Dedekind sums: a geometric viewpoint

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We define a *generalized Dedekind sum* as an expression of the form

$$\sum_{\lambda^a=1} \frac{\lambda^t}{(1-\lambda^{a_1})\cdots(1-\lambda^{a_n})} \, .$$

Here the sum is taken over all a'th roots of unity for which the summand is not singular. Sums of this type have intrigued mathematicians from various areas such as Number Theory, Topology, and Combinatorial Geometry since their introduction by Dedekind in 1892. Our definition, which is due to Gessel, includes as special cases the classical Dedekind sum (essentially the case $n = 2$, $t = 0$) and its generalizations due to Rademacher ($n = 2$, arbitrary t), and Zagier ($t = 0$, arbitrary n). Our interest in these sums stems from the appearance of Dedekind's and Zagier's sums in lattice point count formulas for polytopes. Through an interplay of complex integration and generating functions, we show that generalized Dedekind sums are natural ingredients for such formulas. As corollaries to our formulas, we recover and extend obtain "reciprocity theorems" of Dedekind, Zagier, and Gessel.