U^{(i_1)}) \subset \operatorname{int}(B(x, \varepsilon))$.

Now choose $w_2 \in W_N(G, i, i_2)$ with $w_2 \neq w_1$ and $F_{w_2}(S_{i_2}) \cap F_{w_1}(C_{i_1}) \neq \emptyset$.

Note that the unions in (1) are non-overlapping. From this it is easy to see that the generalized open set condition is fulfilled by the open sets $(S_1^\circ, \ldots, S_q^\circ)$. This implies together with the connectedness of S_{i_2} that $F_{w_2}(S_{i_2}) \subset F_{w_1}(C_{i_1})$ (note that $F_{w_1}(C_{i_1})$ is a component of the complement of $F_{w_1}(U^{(i_1)}) \subset F_{w_1}(S_{i_1})$). Furthermore,

$$F_{w_2}(C_{i_2}) \cup F_{w_2}(U^{(i_2)}) \subset F_{w_1}(C_{i_1}).$$

Now (since S_i is connected) by induction we can choose $|W_N(G,i)|$ distinct $w \in W_N(G,i)$ such that

$$\bigcup_{\ell=1}^{q} \bigcup_{w \in W_N(G,i,\ell)} F_w(U^{(\ell)}) \subset B(x,\varepsilon)$$

But this would imply that $S_i = \bigcup_{w \in W_N(G,i)} F_w(S_j)$ were contained in $B(x,2\varepsilon)$, a contradiction.

The analogue of Corollary 2.5 for GIFS with strongly connected graphs is proved in a very similar way as Corollary 2.5 itself. The remaining part of the proof is exactly the same as the proof of Theorem 1.1.

Now we turn to the connectedness of GIFS. The result we shall prove is a generalization of Hata [14, Theorem 4.6] and Kirat-Lau [22, Theorem 1.2]. To this end, we need the following notations. For a path w in the directed graph $G(V,E)$ we denote by $t(w)$ the terminal vertex of w. Let $E_{i\cdot}$ denote the set of all edges in G leading away from the vertex i. Furthermore, we define the set

$$\mathcal{E} := \left\{ (e_1, e_2) \in E \times E \mid F_{e_1}(S_{t(e_1)}) \cap F_{e_2}(S_{t(e_2)}) \neq \emptyset \right\}.$$

Theorem 4.1. *Let $S = S_1 \cup \ldots \cup S_q$ be a GIFS with graph $G(V,E)$ and contractions F_e $(e \in E)$. Then S_j is connected for each $j \in \{1, \ldots, q\}$ if and only if the following "$\mathcal{E}$-connectedness" property is true for each $i \in \{1, \ldots, q\}$:*

For each pair $e_1, e_2 \in E_{i\cdot}$ there exists a subset $\{e_{j_1}, \ldots, e_{j_k}\} \subset E_{i\cdot}$ such that $e_1 = e_{j_1}$, $e_2 = e_{j_k}$ and $(e_{j_\ell}, e_{j_{\ell+1}}) \in \mathcal{E}$ for $1 \leq \ell \leq k$.

If this condition holds, each S_j is a locally connected continuum or a single point. In particular, they are arcwise connected.

Proof. By iterating the GIFS subdivision we easily see that

$$S_i = \bigcup_{j=1}^{q} \bigcup_{e \in E_{ij}} F_e(S_j) = \bigcap_{k \geq 1} \bigcup_{w \in W_k(G,i)} F_w(B).$$

Here B is a ball which is large enough to fulfill $S_j \subset B$ for all $j \in \{1, \ldots, q\}$ and $F_e(B) \subset B$ for all $e \in E$. Trivially, $F_w(B)$ is connected. We want to prove by induction that $\bigcup_{w \in W_k(G,i)} F_w(B)$ is connected for arbitrary k. For $k = 1$ this follows from the $\mathcal{E}_i$-connectedness property. Suppose now that

$\bigcup_{w\in W_k(G,i)} F_w(B)$ is connected for each $i \in \{1,\ldots,q\}$. In order to perform the induction step note that

$$\bigcup_{w\in W_{k+1}(G,i)} F_w(B) = \bigcup_{e\in E_{i\cdot}} F_e\left(\bigcup_{w\in W_k(G,t(e))} F_w(B)\right).$$

The inner union is a connected set which contains $S_{t(e)}$. The connectedness now follows by applying again the $\mathcal{E}_i$ connectedness property.

Thus we have seen that each S_j is a non-empty connected compact set. In view of the Theorem of Hahn and Mazurkiewicz, it remains to show that these sets are locally connected. This is done in the next proposition.

Proposition 4.2. *Let $S = S_1 \cup \ldots \cup S_q$ be a GIFS with graph $G(V,E)$ and contractions F_e $(e \in E)$. If S_i is connected for each $i \in \{1,\ldots,q\}$ then all the S_i are locally connected. Furthermore, S is locally connected.*

Proof. Choose $\varepsilon > 0$ arbitrary. Then there exists an N such that $\mathrm{diam} F_w(S_j) < \varepsilon$ holds for all $w \in W_N(G,i,j)$ and for all $i,j \in \{1,\ldots,q\}$. Note that each element of S_i is contained in one of these sets and that each of these sets is connected. Thus S_i $(1 \le i \le q)$ as well as S is a finite union of connected sets of diameter less than ε. Since ε was arbitrary, the assertion follows from [23, §50, II, Theorem 2].

The following example shows that the connectedness of the GIFS $\{S_1,\ldots,S_q\}$ is essential for its local connectedness.

Example 5. Let $S := S_1 \cup S_2 \cup S_3$ with

$$\begin{aligned}
S_1 &= (\{0\}\times[0,2]) \cup \bigcup_{n\ge 1}\left(\left[\frac{3}{2^n},\frac{4}{2^n}\right]\times[0,2]\right)\\
S_2 &= [3,4]\times[0,2]\\
S_3 &= [0,4]\times[-1,0].
\end{aligned}$$

Then S_2, S_3 and S as well as their boundaries are connected, but S_1 is disconnected (otherwise we had a contradiction to Proposition 4.2). It is easy to see that neither S nor ∂S is locally connected.

Now let

$$\begin{aligned}
F_1(x,y) &:= \frac{1}{2}(x,y).\\
F_2(x,y) &:= \frac{1}{2}(x,y) + (0,1).
\end{aligned}$$

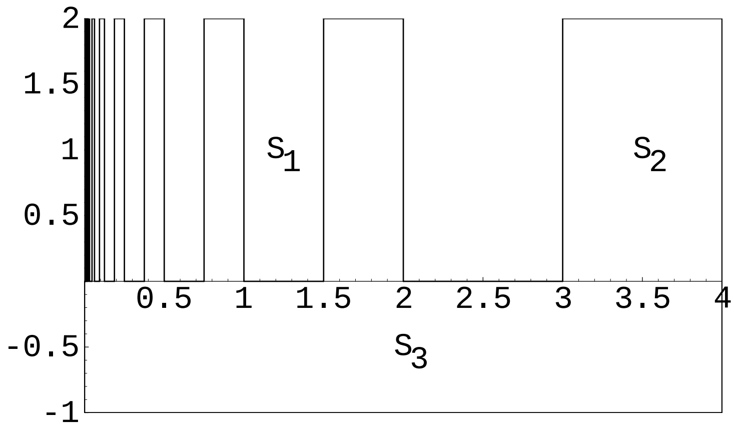

FIGURE 4. A GIFS attractor which is not locally connected

Then

$$\begin{aligned} S_1 &= F_1(S_1) \cup F_2(S_1) \cup F_1(S_2) \cup F_2(S_2), \\ S_2 &= \text{easily seen to be a union of contracted copies of } S_2 \text{ and } S_3, \\ S_3 &= \text{easily seen to be a union of contracted copies of } S_2 \text{ and } S_3. \end{aligned}$$

Thus $S = S_1 \cup S_2 \cup S_3$ is a GIFS. It is easily seen that S and ∂S have the required properties.

The graph G directing this GIFS is not primitive and ∂S is arcwise connected. However, with some more effort one can construct an example with primitive graph and not arcwise connected ∂S.

5. Concluding remarks

Let T be an IFS attractor. In the present paper we showed criteria which ensure the connectedness of the boundary ∂T of T. A next step would be to investigate if ∂T is even arcwise or locally connected under these conditions or if additional assumptions are needed in order to get these properties. We think that there are two ways which can lead towards results of this type. First one could try to apply methods from topology in order to get arcwise and local connectedness of ∂T directly. This approach may be promising especially for the case of plane attractors $T \subset \mathbb{R}^2$ since one can use facts from the well-studied plane topology.

Another approach is of a different flavor. Recently, Dekking-van der Wal [10] showed for a class of self similar GIFS attractors with non-empty interior that their boundary is also a GIFS attractor. Since we have shown criteria for the connectedness of the boundary of an attractor, results on its arcwise and local connectedness follow from our Proposition 4.2 for this class of attractors. If the class of attractors T for which ∂T is a GIFS can be enlarged, this would yield to results on the arcwise and local connectedness of ∂T even for higher dimensional attractors.

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