

# Multiple sine functions and zeta functions

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Multiple sine functions are generalizations of the usual sine function. The purpose of this lecture is to explain their basic properties and applications. Multiple sine functions have (quasi-) periodicity and multiplication formulas. In some cases, they satisfy algebraic differential equations. This property implies that the gamma factor of the Selberg zeta function is explicitly written via the multiple gamma function discovered by Barnes 100 years ago.

Another important theme is the nature of division values of the multiple sine function. Such values seem to be nearly algebraic and they would provide a solution to Kronecker's Jugendtraum constructing class fields explicitly by special values of a suitable function associated to a number field. It turns out that these values are related to difficult special values of zeta and L functions and Mahler measures.

Moreover, we report progresses in the theory of multiple zeta functions (absolute tensor products). These functions are further generalizations of multiple sine functions and the usual zeta functions. They give an important family of functions of higher order. We survey the following properties: zeros, poles, the functional equation and special values. We explain unsolved problems also.

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