Character sums with Beatty sequences

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For fixed real numbers α and β, the associated *Beatty sequence* is the sequence of integers defined by $\{\lfloor \alpha n + \beta \rfloor \ : \ n = 1, 2, 3, \dots\}$.

Beatty sequences appear in a variety of mathematical settings and their arithmetic properties have been extensively studied by A. G. Abercrombie, A. S. Fraenkel and R. Holzman, T. Komatsu, K. O'Bryant, R. Tijdeman and many others. We estimate exponential sums of the form

$$S_p(\alpha, \beta, \chi; N) = \sum_{n \le N} \chi(\lfloor \alpha n + \beta \rfloor),$$

where α is *irrational*, and χ is a *nontrivial multiplicative character* modulo a prime number p. In particular, our bounds imply that for fixed α, β and a small real number $\varepsilon > 0$, if p is sufficiently large and $N \ge p^{1/3+\varepsilon}$, then among the first N elements of the Beatty sequence, there are $N/2 + o(N)$ *quadratic non-residues* modulo p. In the case that α is not *Liouville* (which includes all algebraic irrationals and almost all real numbers), our results yield explicit bounds on the error term.

The method we use is similar to those used to estimate short double sums in modern proofs of the *Burgess bound*. Our underlying approach is based on some ideas of A. A. Karatsuba.

Various bounds on the size of the least quadratic non-residue modulo p in the sequence $\{\lfloor \alpha n \rfloor \ : \ n = 1, 2, 3, \dots\}$ have been obtained by M. Z. Garaev and by S. N. Preobrazhenskiĭ. For example, M. Z. Garaev shows that for any real $\alpha > 0$ and any prime p, there is a positive integer $n \le p^{(1+e^{-1/2})/4+o(1)}$ such that $\lfloor \alpha n \rfloor$ is a quadratic non-residue modulo p. The bounds of S. N. Preobrazhenskiĭare stronger but require certain restrictions on α.

Our results improve these bounds on the size of the least quadratic non-residue and also apply to studying primitive roots in Beatty sequences, the question which cannot be approached by previously known methods.

We also indicate how our approach can lead to some asymptotic formulas for average values of various arithmetic functions on Beatty sequences.