Algebraic independence in the p-adic domain

Yu. V. Nesterenko, Moscow State University

Let p be a prime number and $\mathbb{C}_p$ be the completion of the algebraic closure of the field of p-adic numbers $\mathbb{Q}_p$. In 1932 and 1935 K. Mahler proved p - adic analogues of Lindemann theorem and 7-th Hilbert problem. Some generalizations of these results were proved in following articles of G. Veldkamp, A. Gunther, A.O. Gelfond and other mathematicians. W.W. Adams, T.N Shorey and P. Philippon stated some results about algebraic independence of the values of exponential function in the p - adic domain. These results are similar to theorems proved in the field $\mathbb{C}$. Nevertheless there are no p - adic analogues of Lindemann-Weierstrass theorem, Gelfond's theorem about algebraic independence of $\alpha^\beta, \alpha^{\beta^2}$ for algebraic $\alpha \neq 0, 1$ and cubic β. In the talk we will discuss some recent results in this area.

Theorem 1. *Let $a_1, \ldots, a_u$ and $b_1, \ldots, b_v$ belong to the field $\mathbb{C}_p$ and satisfy some technical conditions and inequalities $|a_i|_p < p^{-1/(p-1)}, \quad |b_j|_p \leq 1$. Then*

$$\text{tr deg}_{\mathbb{Q}}\, \mathbb{Q}\left(a_1, \ldots, a_u, b_1, \ldots, b_v, e^{a_1 b_1}, \ldots, e^{a_u b_v}\right) \geq \frac{uv}{u+v}.$$

Corollary 1. *Let $\alpha, \beta \in \mathbb{C}_p$ be algebraic numbers, $|\alpha - 1|_p < p^{-1/p}$ and $|\beta|_p \leq 1$, $\deg \beta = d \geq 2$. Then*

$$\text{tr deg}_{\mathbb{Q}}\, \mathbb{Q}\left(\log\alpha,\, \alpha^\beta,\, \ldots,\, \alpha^{\beta^{d-1}}\right) \geq \frac{d}{2}.$$

The next corollary is "half" of Lindemann-Weierstrass theorem.

Corollary 2. *Let $\beta_1, \ldots, \beta_d \in \mathbb{C}_p$ be algebraic numbers that form a basis of some finite extension of $\mathbb{Q}$ and satisfy inequalities $|\beta_j|_p < p^{-1/(p-1)}$. Then at least $d/2$ of numbers*

$$e^{\beta_1},\, e^{\beta_2},\, \ldots,\, e^{\beta_d}$$

are algebraically independent over $\mathbb{Q}$.

Corollary 3. *For any natural n the set*

$$e^p, e^{pe^p}, e^{pe^{2p}}, \ldots, e^{pe^{np}}$$

contains at least $\left[\frac{n+5}{4}\right]$ algebraically independent over $\mathbb{Q}$ numbers.

The following result has quantitative nature.

Theorem 2. *Let $\alpha \in \mathbb{C}_p$ be algebraic number satisfying $0 < |\alpha|_p < p^{-1/(p-1)}$. Then for any polynomial $A(x) \in \mathbb{Z}[x]$, $A \neq 0$, we have*

$$|A(e^\alpha)|_p \geq e^{-ct(A)^3(\ln t(A))^3}$$

with a positive constant c depending only on α.