COMPLETE ASYMPTOTIC EXPANSIONS ASSOCIATED WITH EPSTEIN ZETA-FUNCTIONS

Masanori KATSURADA

Mathematics, Hiyoshi Campus, Keio University

Throughout the following, $s=\sigma+it$ denotes the complex variable and $z=x+iy$ the complex parameter with $y>0$. The main object of this talk is the Epstein zeta-function $\zeta_{\mathbb{Z}^2}(s;z)$ (attached to the quadratic form $|u+vz|^2$) defined by

$$\zeta_{\mathbb{Z}^2}(s;z)=\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}}|m+nz|^{-2s}\qquad(\sigma=\operatorname{Re}s>1),$$

and its meromorophic continuation over the whole s-plane. Let α, β be complex numbers which will be fixed later, and $\Gamma(s)$ the gamma function. We introduce the Laplace-Mellin and the Riemann-Liouville (or the Erdélyi-Kober) transforms of $\zeta_{\mathbb{Z}^2}(s;x+iy)$ (with the normalization multiples) as

$$\mathcal{LM}^{\alpha}_{y;Y}\,\zeta_{\mathbb{Z}^2}(s;x+iy)=\frac{1}{\Gamma(\alpha)}\int_0^{\infty}\zeta_{\mathbb{Z}^2}(s;x+iyY)y^{\alpha-1}e^{-y}dy,\tag{1}$$

$$\mathcal{RL}^{\alpha,\beta}_{y;Y}\,\zeta_{\mathbb{Z}^2}(s;x+iy)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^{1}\zeta_{\mathbb{Z}^2}(s;x+iyY)y^{\alpha-1}(1-y)^{\beta-1}dy.\tag{2}$$

These can be regarded as the weighted mean values of $\zeta_{\mathbb{Z}^2}(s;x+iy)$; the factor $y^{\alpha-1}$ is inserted to secure the convergence of the integrals as $y\to+0$, while the functions e^{-y} and $(1-y)^{\beta-1}$ have effects to extract the parts corresponding to $y=O(Y)$ of $\zeta_{\mathbb{Z}^2}(s;z)$ with their respective weights. Note that the *confluence* operation

$$\mathcal{RL}^{\alpha,\beta}_{y,\beta Y}\,\zeta_{\mathbb{Z}^2}(s;x+iy)\longrightarrow\mathcal{LM}^{\alpha}_{y;Y}\,\zeta_{\mathbb{Z}^2}(s;x+iy)\qquad(\beta\to+\infty)\tag{3}$$

is valid since $\zeta_{\mathbb{Z}^2}(s;x+iy)=O(y^{\max(0,1-2\sigma)})$ as $y\to+\infty$.

In this talk we shall present a complete asymptotic expansion of $\zeta_{\mathbb{Z}^2}(s;x+iy)$ when $y\to+\infty$, and further show that similar asymptotic series still exist for (1) and (2) both when $Y\to+\infty$. It should be noted that various (confluent) hypergeometric functions appear and work in the proofs of these expansions; especially their summation and transformation properties play crucial rôles in the analysis of the remainder terms. Furthermore, one can observe that the asymptotic expansion of (2) precisely reduces to that of (1) through the *confluence* operation (3).

4–1–1 Hiyoshi, Kouhoku-ku, Yokohama 233–8521, Japan

E-mail address: `katsurad @ hc.cc.keio.ac.jp`; `masanori @ math.hc.keio.ac.jp`