Dual systems of algebraic iterated function systems

Hui RAO

RIMS, Kyoto University

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§1. Algebraic IFS

Graph-directed Iterated function system

Let $G = (V, \Gamma)$ be a directed graph with vertex set $V = \{1, \cdots, N\}$ and directed-edge set Γ.

Definition (Graph IFS)

A graph-directed IFS is a sequence of of contractions

$$(f_\gamma)_{\gamma\in\Gamma} \tag{1}$$

where

$$f_\gamma : \mathbb{R}^d \mapsto \mathbb{R}^d.$$

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Invariant sets

Γ_{ij}: edges from vertex i to j.

$(E_1, \ldots, E_N)$ non-empty compact sets satisfying

$$E_i = \bigcup_{j=1}^{N} \bigcup_{\gamma \in \Gamma_{ij}} f_\gamma(E_j), \quad 1 \leq i \leq N. \tag{2}$$

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A separation condition

Graph IFS (1) is said to satisfy the open set condition (OSC), if there exist open sets $U_1, \dots, U_N$ such that

$$\bigcup_{j=1}^{N} \bigcup_{\gamma\in\Gamma_{ij}} f_\gamma(U_j) \subset U_i, \quad 1 \le i \le N,$$

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IFS with algebraic parameters

Definition (Algebraic IFS)

Graph IFS (1) is called an algebraic graph IFS, if

- $f_\gamma : \mathbb{R} \to \mathbb{R}$ *has the form*

$$f_\gamma(x) = \frac{x + b_\gamma}{\beta} \tag{3}$$

- *Expanding factor:* $\beta > 1$ *is a real algebraic number.*
- $b_\gamma \in \mathbb{Q}(\beta)$.

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IFS with algebraic parameters

§2. Background and motivation

Background

The tiling systems arising from

- Pisot substitutions: initialed by Rauzy in 1982
- β-numeration systems: initialed by Thurston in 1989

Tiling theory, Substitution dynamical system,
Markov partition, Number theory,
Spectral theory, Fractal geometry
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Studies of substitutions:

- Irreducible Pisot case
- reducible Pisot case
- non-Pisot case

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Kalle and Steiner: Trans. AMS. 2012 studies tilings arising from

- symmetric β-expansion: Akiyama and Scheicher 2007.
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§3. Dual system of algebraic IFS

Reverse graph

Let $G = (V, \Gamma)$ be a directed graph.

Reverse graph $G' = (V, \Gamma')$ of G is obtained by reverse the direction of every edge in G.

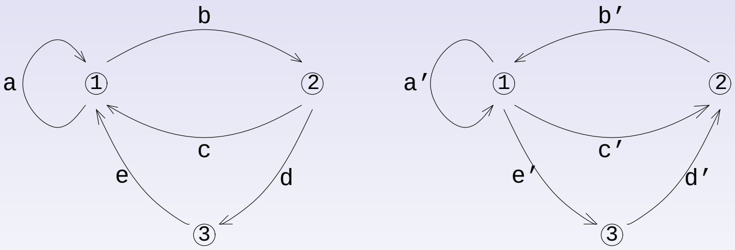

Figure: 1. Left: Graph of Rauzy substitution. Right: Reverse graph.

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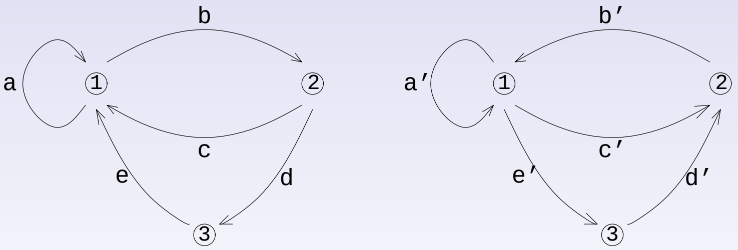

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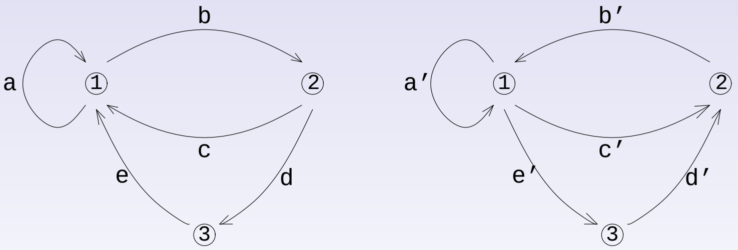

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Dual in field $\mathbb{Q}(\beta)$

Algebraic conjugates of β: $1 > |\beta_2| \geq \cdots |\beta_d|$.

For

$$x = \sum_{j=0}^{d-1} x_j \beta^j \in \mathbb{Q}(\beta),$$

define

$$x^* = \sum_{j=0}^{d-1} x_j (\beta_2^j, \cdots, \beta_d^j).$$

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Dual map

Dual contraction matrix:

$$B^* = \begin{pmatrix} \beta_2 & & & \\ & \beta_3 & & \\ & & \ddots & \\ & & & \beta_d \end{pmatrix}.$$

Definition (Dual map)

The dual map of $f(x) = \frac{x+b}{\beta}$ *is defined to be* $f^* : \mathbb{R}^s \to \mathbb{R}^s$ *as*

$$f^*(y) = B^* y + b^*. \tag{4}$$

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Dual IFS

§4. An Example

A system related to Rauzy substitution

Let $\beta : \beta^3 = \beta^2 + \beta + 1$.

The following graph IFS is arising from Rauzy substitution

$$\begin{cases} \beta I_1 = I_1 \cup (I_2 + 1) \\ \beta I_2 = I_1 \cup (I_3 + 1) \\ \beta I_3 = I_1. \end{cases}$$

The invariant sets are

$$I_1 = [0, 1],\ I_2 = [0, 1/\beta + 1/\beta^2],\ I_3 = [0, 1/\beta].$$

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The maps corresponding to the edges of Γ are

$$f_a(x) = \frac{x}{\beta}, \quad f_b(x) = \frac{x+1}{\beta}, \quad f_c(x) = \frac{x}{\beta},$$
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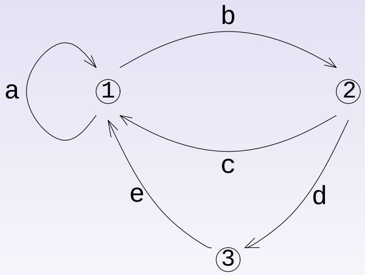

Figure: 2. Graph of original IFS.

A system related to Rauzy substitution

Hui RAO

1. Algebraic IFS

2. Background and motivation

3. Dual system of algebraic IFS

4. An Example

5. Algebraic IFS arising from substitutions

6. Algebraic IFS arising from numeration systems

Separation condition: a fundamental result

Rauzy-Thurston tiling

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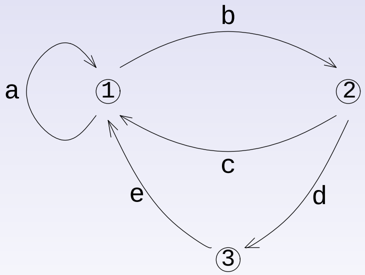

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Dual parameters

- Conjugates: $\beta_2 \approx 0.419 + 0.606i$, $\beta_3 \approx 0.419 - 0.606i$.
- Dual contraction:

$$B^* = \begin{pmatrix} 0.419 + 0.606i & 0 \\ 0 & 0.419 - 0.606i \end{pmatrix},$$

or equivalently,

$$B^* = \begin{pmatrix} 0.419 & -0.606 \\ 0.606 & 0.419 \end{pmatrix}.$$

- Dual translations: The dual of 1 is the vector $\mathbf{e}_1$.

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The dual graph IFS is:

$$g_{a'}(y) = B^*y, \quad g_{b'}(y) = B^*y + \mathbf{e}_1, \quad g_{c'}(y) = B^*y,$$
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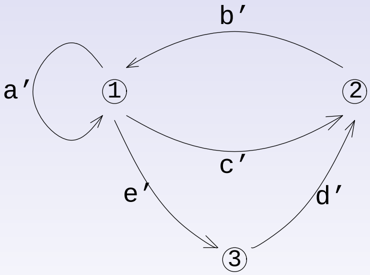

Figure: 3. Graph of dual IFS.

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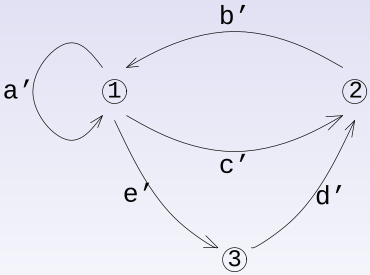

Figure: 3. Graph of dual IFS.

Dual parameters

An equivalent expression of the dual graph IFS is:

$$\begin{cases} (B^*)^{-1}X_1 = X_1 \cup X_2 \cup X_3 \\ (B^*)^{-1}X_2 = X_1 + (B^*)^{-1}\mathbf{e}_1 \\ (B^*)^{-1}X_3 = X_2 + (B^*)^{-1}\mathbf{e}_1. \end{cases} \tag{6}$$

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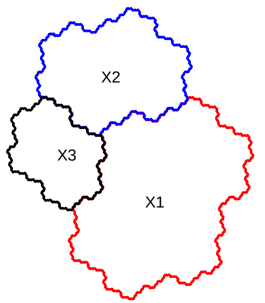

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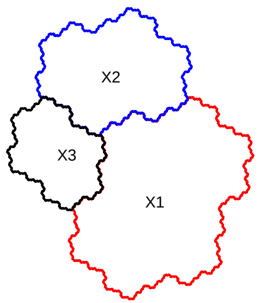

Figure: 4. Invariant sets

One-dimensional IFS arising from substitutions

5. Algebraic IFS arising from substitutions

Simple.

One-dimensional IFS arising from substitutions

5. Algebraic IFS arising from substitutions

Simple.

6. Algebraic IFS arising from numeration systems

Piecewise linear dynamical systems with uniform expanding ratio

- A numeration system is typically a piecewise linear map T from an interval I to itself.
- Markov condition: the orbits of the discontinuous points of T are all eventually periodic.
- Basic interval: Discontinuous points and their obits cut I into small intervals

$$I = I_1 \cup I_2 \cup \cdots \cup I_m.$$

- Each $T(I_j)$ is a union of basic intervals, and this gives us a one-dimensional graph-directed IFS.

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Symmetric β-expansions

In Akiyama and Scheicher (2007) introduced symmetric β-transformation for $1 < \beta < 3$ as

$$S_\beta : [-1/2, 1/2] \to [-1/2, 1/2],$$

$$S_\beta(x) = \beta x - \lfloor \beta x + \frac{1}{2} \rfloor. \tag{7}$$

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Symmetric β-expansions

Example: β is same as in Rauzy substitution.
The discontinues orbits of the transformation cut the interval into 7 basic intervals:

$$I_1, I_2, I_3, I_0, J_3, J_2, J_1$$

from left to right.

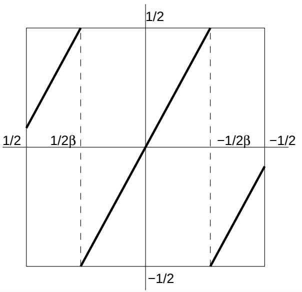

Figure: 5. Symmetric β-transformation

Symmetric β-expansions

The set equations are:

$$\beta I_0 = I_3 \cup I_0 \cup J_3$$

and $I_1, I_2, I_3, J_1, J_2, J_3$ satisfying the set equations

$$\begin{cases} \beta I_1 = (J_3 \cup J_2 \cup J_1) - 1 \\ \beta I_2 = I_1 \\ \beta I_3 = I_2 \\ \beta J_1 = (I_1 \cup I_2 \cup I_3) + 1 \\ \beta J_2 = J_1 \\ \beta J_3 = J_2. \end{cases} \tag{8}$$

We omit I_0 since it is a non-primitive member and its dual is a single point.

Symmetric β-expansions

The six invariant sets of the dual IFS are

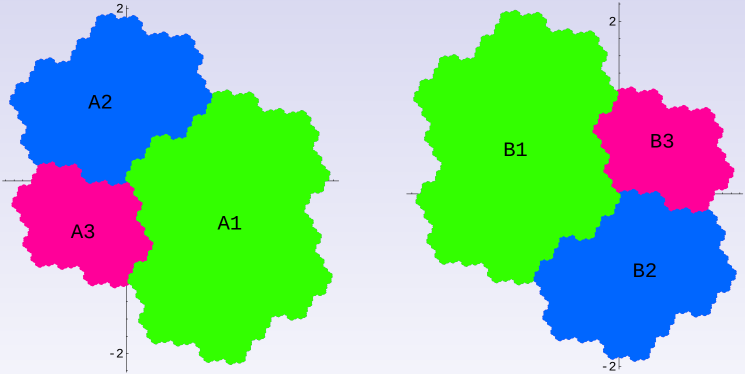

Figure: 6. Tiles of symmetric β-transformation.

Minimal weight expansion

The minimal weight expansion is introduced by Frougny and Steiner (2008).

W_β is defined on the interval $I = [-\frac{\beta}{\beta+1}, \frac{\beta}{\beta+1}]$ as

$$W_\beta(x) = \begin{cases} \beta x + 1, & \text{if } x \in [-\frac{\beta}{\beta+1}, -\frac{1}{\beta+1}] \\ \beta x, & \text{if } x \in [-\frac{1}{\beta+1}, \frac{1}{\beta+1}] \\ \beta x - 1, & \text{if } x \in [\frac{1}{\beta+1}, -\frac{\beta}{\beta+1}]. \end{cases}$$

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Hui RAO

The minimal weight expansion is introduced by Frougny and Steiner (2008).
W_β is defined on the interval $I = [-\frac{\beta}{\beta+1}, \frac{\beta}{\beta+1}]$ as

$$W_\beta(x) = \begin{cases} \beta x + 1, & \text{if } x \in [-\frac{\beta}{\beta+1}, -\frac{1}{\beta+1}] \\ \beta x, & \text{if } x \in [-\frac{1}{\beta+1}, \frac{1}{\beta+1}] \\ \beta x - 1, & \text{if } x \in [\frac{1}{\beta+1}, -\frac{\beta}{\beta+1}]. \end{cases}$$

Minimal weight expansion

The partition is:

$$I = I_1 \cup I_2 \cup I_0 \cup J_2 \cup J_1.$$

Set equation is

$$\begin{cases} W_\beta(I_1) = I_0 \cup J_2 \\ W_\beta(I_2) = I_1 \\ W_\beta(I_0) = I_2 \cup I_0 \cup J_2 \\ W_\beta(J_2) = J_1 \\ W_\beta(J_1) = I_2 \cup I_0. \end{cases} \quad or \quad \begin{cases} \beta I_1 = (I_0 \cup J_2) - 1 \\ \beta I_2 = I_1 \\ \beta I_0 = I_2 \cup I_0 \cup J_2 \\ \beta J_2 = J_1 \\ \beta J_1 = (I_2 \cup I_0) + 1. \end{cases} \tag{9}$$

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Minimal weight expansion

Minimal weight transformation and invariant sets of its dual system

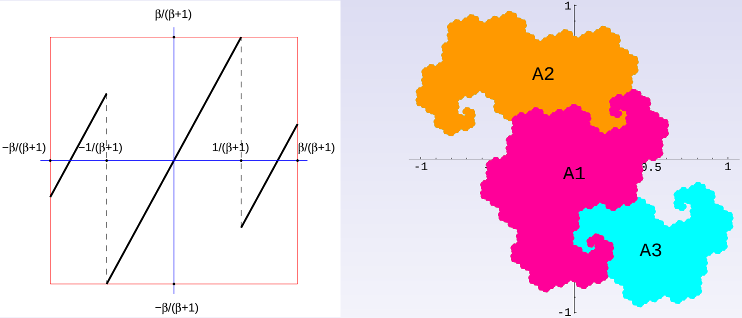

Figure: 7. Minimal weight expansion

Open set condition

§7. Separation condition:

Open set condition

Theorem (Separation condition)

An algebraic graph IFS and its dual system satisfy the open set condition simultaneously.

Consequently, a self-affine tiling system is defined by the dual system via dilation and subdivision.

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Self-replicating tiling

§8. Rauzy-Thurston tiling.

The ideal generated by digit sets

- Denote

$$\mathbb{Z}[\beta] = \left\{ \sum_{j=0}^{d-1} m_j \beta^j : m_j \in \mathbb{Z} \right\}.$$

We assume that the elements of $\mathcal{D}$ are all in $\mathbb{Z}[\beta]$.

- Let $\mathcal{K}$ be the ideal of $\mathbb{Z}[\beta]$ generated by the elements of $\mathcal{D}$.

In other words, $\mathcal{K}$ is the smallest module which is invariant by multiplying β.

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Rauzy-Thurston Tiling

Rauzy-Thurston tiling:

$$\mathcal{J} = \bigcup_{j=1}^{N} \{X_j + b^*;\ b \in I_j \cap \mathcal{K}\} \tag{10}$$

To show that $\mathcal{J}$ is self-replicating or quasi-periodic, we need to assume one of the following conditions:

- I_j are finite unions of intervals;
- $\partial I_j \cap \mathcal{K} = \emptyset$.

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Rauzy-Thurston Tiling

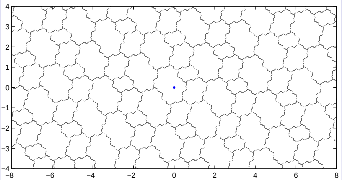

Figure: 8. Rauzy-Thurston Tiling of Rauzy substitution

Branches of Rauzy-Thurston tiling

Theorem

If one of the above conditions holds, then there is an $m \geq 1$ such that almost every point of $\mathbb{R}^s$ is covered by exactly m tiles.

Theorem

$\mathcal{J}$ can be written as $\mathcal{J} = \bigcup_{j=1}^{m} \mathcal{J}_j$, where $\mathcal{J}_j$ are tilings of $\mathbb{R}^s$.

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Branches of Rauzy-Thurston tiling

The Rauzy-Thurston tiling of the symmetric β-transformation contains two pages, as indicated below.

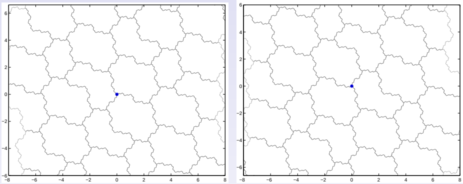

Figure: 9. Two pages of Rauzy-Thurston tiling

Pisot Spectrum Conjecture

§9. Pisot Spectrum Conjecture

Pisot Spectrum Conjecture

Pisot Spectrum Conjecture:

- $\mathcal{J}$ is a tiling.
- $$\sum_{j=1}^{N} L^1(I_j)L^s(X_j) = \text{norm of } \mathcal{K}.$$

We say the system is tight if the above equality holds.

An reducible substitution

Rauzy substitution and its variation:

$$\sigma: \quad \begin{aligned} 1 &\mapsto 12 \\ 2 &\mapsto 13 \\ 3 &\mapsto 1, \end{aligned} \qquad \Sigma: \quad \begin{aligned} 1 &\mapsto 1'2 \\ 2 &\mapsto 1'3 \\ 3 &\mapsto 1' \\ 1' &\mapsto 12' \\ 2' &\mapsto 13' \\ 3' &\mapsto 1. \end{aligned}$$

Clearly

$$I_1' = I_1, I_2' = I_2, I_3' = I_3,$$

$$X_1' = X_1, X_2' = X_2, X_3' = X_3.$$

Hence the Rauzy-Thurston tiling of Σ contains two pages and these two pages are identical!

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The End

Thank you!