On approximation measures of certain $q$-continued fractions

Ville Merilä
University of Oulu, Finland

Abstract

Let $K$ be an algebraic number field of degree $\kappa$ over $Q$. Let

\[ || * ||_v = | * |_v^{\kappa_v / \kappa}, \quad \kappa_v = [K_v : Q_v], \]

be the normalized valuation of $K$ and denote $\lambda = \lambda_q = \log H(q) / \log ||q||_v$, where $H(q)$ is the height of $q \in K^*$ satisfying $|q|_v < 1$. Then the (proper) $q$-continued fraction

\[ G(q) = K_{n=1}^\infty \frac{q^{s(n-1)}(S_0 + S_1q^{n-1} + \ldots + S_hq^{h(n-1)})}{T_0 + T_1q^n + \ldots + T_lq^{ln}}, \quad S_i, T_i \in K, \ S_0T_0 \neq 0, \]

where $s \geq 1$ and $s + \lambda A > 0$, has an approximation measure (exponent) $\mu = s\kappa / \kappa_v(s + A\lambda)$ where $A = \max\{l, (s + h)/2\}$.

The results imply, for example, irrationality measures $\mu = 3/(3 + 2\lambda)$ for the famous Ramanujan-Selberg continued fractions

\[ S_1(q) = \frac{(-q^2; q^2)_{\infty}}{(-q; q^2)_{\infty}} = \frac{1}{1} + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \frac{q^4}{1} + \ldots, \]

\[ S_2(q) = \frac{(q; q^8)_{\infty}(q^7; q^8)_{\infty}}{(q^3; q^8)_{\infty}(q^5; q^8)_{\infty}} = \frac{1}{1} + \frac{q}{1} + \frac{q^4}{1} + \frac{q^3 + q^6}{1} + \frac{q^8}{1} + \ldots, \]

and for Eisenstein’s continued fraction

\[ E_1(q) = \sum_{n=0}^{\infty} q^{n^2} = \frac{1}{1 - q} + \frac{q^2 - q}{1} - \frac{q^5 - q^3}{1} - \ldots, \]

related to the Jacobi Theta functions.