Aztec diamond

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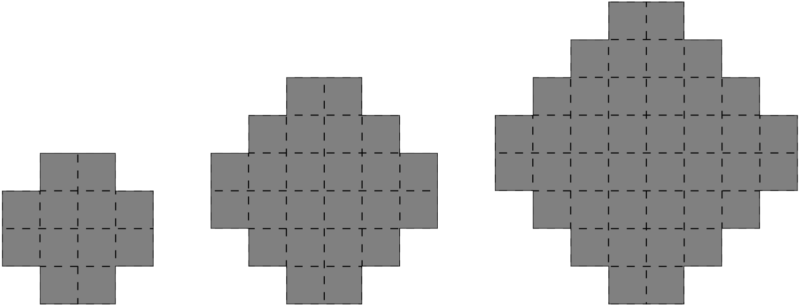

Domino Tilings

The number of domino tilings of aztec diamond of order n is $2^{\frac{n(n+1)}{2}}$

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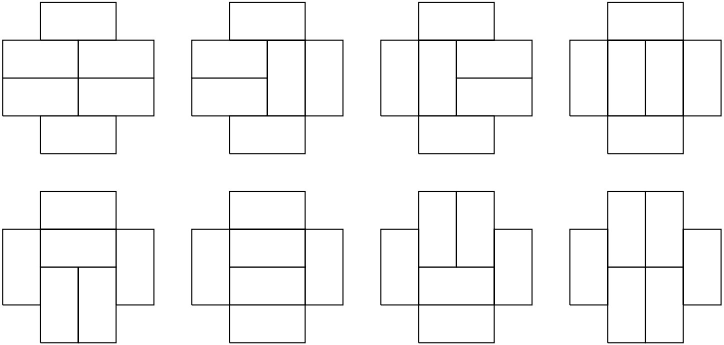
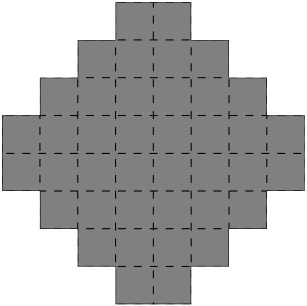
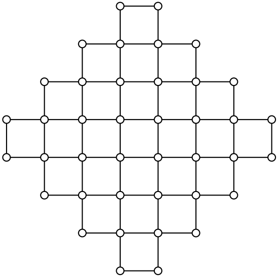


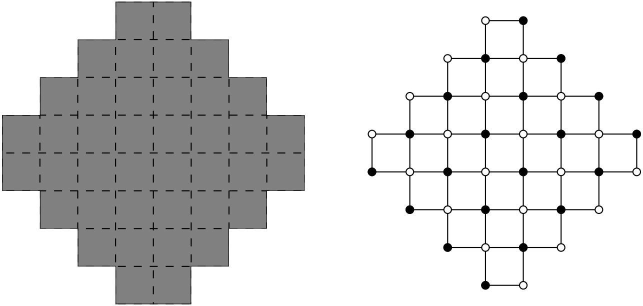
Arctic circle phenomenon

Tilings as matchings

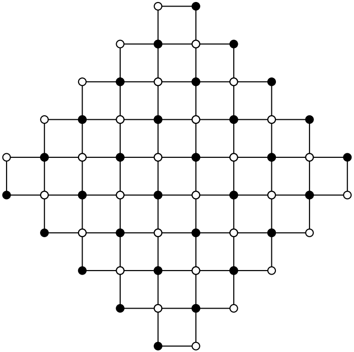




Tilings as matchings



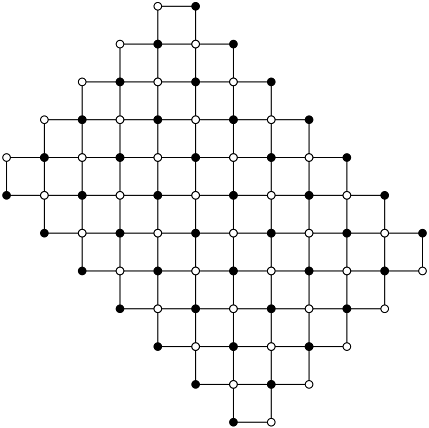
Aztec rectangle (Propp's problem)



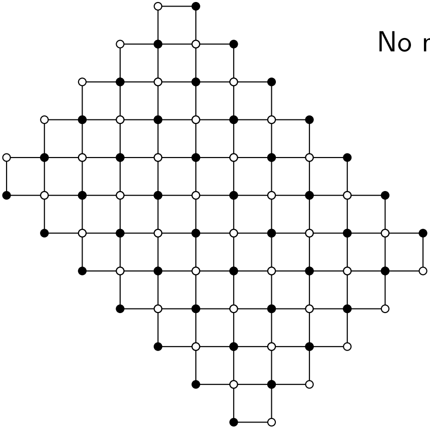
diamond of order 5

Aztec rectangle (Propp's problem)

5×7 rectangle



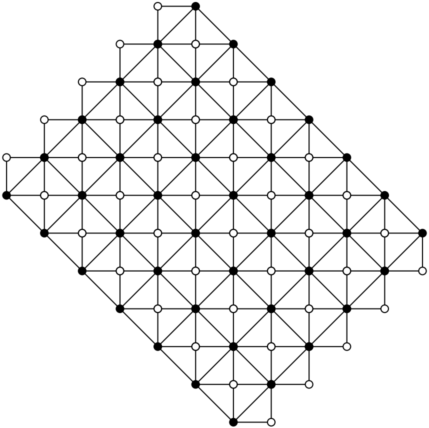
Aztec rectangle (Propp's problem)



5×7 rectangle

No matchings

Aztec rectangle (Propp's problem)

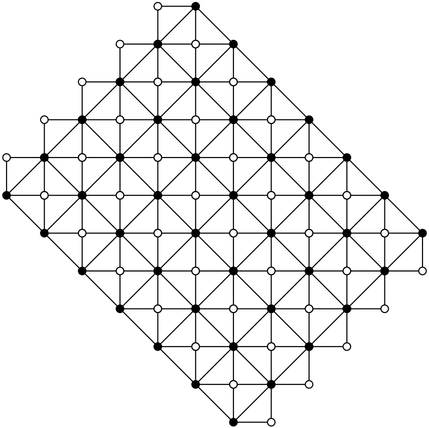


5×7 rectangle

Add diagonal edges connecting nearest black vertices

Aztec rectangle (Propp's problem)

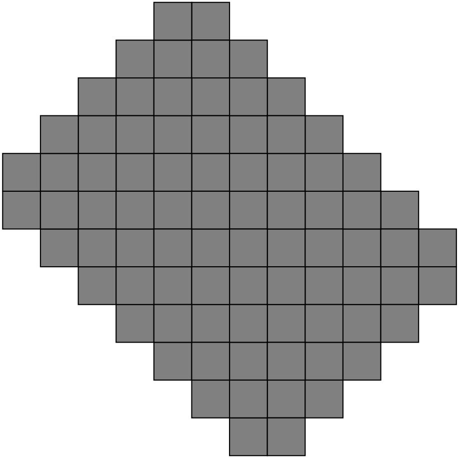
5×7 rectangle



Add diagonal edges connecting nearest black vertices

A matching of $m \times n$ rectangle contains $\frac{|m-n|}{2}$ diagonal edges.

Aztec rectangle (Propp's problem)



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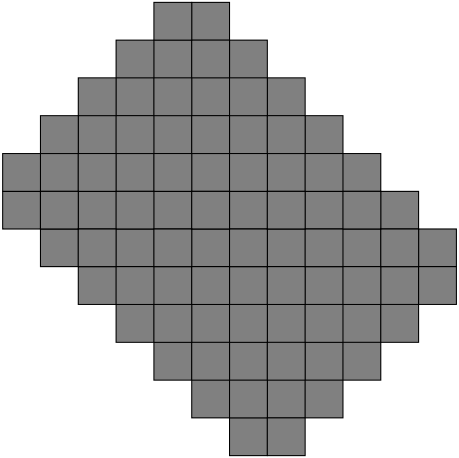

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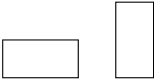
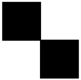
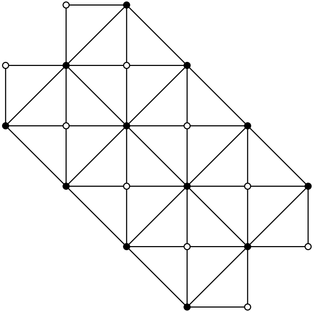


$2mn + m + n - \frac{|m-n|}{2}$ dominos

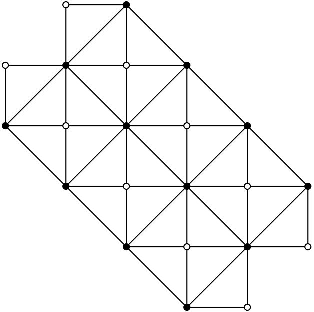
$\frac{|m-n|}{2}$

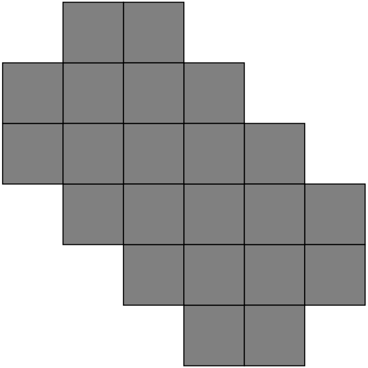


Matchings and Tilings

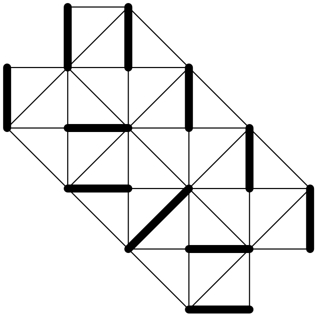


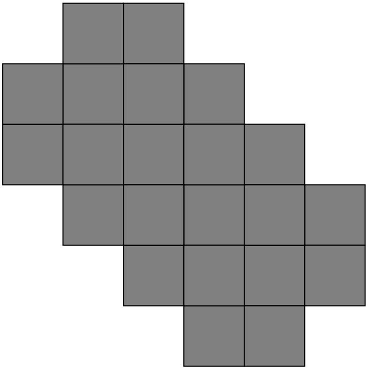
Matchings and Tilings





Matchings and Tilings





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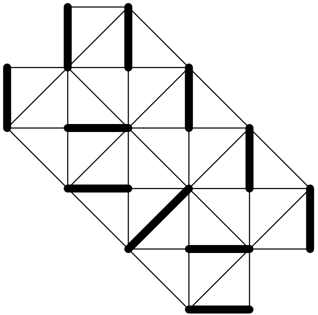
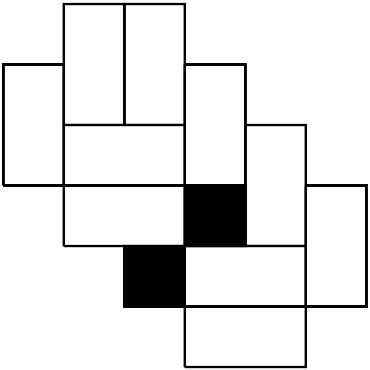
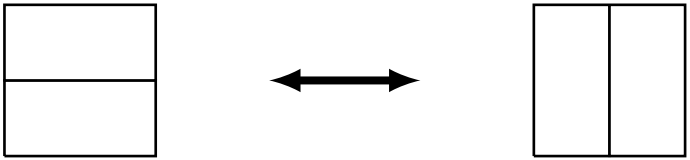




Our first result: Local transformation rule



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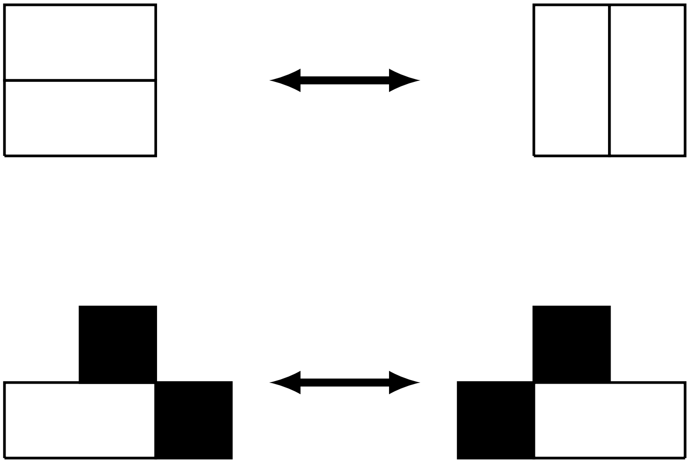

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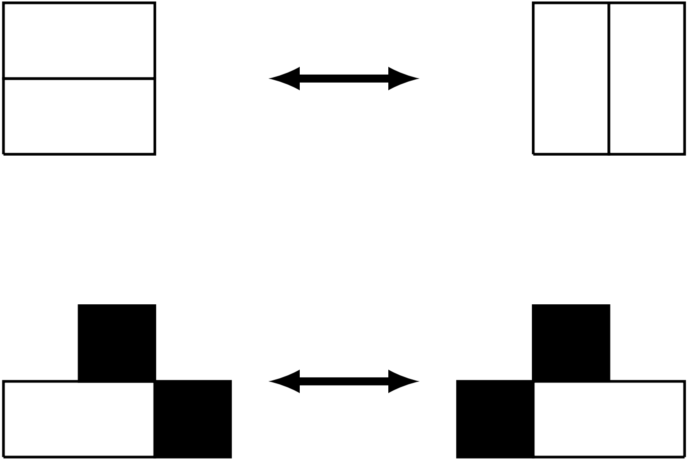
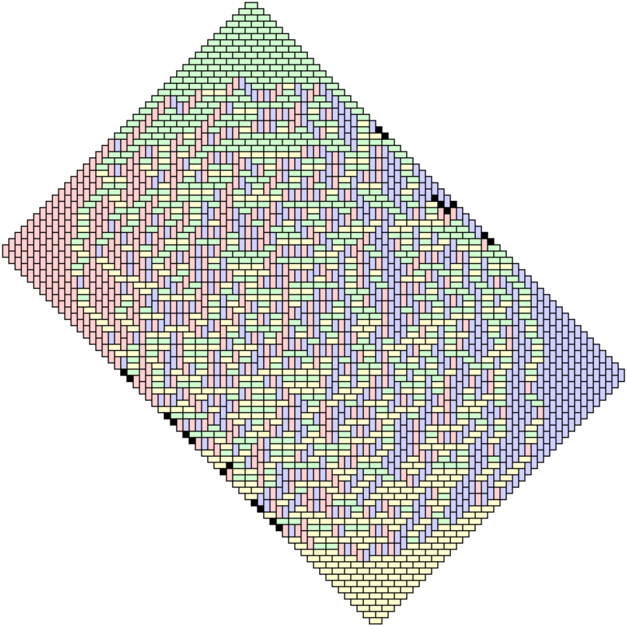
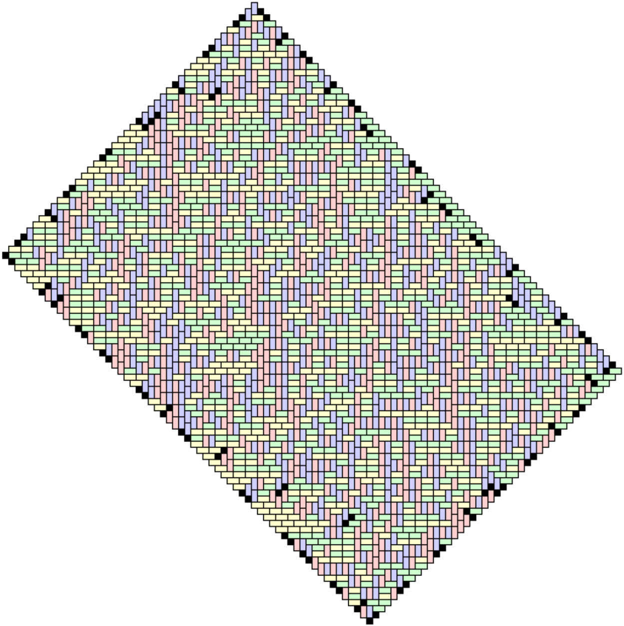


These local moves connects all tilings.
We can construct an ergodic Markov chain whose state space is the set of all tilings.

Typical domino tiling of an Aztec rectangle



Typical domino tiling of another rectangle



Our result 2

Theorem

For any simply connected subgraph G of the square lattice graph with a prescribed special vertex r called root, there exists a simply connected area R_G which can be tiled with dominos and one diagonal impurity.
The probability of finding the impurity at a given position can be explicitly represented in terms of the probability concerning the simple random walk on G.

Our result 2

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Remark

Always possible

$$G \longmapsto R_G$$

Our result 2

Theorem

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Remark

Always possible

$$G \longmapsto R_G$$

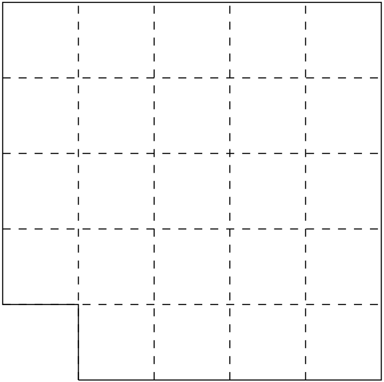
Not always possible

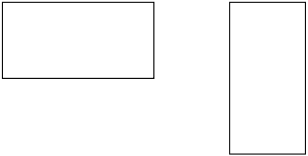
$$R \longmapsto G_R$$

Example

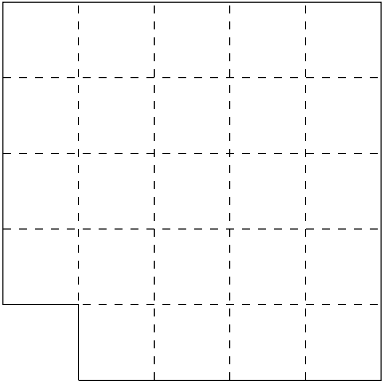
Tile the 5×5 rectangle with a corner removed

with 1×2 rectangles.





Example



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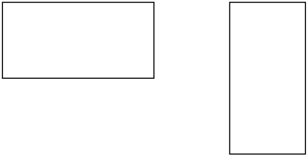

192 tilings exist.

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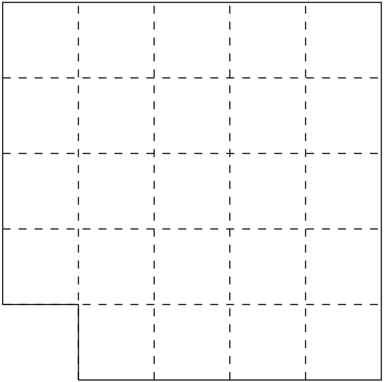
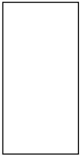
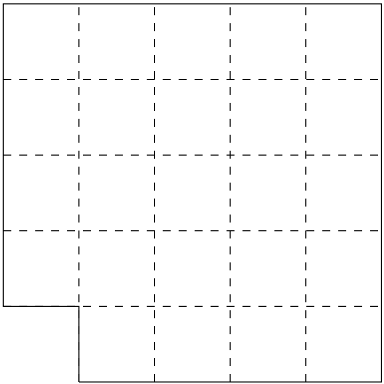




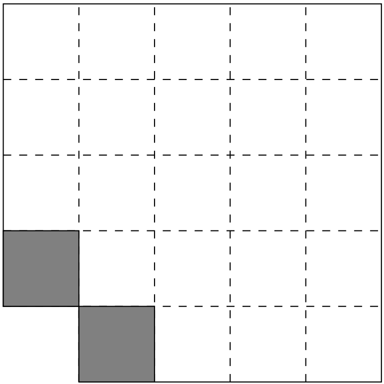
192 tilings exist.

$$192 = \frac{1}{9} \prod_{\substack{(i,j) \in \{0,1,2\}^2 \\ (i,j) \neq (0,0)}} \left(4 - 2\cos\frac{j\pi}{3} - 2\cos\frac{k\pi}{3}\right)$$

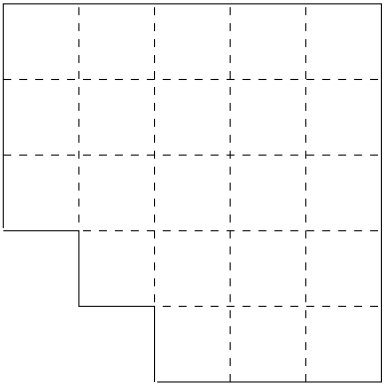
Example



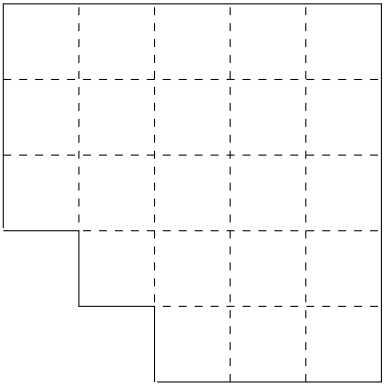
Example



Example



Example

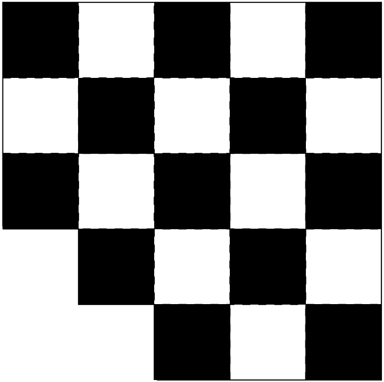


No tilings

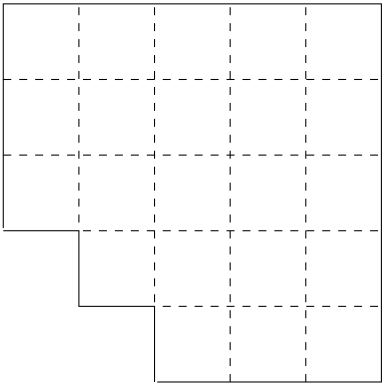
Example

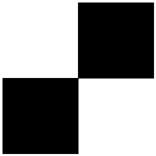
No tilings

There are two more blacks.



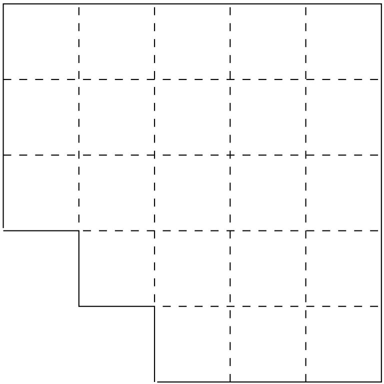
Example

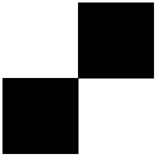




Use a special type of tile.

Example

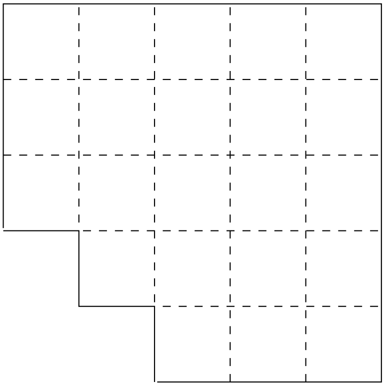


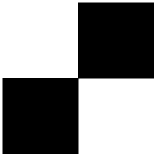


Use a special type of tile.

Where should it be put?

Example



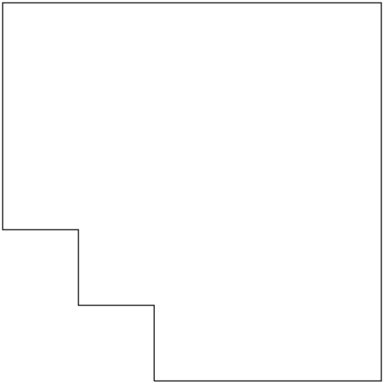


Use a special type of tile.

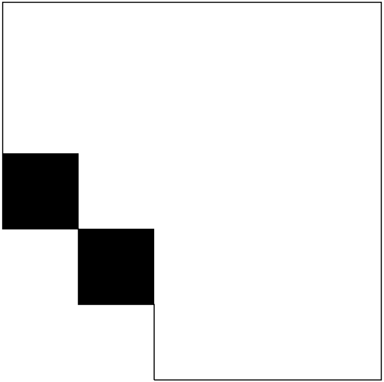
Where should it be put?

The place with the most tilings?

Example

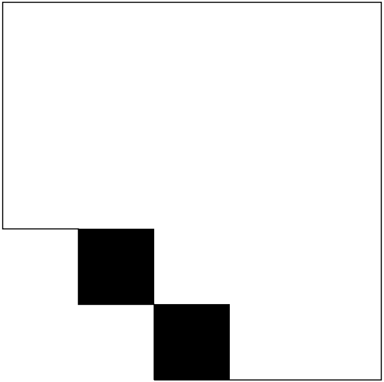


Example



56 tilings

Example



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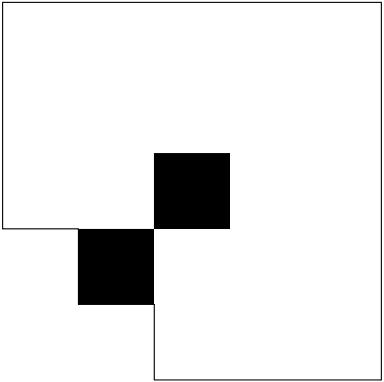

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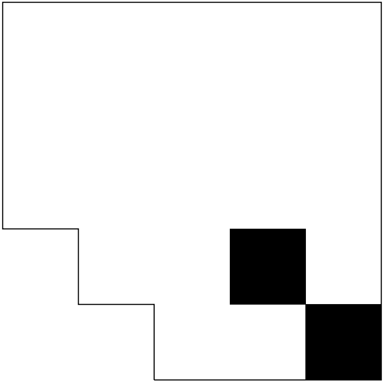
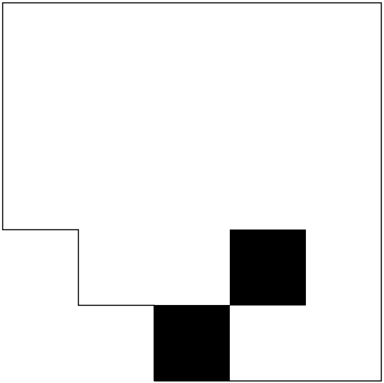
Example



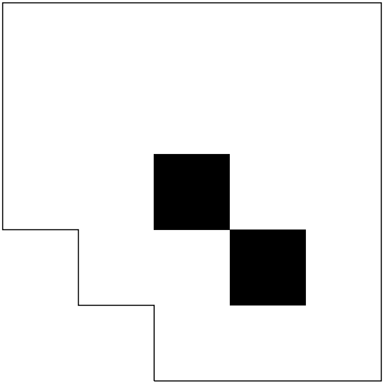
16 tilings

Example

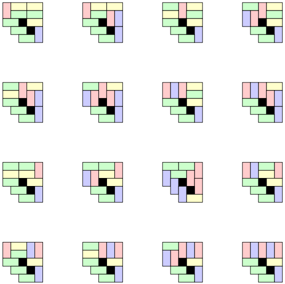


16 tilings

Example



16 tilings



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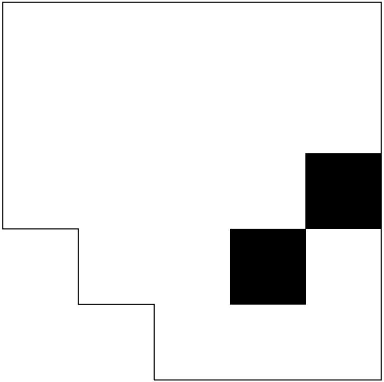
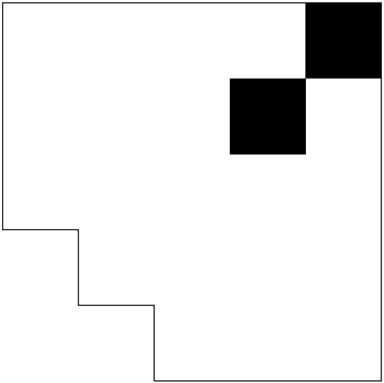
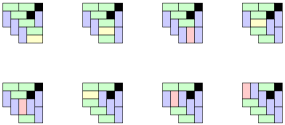


16 tilings

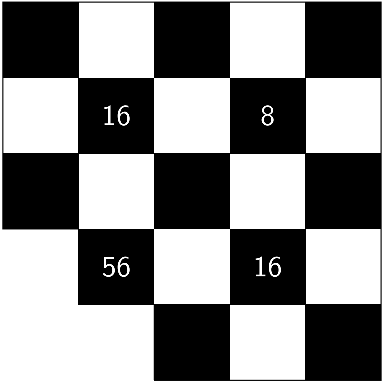
Example



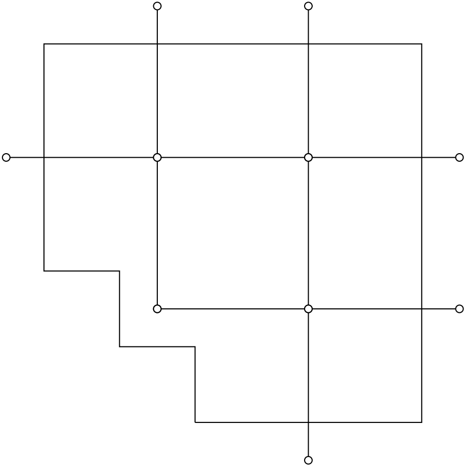
8 tilings



Example



Simple Random Walks and Tilings



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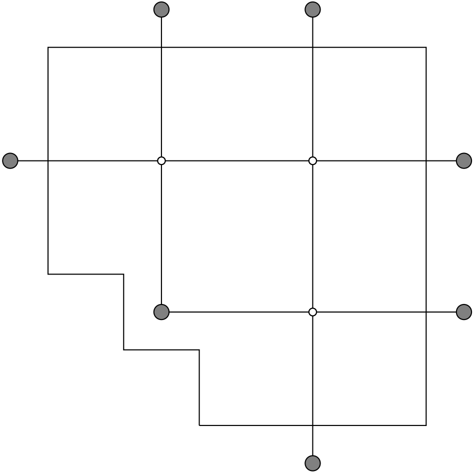

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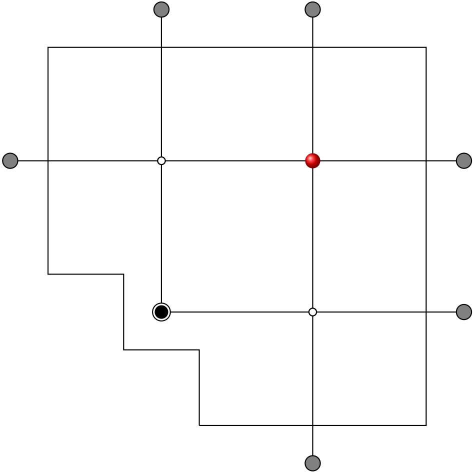
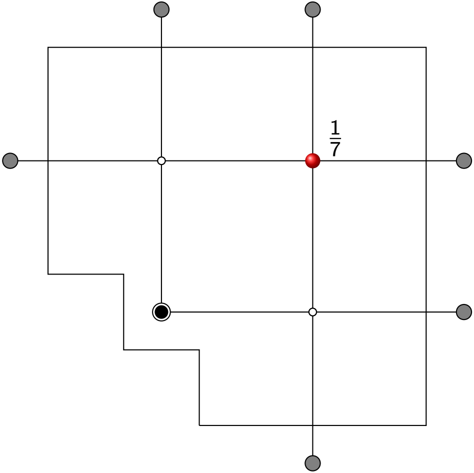



absorbing state

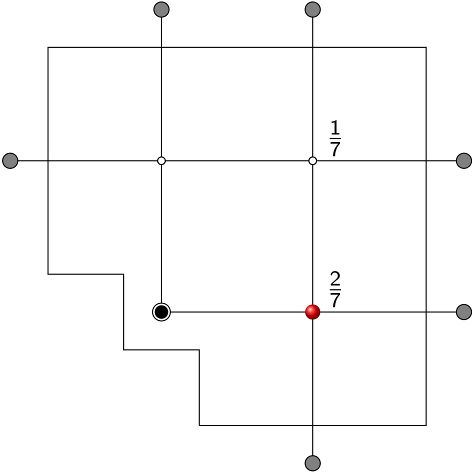
Simple Random Walks and Tilings



absorbing state

Probability of arriving at ◉

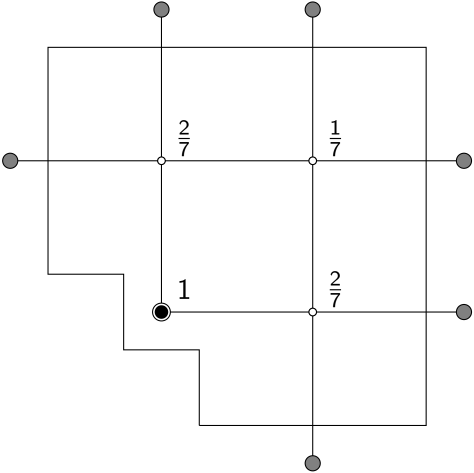
Simple Random Walks and Tilings



absorbing state

Probability of arriving at

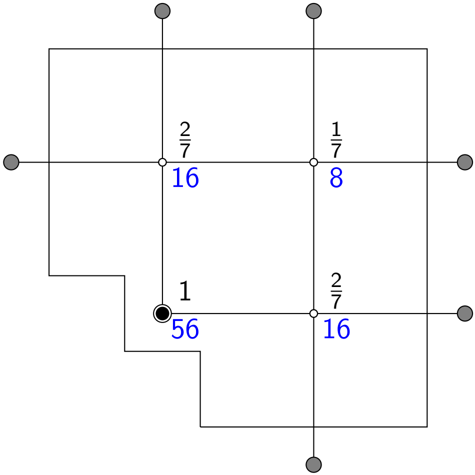
Simple Random Walks and Tilings



absorbing state

Probability of arriving at

Simple Random Walks and Tilings

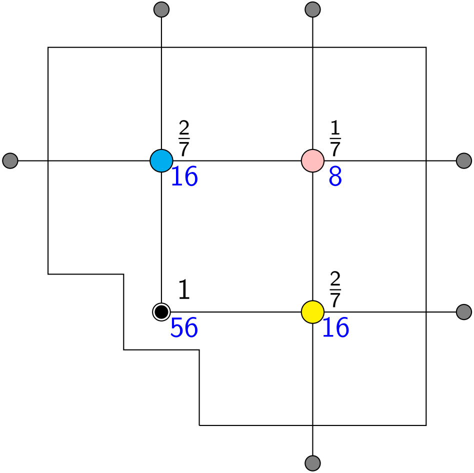


absorbing state

Probability of arriving at

Simple Random Walks and Tilings

absorbing state

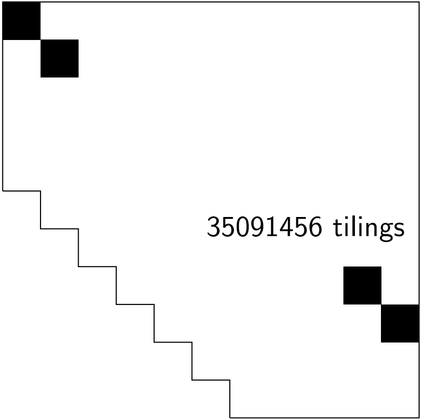


Probability of arriving at

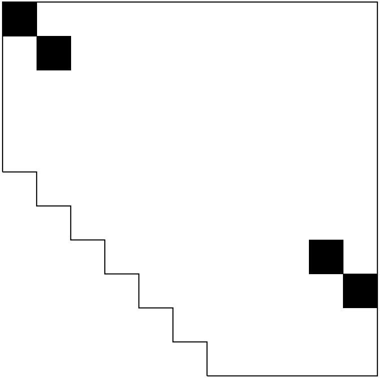
Combinatorial Laplacian

$$56 = \det \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix}$$

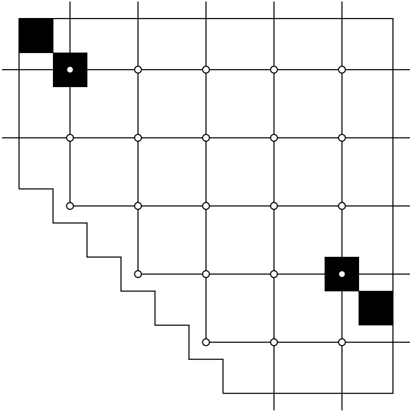
Enumerations of domino tilings with fixed impurities



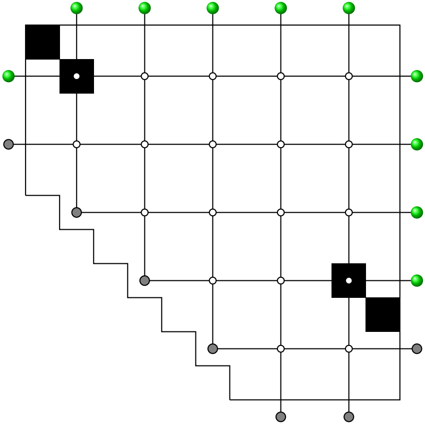
Random walk and hitting matrix



Random walk and hitting matrix

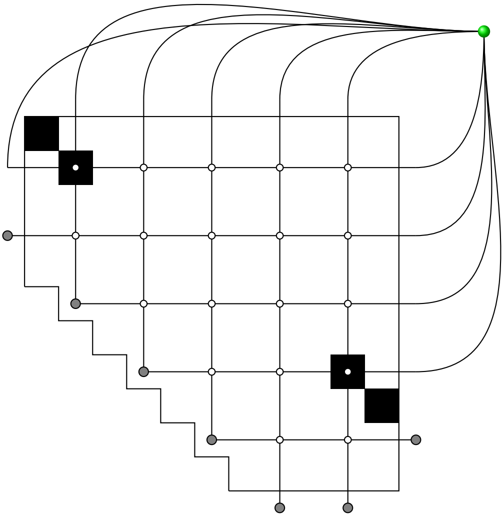


Random walk and hitting matrix



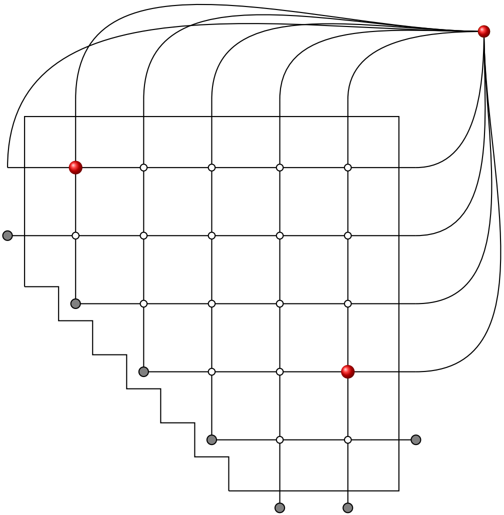
absorbing states

Random walk and hitting matrix



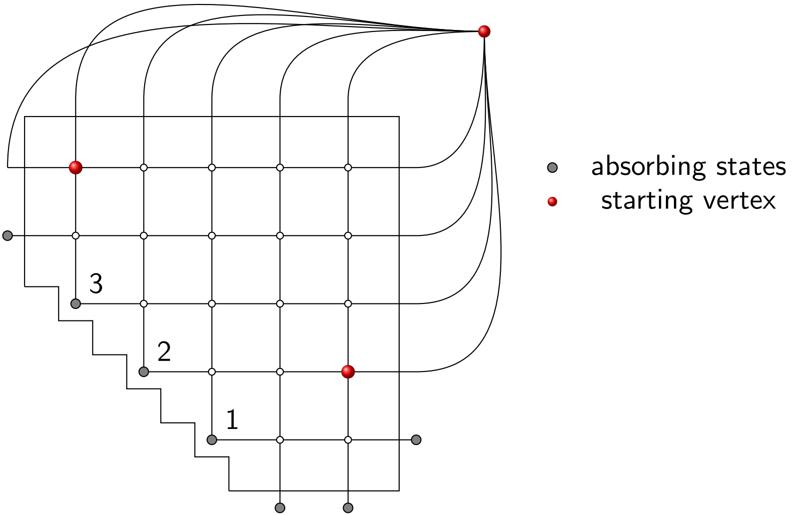
- absorbing states
- identified

Random walk and hitting matrix

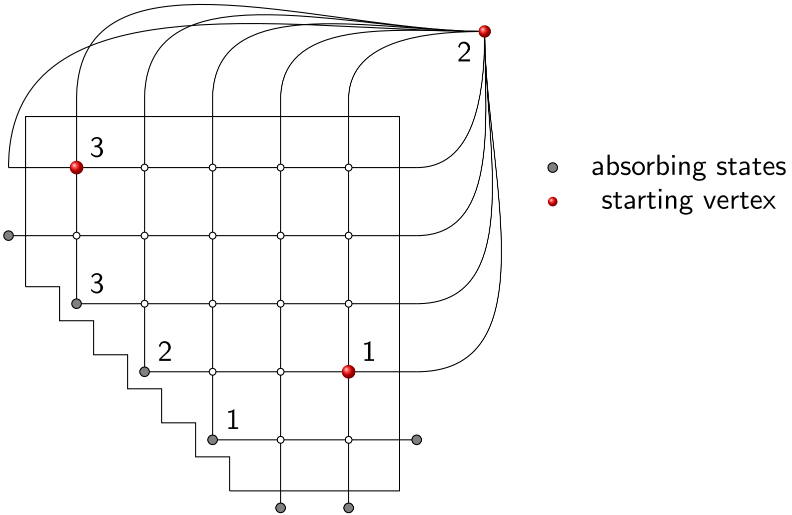


- absorbing states
- starting vertex

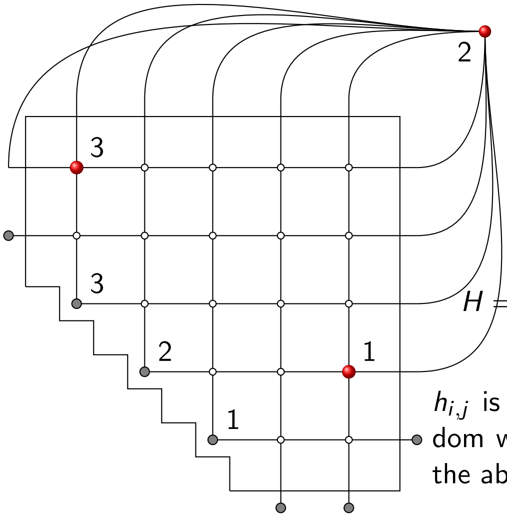
Random walk and hitting matrix



Random walk and hitting matrix



Random walk and hitting matrix



● absorbing states

● starting vertex

$$H = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix} \end{array}$$

$h_{i,j}$ is the probability of a random walk starting at i reaches the absorbing state j.

$$H = \begin{pmatrix} \frac{99562363}{1214591040} & \frac{828541}{5190560} & \frac{55394}{421733} \\ \\ \frac{3345817}{75911940} & \frac{17097}{162205} & \frac{55512}{421733} \\ \\ \frac{1790413}{50607960} & \frac{28529}{324410} & \frac{52981}{421733} \end{pmatrix}$$

Combinatorial Laplacian

$$L = (l_{ij}), \quad l_{ij} = \begin{cases} \deg(i) & i = j \\ -\sharp\text{edges between } i \text{ and } j & i \neq j, \end{cases}$$

where i's and j's are non-absorbing states.

$$\det H \times \det L = 0.000451431 \cdots \times 77733826560 = 35091456 \in \mathbb{Z}$$

Proof Sketch

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- Modification of Temperley bijection (中野理論)

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- Modification of Temperley bijection (中野理論)
- S. Fomin's results on LERW.