Spectral sets of certain functions associated with Dirichlet series.

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The theory of almost periodic functions was established by H. Bohr. Almost periodic functions are a natural extension of periodic functions. One of important results in Bohr's theory is that the class of almost periodic functions φ is identical with the closure of the linear span of $\{e^{i\lambda t}\}_{\lambda\in\Lambda}$ in the sense of the L^∞ norm, where Λ is a countable set in $\mathbf{R}$ defined as a support of a certain transform of φ.

One of interesting examples of almost periodic functions comes from the Riemann zeta-function $\zeta(s)$, where s is a complex variable with $s=\sigma+it$. Let $\zeta_\sigma(t)$ be the function defined by $\zeta_\sigma(t)=\zeta(\sigma+it)$. Then, for $\sigma>1$, $\zeta_\sigma(t)$ is an almost periodic function with $\Lambda=\{-\log n\}_{n=1}^\infty$.

A. Beurling studied almost periodic functions φ from a point of view of spectral sets $S(\varphi)$. The concept of spectral sets is defined as a support of a certain transform of φ which is a generalization of the Fourier transform. Beurling's result is this: *Let φ be a uniformly continuous and bounded function on $\mathbf{R}$. If $S(\varphi)$ is a countable set which does not accumulate to a finite value, then φ is in the L^∞ norm closure of the linear span of $\{e^{i\lambda t}\}_{\lambda\in S(\varphi)}$, and consequently, an almost periodic function.*

It is a natural motivation to extend Beurling's result to ones for unbounded functions. This is a difficult problem and should be tried. The present talk is concerned with this motivation from a point of view of the Riemann zeta-function. For $\sigma<1$ it is known that ζ_σ is unbounded, and so, it is no longer an almost periodic function in the sense of Bohr. So, we firstly study its spetral set $S(\zeta_\sigma)$ for $\sigma<1$. A result is that $S(\zeta_\sigma)=\mathbf{R}$ for σ with $\sigma<1$. The result $S(\zeta_\sigma)=\mathbf{R}$ might suggest that $S(\zeta_\sigma)$ is consisted of the discrete spectrum $\{-\log n\}_{n=1}^\infty$ and the continuous spectrum $\mathbf{R}$ in a sense.

Apart from ζ_σ, we discuss spectral sets of functions which are expressed by Dirichlet series on a half plane. For example, we see that $S(\zeta_\sigma^k)=\mathbf{R}$ for σ with $\sigma<1$, where $k\in\mathbf{N}$, and $S(L_\sigma)=\{-\log n | n\in\mathbf{N},\ (n,q)=1\}$ for σ with $\sigma\le 1$, where $L_\sigma(t)$ is the function defined by using Dirichlet L-function $L(s,\chi)$ with a primitive character χ mod q.