Multiple Euler factors

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In 1992 Kurokawa originally defined the absolute tensor product, which constructs a new zeta function from ordinary zeta functions. Zeros and poles of the new zeta function are located at sums of zeros or poles of ordinary zeta functions. We expect that the new zeta function has similar properties to those of ordinary zeta functions. For example:

- If $Z_j(s)$ have Euler product expressions running through sets P_j of generalized primes for any j, the absolute tensor product $Z_1(s) \otimes \cdots \otimes Z_r(s)$ might have the multiple Euler product expression, which is a product expression through $P_1 \times \cdots \times P_r$.

- If $Z_j(s)$ have the functional equation between $s \leftrightarrow d_j - s$ for any j, $Z_1(s) \otimes \cdots \otimes Z_r(s)$ might have the functional equation between s and $d_1 + \cdots + d_r - s$.

When $Z_j(s)$ have Euler product expressions for any j, we also predict that there exist relations between the multiple Euler product for $Z_1(s) \otimes \cdots \otimes Z_r(s)$ and absolute tensor products of Euler factors for $Z_j(s)$.

We consider absolute tensor products of Euler factors $\zeta_p(s) := (1-p^{-s})^{-1}$ for the Riemann zeta function. $\zeta_p(s)$ can be regarded as a zeta function having a Euler product expression (through the set $\{p\}$ of one point) and a functional equation $\zeta_p(-s) = -p^{-s}\zeta_p(s)$.

In this talk we will report the expression for $\zeta_{p_1}(s) \otimes \cdots \otimes \zeta_{p_r}(s)$ similar to $\zeta_p(s) = \exp(\sum_{m=1}^{\infty} m^{-1} p^{-ms})$. From Kurokawa and Wakayama's result this expression can be described by the formal series (generalized polylogarithm) of r-variable with the constant term 1 when $p_1, \ldots, p_r$ are distinct prime numbers. On the other hand, when $p_j = p_k$ for some $j, k (j \neq k)$, we will see the degenerate form of the formal series.

We will also report the functional equation associated to $\zeta_{p_1}(s) \otimes \cdots \otimes \zeta_{p_r}(s)$ between s and $-s$. The asymmetry of the distribution of zeros and poles will appear in the functional equation.