Some aspects of rigidity for self-affine tilings

Boris Solomyak

University of Washington

December 21, 2011, RIMS

What is rigidity?

- A structure is "rigid" if it doesn't allow a small perturbation
- This often means some algebraic constraints
- Another aspect: weak form of equivalence implies strong form of equivalence
- Substitution (self-affine) tilings exhibit both kinds of rigidity

Plan of the talk

1. Preliminaries
2. Which expansions are possible for self-affine tilings? (Thurston, Kenyon, Kenyon & S.)
3. When does the tiling have large discrete spectrum (diffraction or dynamical)? (J.-Y. Lee & S.)
4. Topological and dynamical rigidity (J. Kwapisz)

I. Preliminaries on tilings

- Prototile set: $\mathcal{A} = \{A_1, \ldots, A_N\}$, compact sets in $\mathbb{R}^d$, which are closures of its interior; interior is connected.
- May have "colors" or "labels"
- A tiling of $\mathbb{R}^d$ with the prototile set $\mathcal{A}$: collection of tiles whose union is $\mathbb{R}^d$ and interiors are disjoint. All tiles are translates of the prototiles.
- A **patch** is a finite set of tiles. $\mathcal{A}^+$ denotes the set of patches with tiles from $\mathcal{A}$.
- Assume **translational** finite patch complexity (or finite pattern condition) (FPC).

Tile-substitutions in $\mathbb{R}^d$

Let $\phi : \mathbb{R}^d \to \mathbb{R}^d$ be an expanding linear map, that is, all its eigenvalues are greater than 1 in modulus.

Definition. Let $\{A_1, \ldots, A_m\}$ be a finite prototile set. A **tile-substitution** with expansion ϕ is a map $\omega : \mathcal{A} \to \mathcal{A}^+$, where each $\omega(A_i)$ is a patch made of translates of A_j, such that

$$\mathrm{supp}(\omega(A_i)) = \phi(A_i), \ \ i \le m.$$

The substitution is extended to all translates of prototiles by $\omega(x + A_j) = \phi x + \omega(A_j)$, and to patches and tilings by

$$\omega(P) = \bigcup \{\omega(T) : \ \ T \in P\}.$$

Self-similar and self-affine tilings

- Substitution matrix: $S = (S_{i,j})_{i,j \le m}$, where $S(i,j) = \#$ tiles of type i in $\omega(A_j)$
- Tile-substitution is **primitive** if S is primitive, that is, some power of S has only positive entries (equivalently, $\exists\, k \in \mathbb{N}$, $\forall\, i \le m$, the patch $\omega^k(A_i)$ contains tiles of all types).
- We say that $\mathcal{T}$ is a **fixed point** of the tile-substitution ω if $\omega(\mathcal{T}) = \mathcal{T}$. Such tilings are called **self-affine**.
- If ϕ is a similarity map (i.e. $|\phi(x)| = r|x|$ for some $r > 1$ and all $x \in \mathbb{R}^d$), then $\mathcal{T}$ is called **self-similar**.

Self-similar tilings: special cases

- ϕ is a pure dilation: $\phi(x) = \lambda x$, for $\lambda > 1$, in $\mathbb{R}^d$
  λ is the **real expansion constant**

- $\phi(z) = \lambda z$ for $\lambda \in \mathbb{C}$, for $|\lambda| > 1$, in $\mathbb{C} \approx \mathbb{R}^2$
  λ is the **complex expansion constant**

Example: chair tiling

Examples are shamelessly taken from "Tiling Encyclopedia", see http://tilings.math.uni-bielefeld.de/

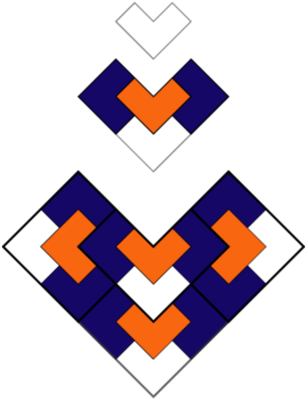

Figure: tile-substitution, real expansion constant $\lambda = 2$

Example: chair tiling

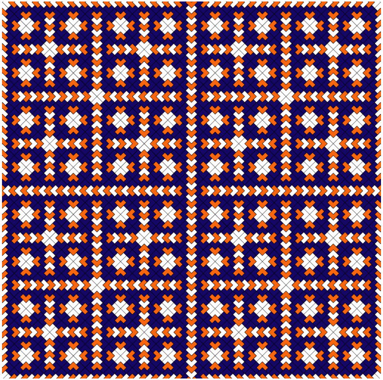

Figure: patch of the tiling

Example: non-FLC tiling (Kenyon)

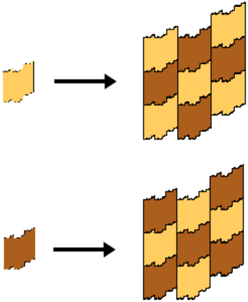

Figure: tile-substitution, real expansion constant $\lambda = 3$

Example: non-FLC tiling (Kenyon)

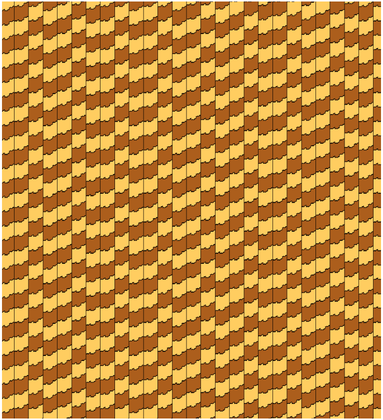

Figure: patch of the tiling

Example: non-FLC tiling (Kenyon)

To get the "self-similar" one, need to iterate and rescale...

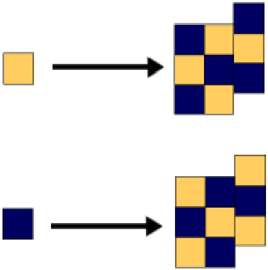

Figure: tile-substitution for the "pre-self-similar" tiling

Example: non-FLC tiling (Kenyon)

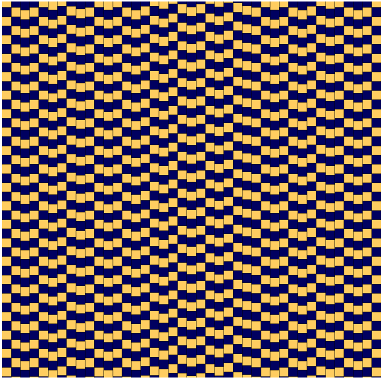

Figure: patch of the "pre-self-similar" tiling (note optical illusion!)

Example: self-similar tiling of the plane with fractal boundary

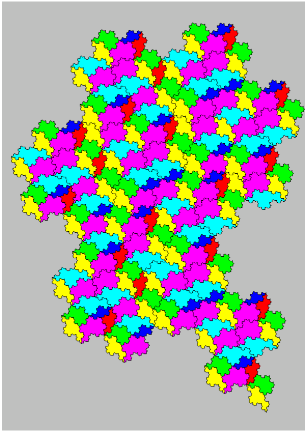

Figure: Self-similar tiling with complex expansion constant λ, $\lambda^4 + \lambda + 1 = 0$ (R. Kenyon)

Example: self-similar tiling of the plane with fractal boundary

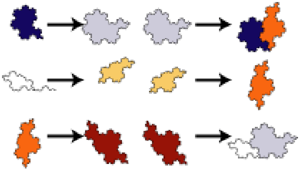

Figure: Substitution rule

Example: self-affine of the plane

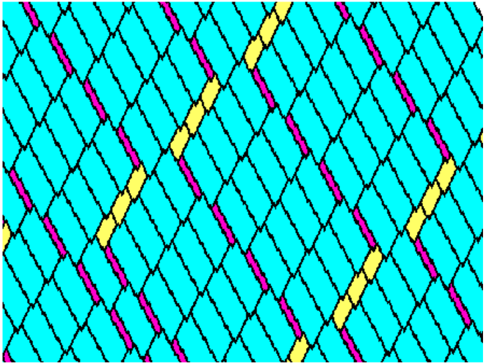

Figure: A self-affine tiling in the plane with diagonal expansion matrix $\mathrm{Diag}[x_1, x_2]$ where $x_1 \approx 2.19869$, $x_2 \approx -1.91223$ are roots of $x^3 - x^2 - 4x + 3 = 0$.

Example: self-affine of the plane (cont.)

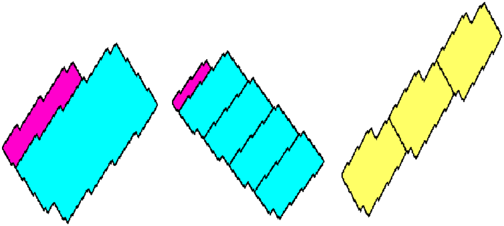

Figure: Subdivision rule: $1 \to \{3, 2\}$, $2 \to \{3, 2, 2, 2, 2\}$, $3 \to \{1, 1, 1\}$.
Construction uses free group $F(a, b, c)$, with a, b, c corresponding to vectors $(1, 1), (x_1 - 1, x_2 - 1), (x_1^2 - x_1, x_2^2 - x_2)$ in $\mathbb{R}^2$, the endomorphsm $\psi(a) = ab, \psi(b) = c, \psi(c) = ab^4$ and tiles $[b, a], [b, c], [a, c]$.

(Some) constructions of tilings with fractal boundary

- Free group endomorphisms, iterating and rescaling, substitutions of higher-dimensional faces: [Dekking 1982], [Ito & Kimura 1991], [Ito & Ohtsuki 1991, 1993], [Kenyon 1990, 1996], [Arnoux & Ito 2001], [Sano, Arnoux & Ito 2001], [Ei & Ito 2005], [Ito & Rao 2006], [Ei, Ito & Rao 2006], [Furukado 2006], [Furukado, Ito & Robinson 2006], ...
- Finite state automata and numeration systems with a complex Pisot base:
  [Thurston 1989], [Petronio 1994], [Akiyama & Sadahiro 1998], [Akiyama 2002], [Berthé & Siegel 2005], ...
- Connections with Markov partitions:
  Many of the same papers + [Praggastis 1992, 1999], [Kenyon & Vershik 1998], [Enomoto 2008], ...

Lemma. *For any primitive tile-substitution ω, there exists $n \in \mathbb{N}$ such ω^n has a fixed point.*

Definition A tiling $\mathcal{T}$ of $\mathbb{R}^d$ is repetitive if every patch $P \subset \mathcal{T}$ appears relatively dense in $\mathbb{R}^d$; more precisely, there exixts $R = R(P)$ such that every ball of radius R contains a translated copy of P.

Lemma. *Let $\mathcal{T}$ be an FLC fixed point of a tile-substitution with substitution matrix S.*
$\mathcal{T}$ is repetitive if and only if S is primitive and a $\mathcal{T}$-patch, containing the origin in the interior of its support, occurs in $\omega^n(T)$ for some $T \in \mathcal{T}$ and some $n \in \mathbb{N}$ (in particular, if the origin lies in the interior of a tile).

Definition A tiling $\mathcal{T}$ of $\mathbb{R}^d$ is aperiodic if

$$\mathcal{T} + x = \mathcal{T} \quad \Rightarrow \quad x = 0.$$

Lemma. *Let $\mathcal{T}$ be a self-affine tiling with expansion map ϕ whose rotational part is of infinite order (in $\mathbb{R}^2$ this means that the rotation is irrational modulo π). Then the tiles cannot be polyhedral, or even piecewise smooth.*

Lemma. *For any tile T of a self-affine tiling, $\mathrm{Vol}(\partial T) = 0$.*

Corollary. *The PF eigenvalue of the substitution matrix of a self-affine tiling with expansion map ϕ is $|\det \phi|$. The vector $(\mathrm{Vol}(A_j))_1^m$ is a left PF eigenvector.*

Corollary. $|\det \phi|$ *is a* **Perron number**, *i.e. an algebraic integer $\theta > 1$ whose Galois conjugates are strictly less than θ in modulus.*

Self-affine tilings and iterated function systems

There exist finite sets $\mathcal{D}_{ij} \subset \mathbb{R}^d$, $i, j \le m$:

$$\omega(A_j) = \{u + A_i : \; u \in \mathcal{D}_{ij}, \; i = 1, \ldots, m\}, \; j \le m,$$

with

$$\phi A_j = \bigcup_{i=1}^{m} (\mathcal{D}_{ij} + A_i), \; j \le m. \tag{1}$$

Here all the sets in the right-hand side must have disjoint interiors; it is possible for some of the $\mathcal{D}_{ij}$ to be empty. Note that $S(i,j) = \#\mathcal{D}_{ij}$.

Self-affine tilings and iterated function systems (cont.)

Rewrite the system of set equations (1):

$$A_j = \bigcup_{i=1}^{m} (\phi^{-1} A_i + \phi^{-1} \mathcal{D}_{ij}), \ j \le m.$$

ϕ^{-1} is a contraction, so there is always a unique nonempty compact solution $\{A_1, \ldots, A_m\}$ (attractor of a graph-directed IFS). The difficulty is to have A_j with nonempty interiors.

Theorem. *There is a self-similar tiling of the line* $\mathbb{R}$ *with expansion* λ *if and only if* $|\lambda|$ *is a Perron number.*

We already know the necessity. Sufficiency follows from Lind's Theorem: *every Perron number is the PF eigenvalue of a primitive matrix*.

Theorem [Lind '84] *If* $\lambda > 1$ *is a Perron number, then there is a primitive non-negative integral matrix* M *with the PF eigenvalue equal to* λ*. Moreover,* M *can be chosen so that* $M_{11} > 0$ *and* $\sum_{i=1}^{m} M_{i1} \geq 3$*.*

Characterization of expansions II

Theorem (?) [Thurston '89] *In $\mathbb{R}^2 \approx \mathbb{C}$, $\phi(z) = \lambda z$ with complex λ, is an expansion of a self-similar tiling* **if and only if** *λ is a complex Perron number, i.e. an algebraic integer of modulus > 1 whose Galois conjugates, except the complex conjugate, $\overline{\lambda}$, are strictly less than $|\lambda|$ in modulus.*

Sufficiency in the theorem above was claimed by Thurston and proved by Kenyon in his GAFA 1996 paper, but there still seems to be a gap... In this talk I will focus on the proof of **necessity**.

Remark. $|\lambda|^2$ Perron does not imply that λ is complex Perron. For example: $\lambda = -1 + i\sqrt{1+\sqrt{5}}$ has $|\lambda|^2 = 2 + \sqrt{5}$, which is Perron (even Pisot), but λ has conjugates $\overline{\lambda}$ and $-1 \pm \sqrt{-1+\sqrt{5}}$, one of which is ≈ -2.11179, whereas $|\lambda| \approx 2.05817$.

Theorem [Kenyon-S. 2010] *Let ϕ be an expanding linear map on $\mathbb{R}^d$, which is diagonalizable over $\mathbb{C}$, and suppose there exists a self-affine tiling of $\mathbb{R}^d$ with expansion ϕ. Let λ be an eigenvalue of ϕ, and let γ be a Galois conjugate of λ. Then either $|\gamma| < |\lambda|$, or γ is also an eigenvalue of ϕ, and its multiplicity is $\geq$ multiplicity of λ.*

- The case of non-diagonalizable maps (Jordan blocks) is open.
- Similarity maps are diagonalizable over $\mathbb{C}$, so this covers all self-similar tilings.

I will present a sketch of the proof, skipping (many!) technical details; it will be more complete in the self-similar case.

Control points

Definition. For each $\mathcal{T}$-tile T, fix a tile γT in the patch $\omega(T)$; choose γT with the same relative position for all tiles of the same type. This defines a map $\gamma : \mathcal{T} \to \mathcal{T}$ called the *tile map*. Then define the *control point* for a tile $T \in \mathcal{T}$ by

$$\{c(T)\} = \bigcap_{n=0}^{\infty} \phi^{-n}(\gamma^n T).$$

There are finitely many choices for control points, depending on γ.

Let $\mathcal{C} = \mathcal{C}(\mathcal{T}) = \{c(T) : \ T \in \mathcal{T}\}$.

Properties of control points

1. If $T' \approx T$ in $\mathcal{T}$, that is, $T' = T + x$, then $c(T') = c(T) + x$
2. $\phi(c(T)) = c(\gamma T)$, for $T \in \mathcal{T}$
3. $\phi(\mathcal{C}) \subset \mathcal{C}$

Proof of algebraicity of eigenvalues

Lemma. *Let $\mathcal{T}$ be a self-affine tiling of $\mathbb{R}^d$ with expansion ϕ. Then all the eigenvalues of ϕ are algebraic integers.*

- *Proof:* $J := \langle \mathcal{C} \rangle$, the subgroup of $\mathbb{R}^d$ generated by $\mathcal{C} = \mathcal{C}(\mathcal{T})$. It is a finitely generated abelian group (by FLC).

- By the Structure Theorem for Free Abelian Groups, there exist *free generators* $v_1, \ldots, v_N \in \mathbb{R}^d$, i.e.,

$$\forall\, \xi \in J,\ \exists\, !\, a_1, \ldots, a_N \in \mathbb{Z} :\ \xi = \sum_{j=1}^{N} a_j v_j.$$

- Let $V = [v_1 \ldots v_N]$, a $d \times N$ matrix, and $a(\xi) = [a_1, \ldots, a_N]^T \in \mathbb{Z}^N$. Then $\xi = Va(\xi)$.

Proof of algebraicity of eigenvalues (cont.)

- $\mathrm{rank}(V) = d$ since $\mathcal{C}$ spans $\mathbb{R}^d$, hence $\mathrm{Ker}(V^T) = \{0\}$.
- $\phi(\mathcal{C}) \subset \mathcal{C} \;\Rightarrow\; \phi J \subset J$, hence there exists an integer $N \times N$ matrix M such that

$$\phi V = VM.$$

- Every eigenvalue of ϕ is an eigenvalue of M: let $\mathbf{e}_\gamma$ be an eigenvector of ϕ^T:

$$\phi^T \mathbf{e}_\gamma = \gamma \mathbf{e}_\gamma \;\Rightarrow\; M^T V^T \mathbf{e}_\gamma = V^T \phi^T \mathbf{e}_\gamma = \gamma V^T \mathbf{e}_\gamma.$$

  Thus, $V^T \mathbf{e}_\gamma \neq 0$ is an eigenvector for M^T corresponding to γ, hence γ is an algebraic integer. $\square$

Address map

We call $\xi \mapsto a(\xi)$ the address map. Note that $a : J \to \mathbb{Z}^N$.

$$\begin{tikzcd}
\mathbb{Z}^N \arrow[r, "i"] & \mathbb{R}^N \arrow[r, "M"] \arrow[d, "V"'] & \mathbb{R}^N \arrow[d, "V"] & \mathbb{Z}^N \arrow[l, "i"'] \\
J \arrow[u, "a"'] \arrow[r, "i"] & \mathbb{R}^d \arrow[r, "\phi"] & \mathbb{R}^d & J \arrow[l, "i"'] \arrow[u, "a"']
\end{tikzcd}$$

Proof sketch of the Perron condition

Lemma. *The matrix M is diagonalizable over $\mathbb{C}$.*

Proof in the case of a self-similar tiling of the plane, with a complex expansion constant λ.

- $J = \langle \mathcal{C} \rangle$ is a finitely-generated $\mathbb{Z}$-module. Then $\mathbb{Q} \cdot J$ is a vector space over $\mathbb{Q}$, on which ϕ acts. Note that $\{y_1, \dots, y_N\}$ is a basis, and the matrix of ϕ in this basis is M.

- Note that $\mathbb{Q} \cdot J$ is also a vector space over $\mathbb{Q}(\lambda)$ (the field). Let $\{\zeta_1, \dots, \zeta_r\}$ be a basis of $\mathbb{Q} \cdot J$ over $\mathbb{Q}(\lambda)$ and let n be the degree of λ. Then
$$\{\lambda^s \zeta_k : \ 0 \le s \le n-1, \ 1 \le k \le r\}$$
is a basis for $\mathbb{Q} \cdot J$ over $\mathbb{Q}$.

Proof of the lemma (cont.)

- In this basis, $\phi =$ multiplication by λ has a matrix which is a direct sum of r copies of the companion matrix of λ.
- Each of them is diagonalizable over $\mathbb{C}$, since the minimal polynomial of λ has no repeated roots. Thus, the linear operator given by M is diagonalizable. $\square$

Remark. In fact, $r = 1$, which implies that $\mathcal{C} \subset \mathbb{Z}[\lambda]\zeta$ for some ζ. This is also a kind of "rigidity". This is a special case of Structure Theorem for control points [Kenyon 1994,1996], [J.-Y. Lee & S. 2012].

Proof sketch of the Perron condition (cont.)

- Suppose that γ is a conjugate of λ and $|\gamma| \geq |\lambda| > 1$. Then γ is an eigenvalue of M. We want to show that γ is an eigenvalue of ϕ.
- Let U_γ be the (real) eigenspace for M corresponding to γ. The only eigenvalues of $M_\gamma := M|_{U_\gamma}$ are γ and $\overline{\gamma}$ (if γ is nonreal).
- Since M is diagonalizable over $\mathbb{C}$, there is a projection π_γ from $\mathbb{R}^N$ to U_γ commuting with M.
- Consider the mapping $f_\gamma : \mathcal{C} \to U_\gamma$ given by

$$f_\gamma(\xi) = \pi_\gamma a(\xi), \quad \xi \in \mathcal{C}.$$

Proof sketch of the Perron condition (cont.)

We have a commutative diagram:

$$\begin{tikzcd}
J \arrow[r, "\phi"] \arrow[d, "a"'] & J \arrow[d, "a"] \\
\mathbb{Z}^N \arrow[r, "M"] \arrow[d, "\pi_\gamma"'] & \mathbb{Z}^N \arrow[d, "\pi_\gamma"] \\
U_\gamma \arrow[r, "M_\gamma"] & U_\gamma
\end{tikzcd}$$

Proof sketch of the Perron condition (cont.)

- Let
$$f_\gamma(\phi^{-k}\xi) = M^{-k} f_\gamma(\xi), \quad \xi \in \mathcal{C}.$$
This is well-defined.
- Now we have f_γ defined on a dense set
$$\mathcal{C}_\infty := \bigcup_{k=0}^{\infty} \phi^{-k}\mathcal{C}.$$

Proof sketch of the Perron condition (cont.)

Lemma *The map f_γ is uniformly continuous on $\mathcal{C}_\infty$, and thus it extends by continuity to $\mathbb{R}^d$, satisfying*

$$f_\gamma \circ \phi = M \circ f_\gamma.$$

Proof sketch. Step 1: the address map a, and hence f_γ is uniformly Lipschitz on $\mathcal{C}$ (large-scale):

$$\|a(\xi) - a(\xi')\| \le L_1 \|\xi - \xi'\|, \quad \xi, \xi' \in \mathcal{C}.$$

This is Thurston's argument: one can get "quasi-efficiently" from a control point to a distant control point by moving from neighbor to neighbor, and then use FLC.

Remark. The address map is usually not continuous on $J = \langle \mathcal{C} \rangle$, since J is usually dense, and the map is into $\mathbb{Z}^N$.

Proof sketch of the Perron condition (cont.)

Step 2: The map f_γ is Hölder continuous on $\mathcal{C}_\infty$: there exist $r > 0$ and $L_2 > 0$ such that for all $\xi, \xi' \in \mathcal{C}_\infty$ with $\|\xi - \xi'\| < r$,

$$\|f_\gamma(\xi) - f_\gamma(\xi')\| \le L_2 \|\xi - \xi'\|^\alpha,$$

where

$$\alpha = \frac{\log|\gamma|}{\log|\lambda_{\mathsf{max}}|}, \quad \lambda_{\mathsf{max}} = \text{largest eigenvalue of } \phi.$$

Note: if ϕ is a similarity map, then $|\lambda_{\mathsf{max}}| = |\lambda|$, and $|\gamma| \ge |\lambda|$ by assumption. We then have that $|\gamma| = |\lambda|$ (otherwise $f_\gamma \equiv \mathrm{const}$), and f_γ is Lipschitz on $\mathbb{R}^d$.

Proof sketch of the Perron condition (cont.)

Lemma. *The function f_γ (now defined on all $\mathbb{R}^d$) depends only on the tile type in $\mathcal{T}$ up to an additive constant: if $T, T+x \in \mathcal{T}$ and $\xi \in T$, then*

$$f_\gamma(\xi + x) = f_\gamma(\xi) + \pi_\gamma a(x).$$

Proof sketch. It is enough to check this on the dense set $\mathcal{C}_\infty$ and then it is a straightforward verification.

Proof sketch of the Perron condition (cont.)

Conclusion in the self-similar case

- We have
$$f_\gamma \circ \phi = M \circ f_\gamma \quad \text{on } \mathbb{R}^d.$$
We want to show that f_γ is linear.
- f_γ is Lipschitz $\Rightarrow$ it is is differentiable a.e.
- f_γ is "almost flat" on a small neighborhood of some $x \in \mathbb{R}^d$. Apply ϕ^k and notice that f_γ gets flatter and flatter near $\phi^k x$.
- Now find a patch in some fixed $B_R(0)$ where $\mathcal{T}$ has the same pattern as near $\phi^k x$ for all k (possible by repetitivity).

Proof sketch of the Perron condition (conclusion!)

- By the last lemma, there are points $x_k \in B_R(0)$ near which f_γ is the same as near $\phi^k x$ (up to an additive constant), hence it is almost flat there. By compactness, it must be exactly flat somewhere.
- Using the expansiveness of ϕ and conjugation again, conclude that f_γ is flat everywhere.
- $f_\gamma(0) = 0$ by construction $\Rightarrow f_\gamma$ is linear. It is a surjection onto U_γ, hence $M_\gamma = M|_{U_\gamma}$ is isomorphic, as a linear map, to a restriction of ϕ, therefore, γ is an eigenvalue of ϕ. $\square$

III. Discrete spectrum of self-affine tilings

- Diffraction spectrum and Dynamical spectrum
- Modern definition of a crystal: material with pure point diffraction
- Dworkin (1993) showed that diffraction spectrum is a “part” of dynamical spectrum, so we will be concerned with the latter.
- Question: when is there “large” discrete component of the spectrum?

Partial answer

Theorem [Jeong-Yup Lee & S., 2008, 2012] *Let $\mathcal{T}$ be self-affine with a diagonalizable over $\mathbb{C}$ expansion map ϕ. Suppose that all the eigenvalues of ϕ are algebraic conjugates with the same multiplicity. Then the following are equivalent:*
(i) *the set of eigenvalues of the tiling dynamical system associated with $\mathcal{T}$ (defined below) is relatively dense in $\mathbb{R}^d$;*
(ii) *the spectrum of ϕ is a Pisot family: for every eigenvalue λ of ϕ and its conjugate γ, either $|\gamma| < 1$, or γ is also an eigenvalue of ϕ;*
(iii) *the set of control points $\mathcal{C} = \mathcal{C}(\mathcal{T})$ is a Meyer set, i.e. $\mathcal{C} - \mathcal{C}$ is uniformly discrete.*

Remarks

1. (i) $\Rightarrow$ (iii) is proved in [J.-Y. Lee & S. 2008], whereas (iii) $\Rightarrow$ (i) follows from [Strungaru 2005] and [Dworkin 1993].
2. (i) $\Rightarrow$ (ii) was proved by [E. A. Robinson 2004], using the criterion for eigenvalues in [S. 1997].
3. (ii) $\Rightarrow$ (i), the most technically difficult part, is proved in [J.-Y. Lee & S. 2012].
4. Examples show that the condition of having conjugates with the same multiplicity cannot be omitted, but it is an open question how to handle non-diagonalizable cases, or cases when not all eigenvalues of ϕ are conjugates.

We will sketch the proof of (i) $\Leftrightarrow$ (ii), but first we need to define the tiling dynamical system.

Tiling space

Tiling space, or hull, generated by $\mathcal{T}$:

$$X_{\mathcal{T}} = \overline{\{-g + \mathcal{T} : g \in \mathbb{R}^d\}},$$

where the closure is in the "local" topology: two tilings are close if after a small translation they agree on a large ball around the origin.

More precisely:

$$\begin{aligned} \widetilde{\varrho}(\mathcal{T}_1, \mathcal{T}_2) \quad := \quad & \inf\{r \in (0, 2^{-1/2}) : \ \exists\, g \in B_r : \\ & \mathcal{T}_1 - g \text{ and } \mathcal{T}_2 \text{ agree on } B_{1/r}\}. \end{aligned}$$

Then $\varrho(\mathcal{T}_1, \mathcal{T}_2) := \min\{2^{-1/2}, \varrho(\mathcal{T}_1, \mathcal{T}_2)\}$ is a metric.

Tiling dynamical system

Theorem. *FLC* $\iff X_{\mathcal{T}}$ *is compact.*

$\mathbb{R}^d$ acts by translations: $T^{\mathbf{t}}(\mathcal{S}) = \mathcal{S} - \mathbf{t}$. Topological dynamical system (action of $\mathbb{R}^d$ by homeomorphisms):

$$(X_{\mathcal{T}}, T^{\mathbf{t}})_{\mathbf{t}\in\mathbb{R}^d} = (X_{\mathcal{T}}, \mathbb{R}^d)$$

Definition. A topological dynamical system is *minimal* if every orbit is dense (equivalently, if it has no nontrivial closed invariant subsets).

Theorem. $\mathcal{T}$ *is repetitive* $\iff (X_{\mathcal{T}}, \mathbb{R}^d)$ *is minimal.*

Uniform patch frequencies

For a patch $P \subset \mathcal{T}$ let $L_P(\mathcal{T}, A) :=$

$$\#\{\mathbf{t} \in \mathbb{R}^d : \; -\mathbf{t} + P \subset \mathcal{T}, \; -\mathbf{t} + \mathrm{supp}(P) \subset A\},$$

the number of $\mathcal{T}$-patches equivalent to P that are contained in A.

Definition. A tiling $\mathcal{T}$ has *uniform patch frequencies* (UPF) if for any non-empty patch P, the limit

$$\mathrm{freq}(P, \mathcal{T}) := \lim_{r \to \infty} \frac{L_P(\mathcal{T}, \mathbf{t} + Q_r)}{r^d} \geq 0$$

exists uniformly in $\mathbf{t} \in \mathbb{R}^d$. Here $Q_r = [-\frac{r}{2}, \frac{r}{2}]^d$.

Unique ergodicity for tiling systems

Theorem. *Let $\mathcal{T}$ be a tiling with FLC. Then the dynamical system $(X_{\mathcal{T}}, \mathbb{R}^d)$ is uniquely ergodic, i.e. has a unique invariant probability measure, if and only if $\mathcal{T}$ has UPF.*

Theorem. *Let $\mathcal{T}$ be a self-affine tiling. Then the dynamical system $(X_{\mathcal{T}}, \mathbb{R}^d)$ is uniquely ergodic.*

Denote by μ the unique invariant measure.

Eigenvalues and Eigenfunctions

Definition. $\alpha \in \mathbb{R}^d$ is an eigenvalue for the measure-preserving $\mathbb{R}^d$-action $(X, T^{\mathbf{t}}, \mu)_{\mathbf{t}\in\mathbb{R}^d}$ if $\exists$ eigenfunction $f_\alpha \in L^2(X, \mu)$, i.e., f_α is not 0 in L^2 and for μ-a.e. $x \in X$

$$f_\alpha(T^{\mathbf{t}}x) = e^{2\pi i \langle \mathbf{t}, \alpha\rangle} f_\alpha(x),\ \mathbf{t} \in \mathbb{R}^d.$$

Here $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^d$.

Warning: eigenvalue is a vector! (like "wave vector" in physics)

Theorem. *If $\mathcal{T}$ is a self-affine tiling, then every measurable eigenfunction for the system $(X_{\mathcal{T}}, \mathbb{R}^d, \mu)$ coincides with a continuous function μ-a.e.*

Return vectors for the tiling:

$$\mathcal{Z}(\mathcal{T}) := \{z \in \mathbb{R}^d : \ \exists\, T, T' \in \mathcal{T},\ T' = T + z\}.$$

Theorem [S. 1997] *Let $\mathcal{T}$ be an aperiodic self-affine tiling with expansion map ϕ. Then the following are equivalent for $\alpha \in \mathbb{R}^d$:*
(i) *α is an eigenvalue for the topological dynamical system $(X_{\mathcal{T}}, \mathbb{R}^d)$;*
(ii) *α is an eigenvalue for the measure-preserving system $(X_{\mathcal{T}}, \mathbb{R}^d, \mu)$;*
(iii) *α satisfies the condition:*

$$\lim_{n\to\infty} e^{2\pi i \langle \phi^n z, \alpha\rangle} = 1 \ \ \text{for all } z \in \mathcal{Z}(\mathcal{T}). \tag{2}$$

Proof of $(i) \Rightarrow (iii)$

- Let $z \in \mathcal{Z}(\mathcal{T})$, i.e., for some $T \in \mathcal{T}$ we have $T + z \in \mathcal{T}$. Let ξ be any point in the interior of T. Then $\mathcal{T} - \xi$ and $\mathcal{T} - z - \xi$ agree on some B_ε.

- Applying ω^n we obtain that

$$\omega^n(\mathcal{T} - \xi) = \mathcal{T} - \phi^n \xi$$

and

$$\omega^n(\mathcal{T} - z - \xi) = \mathcal{T} - \phi^n z - \phi^n \xi$$

agree on $\phi^n B_\varepsilon$.

- By the definition of tiling metric,

$$\varrho(\mathcal{T} - \phi^n \xi, \mathcal{T} - \phi^n z - \phi^n \xi) \to 0, \quad \text{as } n \to \infty.$$

Proof (cont.)

- A continuous function on a compact metric space is uniformly continuous, hence

$$|f(\mathcal{T} - \phi^n \xi) - f(\mathcal{T} - \phi^n z - \phi^n \xi)| \to 0, \quad \text{as } n \to \infty.$$

- Using the eigenfunction equation we obtain

$$\left|1 - e^{2\pi i \langle \phi^n z, \alpha \rangle}\right| \to 0, \ \ n \to \infty,$$

proving (2). $\square$

Relatively dense set of eigenvalues implies Pisot family

This uses a generalization of Pisot Theorem, due to I. Körneyi (1986) (a similar result was independently obtained by C. Mauduit (1989)).

Theorem [Körneyi] (partial statement in a special case). *Let* $\lambda_1, \ldots, \lambda_r$ *be distinct complex algebraic integers, with* $|\lambda_j| \geq 1$*, and there exist nonzero* α_j*,* $j \leq r$*, such that*

$$\mathrm{dist}\left(\sum_{j=1}^{r} \alpha_j \lambda_j^n, \mathbb{Z}\right) \to 0, \quad n \to \infty.$$

Then $\{\lambda_1, \ldots, \lambda_r\}$ *is a Pisot family and* $\alpha_j = \frac{p(\lambda_j)}{q(\lambda_j)}$ *for some* $p, q \in \mathbb{Z}[x]$.

Relatively dense set of eigenvalues implies Pisot family (cont.)

- From (2) we obtain

$$\mathrm{dist}(\langle \phi^n z, \alpha \rangle, \mathbb{Z}) \to 0, \quad n \to \infty$$

for a return vector $z \in \mathbb{R}^d$ and an eigenvalue $\alpha \in \mathbb{R}^d$.

- We assumed ϕ is diagonalizable over $\mathbb{C}$, hence there is a basis $\{\mathbf{e}_i\}$ of eigenvectors (need to be a bit careful with complex eigenvalues).
- The set of return vectors is relatively dense, hence we can choose z so that all of its coordinates w.r.t. to $\{\mathbf{e}_i\}$ are nonzero. And we can make sure that $\langle \mathbf{e}_i, \alpha \rangle \neq 0$ for all i.
- Now application of Körneyi's Theorem yields the result. $\square$

Pisot family implies relatively dense set of eigenvalues

To outline the proof, assume for simplicity that

1. ϕ has eigenvalues of multiplicity one (hence the characteristic polynomial is irreducible);
2. all these eigenvalues are real;
3. the tiling $\mathcal{T}$ is aperiodic.

Without loss of generality, we can assume that

$$\phi = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_d \end{bmatrix},$$

and $|\lambda_1| \le |\lambda_2| \le \ldots \le |\lambda_d|$.

Pisot family implies relatively dense set of eigenvalues (cont.)

Let

$$\mathbb{Q}[\phi] := \{p(\phi) : \ p \in \mathbb{Q}[x]\}, \quad \mathbb{Z}[\phi] := \{p(\phi) : \ p \in \mathbb{Z}[x]\}.$$

The crucial step is to determine the structure of the control point set $\mathcal{C} = \mathcal{C}(\mathcal{T})$.

Structure Theorem *Let $\mathcal{T}$ be a self-affine tiling of $\mathbb{R}^d$ with a diagonalizable expansion map ϕ whose eigenvalues are all algebraically conjugate. Suppose that the additional assumptions (1)-(3) are satisfied. Then there exists a vector $\boldsymbol{\alpha} \in \mathbb{R}^d$ such that*

$$\mathcal{C} \subset \mathbb{Z}[\phi]\boldsymbol{\alpha}.$$

Pisot family implies relatively dense set of eigenvalues (cont.)

First we use the Structure Theorem to finish the proof.

- Observe that the set of return vectors satisfies $\mathcal{Z} \subset \mathcal{C} - \mathcal{C} \subset \mathbb{Z}[\phi]\boldsymbol{\alpha}$
- The set of control points $\mathcal{C}$ is relatively dense, hence the vector
$$\boldsymbol{\alpha} := [a_1, \ldots, a_d]^T \quad \text{has all } a_j \neq 0$$
- Consider
$$\boldsymbol{\beta} := [a_1^{-1}, \ldots, a_d^{-1}]^T.$$
We claim that the set $\{\phi^j \boldsymbol{\beta}\}_{j=0}^{d-1}$ is contained in the set of eigenvalues. This set is a basis of $\mathbb{R}^d$ (over $\mathbb{R}$), and since the set of eigenvalues forms an additive group, the proof will be complete.

Pisot family implies relatively dense set of eigenvalues (cont.)

We have for $\mathbf{x} = \phi^i \boldsymbol{\alpha}$, $\boldsymbol{\gamma} = \phi^j \boldsymbol{\beta}$:

$$\langle \phi^n \mathbf{x}, \boldsymbol{\gamma} \rangle = \langle \phi^{n+i} \boldsymbol{\alpha}, \phi^j \boldsymbol{\beta} \rangle = \sum_{k=1}^{d} \lambda_k^{n+i+j} \to 0 \pmod{\mathbb{Z}}.$$

The convergence follows from the Pisot family property. $\square$

Proof sketch of Structure Theorem

- Consider $\langle \mathcal{C} \rangle_{\mathbb{Q}}$, the linear space over $\mathbb{Q}$ generated by the set of control points $\mathcal{C}$. The FLC property implies that this space is finite-dimensional over $\mathbb{Q}$.

- An easy argument shows that $\mathbb{Q}[\phi]$ is a field. Since $\phi(\mathcal{C}) \subset \mathcal{C}$, it follows that
$$\mathcal{H} := \langle \mathcal{C} \rangle_{\mathbb{Q}} = \langle \mathcal{C} \rangle_{\mathbb{Q}[\phi]}.$$
Our result will follow if we show that $\mathcal{H}$ has dimension one as a linear space over $\mathbb{Q}[\phi]$.

- Choose any control point $\boldsymbol{\xi} \in \mathcal{C}$ with all non-zero coordinates. We can define $\sigma : \mathcal{H} \to \mathbb{Q}[\phi]\boldsymbol{\xi}$ as a $\mathbb{Q}[\phi]$-module homomorphism which is identical on $\mathbb{Q}[\phi]\boldsymbol{\xi}$ (basically, a projection commuting with ϕ), and then let σ' be the restriction of σ to the set $\mathcal{C}_\infty = \bigcup_{k=0}^{\infty} \phi^{-k}\mathcal{C}$.

Proof sketch of Structure Theorem (cont.)

- The function σ' is uniformly continuous on the dense set $\mathcal{C}_\infty$, and so can be extended by continuity to $\mathbb{R}^d$. The extension commutes with ϕ. This proceeds essentially following Thurston's and Kenyon's arguments (see part II).
- Next we show that the extension of σ' to $\mathbb{R}^d$, which we also denote σ', is linear over $\mathbb{R}$. This will be sufficient, since σ' is the identity on a relatively dense set, hence it is the identity on all of $\mathbb{R}^d$ whence $\mathcal{C} \subset \mathbb{Q}[\phi]\boldsymbol{\xi}$, as desired.
- Establishing linearity is the hardest part of the proof; it follows the scheme worked out in [Kenyon & S. 2010].

Proof sketch of Structure Theorem (end)

- σ' is Lipschitz on all lines parallel to the eigenvector $\mathbf{e}_1$ of ϕ with the smallest in modulus eigenvalue λ_1.
- By Rademacher's Theorem, this implies that σ' is differentiable almost everywhere in the $\mathbf{e}_1$ direction.
- Another useful property of σ' is that it depends only on the tile type, up to an additive constant.
- Taking points of differentiability, "blowing up" by the expansion ϕ and using the last item we prove that σ' is affine linear on all lines parallel to $\mathbf{e}_1$.
- Projections of $\mathcal{C}$ to the coordinate axes cannot be discrete; this yields linearity on the entire $\mathbb{R}^d$. □

Remark. The last "non-discreteness" claim may seem surprising: why can't we have the tiling $\mathcal{T}$ as a direct product of, say, two tilings corresponding to a partition of the set of eigenvalues of ϕ? The answer is that this is prohibited by the Characterization of Expansions III from [Kenyon & S. 2010]: it is impossible to split the set of eigenvalues of ϕ in such a way that both parts form a Perron family.

IV. Topological rigidity (after J. Kwapisz)

It turns out that the topology of the tiling space already determines dynamics!

Theorem [J. Kwapisz, ETDS, published online 2011] *Suppose that $\mathcal{T}_0$ and $\widetilde{\mathcal{T}_0}$ are self-similar aperiodic tilings of $\mathbb{R}^d$, with expansions ϕ and $\widetilde{\phi}$, repsectively. Let $X = X_{\mathcal{T}_0}$ and $\widetilde{X} = X_{\widetilde{\mathcal{T}_0}}$ be the corresponding tiling spaces. If there is a homeomorphism $h_0 : X \to \widetilde{X}$, then there is a linear isomorphism $A : \mathbb{R}^d \to \mathbb{R}^d$ and a homeomorphism $h : X \to \widetilde{X}$ conjugating the translation action on X to the rescaling of the translation action on $\widetilde{X}$:*

$$h(\mathcal{T} - g) = h(\mathcal{T}) - Ag, \ \mathcal{T} \in X, \ g \in \mathbb{R}^d.$$

Open Question: is the same true for self-affine tilings?

Global linearization

Theorem [J. Kwapisz] *In the context of the last theorem, if additionally* $h_0(\mathcal{T}_0) = \widetilde{\mathcal{T}}_0$, *then* h_0 *is homotopic to some homeomorphism* $h_{\rm lin} : X \to \widetilde{X}$ *such that* $h_{\rm lin}(\mathcal{T}_0) = \widetilde{\mathcal{T}}_0$ *and* $h_{\rm lin}$ *is linear, i.e. for some linear map* L,

$$h_{\rm lin}(\mathcal{T} - g) = h_{\rm lin}(\mathcal{T}) - Lg, \ \ g \in \mathbb{R}^d.$$

Rough sketch of the proof

- Aperiodic tiling spaces are locally: $\mathbb{R}^d \times$ Cantor set.
- A homeomorphism $h_0 : X \to \widetilde{X}$ has to take orbits to orbits:

$$h_0(\mathcal{T} - g) = h_0(\mathcal{T}) - \alpha(\mathcal{T}, g),$$

where $\alpha(\mathcal{T}, g)$ is a cocycle over the $\mathbb{R}^d$-action on X:

$$\alpha(\mathcal{T}, g_1 + g_2) = \alpha(\mathcal{T}, g_1) + \alpha(\mathcal{T} - g_1, g_2).$$

Step 1: Averaging of cocycles

Idea: cocycles are linear on large scales.

Lemma *There exists a linear isomorphism* $A_\alpha : \mathbb{R}^d \to \mathbb{R}^d$ *such that*

$$\lim_{s\to\infty} \frac{\alpha(\mathcal{T}, sv)}{s} = A_\alpha v, \ \forall \mathcal{T} \in X, \ v \in \mathbb{R}^d.$$

The limit is uniform in the sense that

$$\lim_{s\to\infty} \sup_{|v|=1,\, \mathcal{T}\in X} \left| \frac{\alpha(\mathcal{T}, sv)}{s} - A_\alpha v \right| = 0.$$

Step 2: Using that $\phi, \widetilde{\phi}$ are similarities (conformality)

Technical lemma. *For any linear isomorphism A_α, if each of $\phi, \widetilde{\phi}$ is similar to an orthogonal transformation of $\mathbb{R}^d$, then there are sequences $m_k, n_k \to \infty$ of integers and a linear isomorphism A such that*

$$\|\widetilde{\phi}^{-n_k}\phi^{m_k} - I\| \to 0,$$

$$\|\widetilde{\phi}^{-n_k}A_\alpha\phi^{m_k} - A\| \to 0,$$

$$\sup_k \|\phi^{m_k}\|\|\widetilde{\phi}^{-n_k}\| < \infty \quad \textit{and} \quad \sup_k \|\phi^{-m_k}\|\|\widetilde{\phi}^{n_k}\| < \infty.$$

This follows from the compactness of the orthogonal group.

Step 3: "Ironing" homeomorphisms to conjugacies

- Aperiodicity of the tiling $\mathcal{T}_0$ implies that the substitution ("inflate and subdivide") action $\omega : X \to X$ is invertible ("recognizability"); it is hyperbolic in the Smale space sense; similarly for $\widetilde{\omega} : \widetilde{X} \to \widetilde{X}$.
- The next idea goes back to linearization results in hyperbolic dynamics (compare with Thurston's argument!)
- Step 1 (averaging) tells us that h_0 is approximately linear on a large scale, and we attempt to bring this linearity to the microscopic scale by renormalizing h_0 with the aid of high iterates ω^{m_k} and $\widetilde{\omega}^{-n_k}$.

Lemma. *The family of homeomorphisms*

$$h_k := \widetilde{\omega}^{-n_k} \circ h_0 \circ \omega^{m_k} \qquad \text{is equicontinuous.}$$

Conclusion of proof sketch

- Interchanging the roles of ω and $\widetilde{\omega}$ we see that h_k^{-1} is also an equicontinuous family.
- By passing to a subsequence we can ensure that h_k and h_k^{-1} converge uniformly. Thus they must converge to h and h^{-1} respectively, where h is a homeomorphism.
- Then one shows that

$$h(\mathcal{T} - g) = h(\mathcal{T}) - Ag,\ \mathcal{T} \in X,\ g \in \mathbb{R}^d.$$

where A is from the Technical Lemma. $\square$