

Title : Algebraic coincidence and model set in self-affine tilings

5 • If \exists expansive map Q , compact sets A_i 's with $A_i = \overline{A_i^\circ} \neq \emptyset$,

9 finite sets D_{ij} s.t

$$10 QA_j = \bigcup_{i=1}^m (A_i + D_{ij}), \quad 1 \leq j \leq m$$

then "substitution tiling" γ can be constructed.

15 If γ is a primitive substitution tiling in \mathbb{R}^d with FLC,

20 $\Lambda_\gamma = \Lambda = (\Lambda_i)_{i \leq m}$ represents an associated substitution Delone (multi-colour) set.

• A "cut-and-project scheme" is a collection of spaces and mappings for which

$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times H & \xrightarrow{\pi_2} & H \\ & & \downarrow & & \\ L & \leftarrow & \tilde{L} & \rightarrow & L^* \\ & & \downarrow & & \\ x & \leftarrow & (x, x^*) & \rightarrow & x^*, \end{array}$$

35 where (1) H : a locally compact Abelian group

(2) \tilde{L} : a lattice in $\mathbb{R}^d \times H$

40 (3) π_1 and π_2 are canonical projections such that

$\pi_1|_{\tilde{L}}$ is 1-1 and $\pi_2(\tilde{L})$ is dense in H

(Note that π_2 does not need to be 1-1)

Let $\Lambda(w) := \{ \pi_G(x) \in \mathbb{R}^d \mid x \in \tilde{L}, \pi_G(x) \in W \}$, where $W \subset H$.

- $\Lambda(w)$ is a "model set" if $w = \overline{w}^\circ$: compact, $w^\circ \neq \emptyset$
- $\Lambda(w)$ is a "regular model set" if $w = \overline{w}^\circ$: compact, $w^\circ \neq \emptyset$,
 $M(\partial w) = 0$

- Γ is an "inter model set" if $w = \overline{w}^\circ$: compact, $w^\circ \neq \emptyset$,
 $\Lambda(w^\circ) \subset \Gamma \subset \Lambda(w)$.

- Λ is a "Meyer set" if Λ is a subset of a model set

Thm (Lee - Moody - Solomyak, '03)

Λ : Lattice primitive substitution Delone set in \mathbb{R}^d .

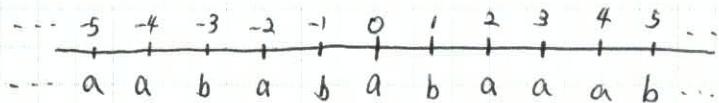
Then Λ admits a modular coincidence $\Leftrightarrow \Lambda$ is a regular model set.

E (constant length substitution)

$$a \xrightarrow{\varphi} aba\bar{a}$$

$$b \rightarrow abab$$

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$$\Lambda_a = (4\Lambda_a \cup (4\Lambda_a + 2) \cup (4\Lambda_a + 3)) \cup (4\Lambda_b \cup (4\Lambda_b + 2))$$

$$\Lambda_b = (4\Lambda_a + 1) \cup ((4\Lambda_b + 1) \cup (4\Lambda_b + 3))$$

$$\begin{pmatrix} \Lambda_a \\ \Lambda_b \end{pmatrix} = \begin{pmatrix} \{4x, 4x+2, 4x+3\} \\ \{4x+1\} \end{pmatrix} \begin{pmatrix} 4x, 4x+2 \\ 4x+1, 4x+3 \end{pmatrix} \begin{pmatrix} \Lambda_a \\ \Lambda_b \end{pmatrix}$$

Note that $4x+2 \subset \Lambda_a$.

Thus $4((\Lambda_a - \Lambda_a) \cup (\Lambda_b - \Lambda_b)) \subset \Lambda_a - 2$.

- Λ : primitive substitution Delone set.

$$\Xi(\Lambda) = \bigcup_{i=1}^m (\Lambda_i - \Lambda_i)$$

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Λ admits an "algebraic coincidence" if $\exists M \in \mathbb{Z}_+$, $\xi \in \Lambda_i, i \leq m$

$$s.t. \quad \xi + Q^M \Xi(\Lambda) \subset \Lambda_i.$$

- (Dynamical hull with local topology)

Λ : FLC.

$$X(\Lambda) := \overline{\{h + \Lambda \mid h \in \mathbb{R}^d\}} \text{ with the usual metric } d.$$

$X(\Lambda)$: compact

$\mathbb{R}^d \curvearrowright X(\Lambda)$. So $(X(\Lambda), \mathbb{R}^d)$: topological dynamical system.

Assume \exists unique invariant probability measure μ .

$$\forall g \in \mathbb{R}^d, \forall f \in L_2(X(\Lambda), \mu), \quad \int g f(\xi) = \int f(\xi - g) \quad \text{for } \xi \in X(\Lambda).$$

Λ has "pure point dynamical spectrum" if the eigenfunctions for the \mathbb{R}^d -action span a dense subspace of $L_2(X(\Lambda), \mu)$.

- Λ_τ admits an "algebraic coincidence"

$\Leftrightarrow \Lambda_\tau$ admits "overlap coincidence"

$\Leftrightarrow \Lambda_\tau$ has pure point dynamical spectrum.

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- (Dynamical hull with an auto correlation topology)

$$\rho(\Lambda', \Lambda'') := \limsup_{n \rightarrow \infty} \frac{\sum_{i=1}^m \#((\Lambda'_i \Delta \Lambda''_i) \cap B_n)}{\text{Vol}(B_n)}$$

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$$A(\Lambda) := \overline{\{h + \Lambda : h \in \mathbb{R}^d\}} \text{ using a uniformity induced by the pseudo metric } \rho.$$

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Thm (Baake - Lenz - Moody '07)

Λ : Delone set in \mathbb{R}^d with UCF & the Meyer property.

Λ : pure point spectrum & all measurable eigenfunctions for $(X(\Lambda), \mu, \mathbb{R}^d)$ can be chosen to be continuous.

$\Rightarrow \exists$ continuous \mathbb{R}^d -map $\beta : X(\Lambda) \rightarrow A(\Lambda)$.

- Λ : Delone set with pure point spectrum.

$$L = \langle \Lambda_i \rangle_{i \in \mathbb{Z}^m}, P_\varepsilon = \{x \in \mathbb{R}^d : \rho(x + \Lambda), \Lambda) < \varepsilon\}$$

Then P_ε : relatively dense.

Define a uniformity on L using the pseudo metric ρ .

Then $\{P_\varepsilon\}$ form a basis for a fundamental system of neighbourhoods of 0 on L in the corresponding topology.

So L becomes a topological group (L_p) .

Let H : a (Hausdorff) completion of L_p .

\exists uniform continuous map $\phi : L_p \rightarrow H$ s.t. $\phi(L_p)$ is dense in H .

$$\mathbb{R}^d \xleftarrow{\pi_1} \mathbb{R}^d \times H \xrightarrow{\pi_2} H$$

$$\begin{array}{ccc} & \cup & \\ L_p & \xleftarrow{\quad} & \xrightarrow{\quad} \phi(L_p) \\ & \cup & \\ x & \xleftarrow{(x, \phi(x))} & \rightarrow \phi(x) \end{array} - (*)$$

where $\tilde{L}_p = \{(\tilde{x}, \phi(x)) : x \in L_p\}$.

- Assume that Λ admits an algebraic coincidence.

Let $X := \{x \in \mathbb{R}^d : \Lambda + x = \Lambda\}$. Then $\{x + \mathbb{Q}^d : n \in \mathbb{Z}_+, x \in L\}$

Serves as a neighbourhood base of the topology on L relative to which L becomes a topological group (L_Q) . In fact, L_Q is topologically isomorphic to L_p .

Thm (Lee - Moody '06)

Λ : Delone set in \mathbb{R}^d with repetitivity

① \exists continuous \mathbb{R}^d -map $\beta : X(\Lambda) \rightarrow A(\Lambda)$

② $\forall i \leq m$, $\lambda_i = \lambda(V_i)$ in $cps(\mathbb{X})$, where \widehat{V}_i : compact, $V_i^\circ \neq \emptyset$, $(\partial V_i)^\circ = \emptyset$

$\Rightarrow \Lambda$: inter model set.

Thm (Solomyak '06)

Λ : primitive substitution Delone set with FLC.

Then all measurable eigenfunctions for $(X(\Lambda), \mu, \mathbb{R}^d)$ can be chosen to be continuous

Thm (Lee '07)

Λ_γ : primitive substitution Delone set with the Meyer property.

Λ_γ admits algebraic coincidence $\Leftrightarrow \Lambda_\gamma$: inter model set.