Explicit algebraic relations for reciprocal sums of even powers of Fibonacci numbers

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Let $\{F_n\}_{n\geq 0}$ and $\{L_n\}_{n\geq 0}$ be Fibonacci numbers and Lucas numbers defined by

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n \geq 0),$$
$$L_0 = 2, \quad L_1 = 1, \quad L_{n+2} = L_{n+1} + L_n \quad (n \geq 0).$$

Duverney, Ke. Nishioka, Ku. Nishioka, and the last named author proved the transcendence of the numbers

$$\sum_{n=1}^{\infty} \frac{1}{F_n^{2s}}, \quad \sum_{n=1}^{\infty} \frac{1}{L_n^{2s}}, \quad \sum_{n=1}^{\infty} \frac{1}{F_{2n-1}^{s}}, \quad \sum_{n=1}^{\infty} \frac{1}{L_{2n}^{s}} \quad (s = 1, 2, \dots)$$

by using Nesterenko's theorem on Ramanujan functions $P(q)$, $Q(q)$, and $R(q)$.

We prove the algebraic independence of the numbers

$$\sum_{n=1}^{\infty} \frac{1}{F_n^{2}}, \quad \sum_{n=1}^{\infty} \frac{1}{F_n^{4}}, \quad \sum_{n=1}^{\infty} \frac{1}{F_n^{6}}$$

and write each

$$\sum_{n=1}^{\infty} \frac{1}{F_n^{2s}} \quad (s = 4,\ 5,\ 6,\ \dots)$$

as a rational function of these three numbers over $\mathbb{Q}$. Similar results are obtained for various series including the reciprocal sums of even powers of Lucas numbers.