On aperiodic monotile by Socolar-Taylor

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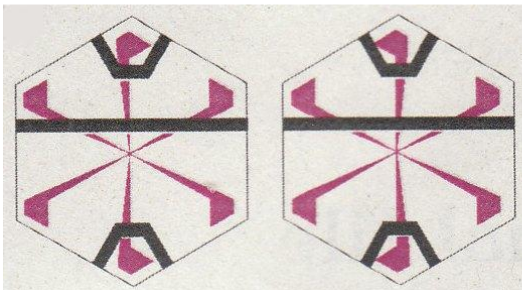
[Main Proposition]
If Socolar-Taylor's tiles cover the plane, then it is not periodic.

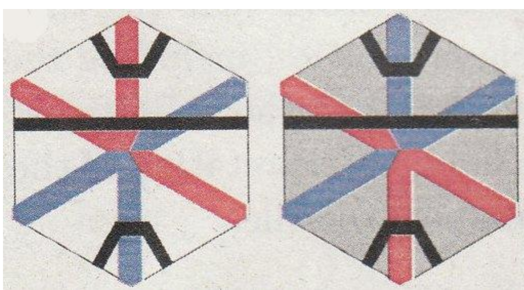
First, we switch to (b) having two shapes for the convenience.

(a)



Left-handed flags→blue stripe
Right-handed flags→red stripe

(b)



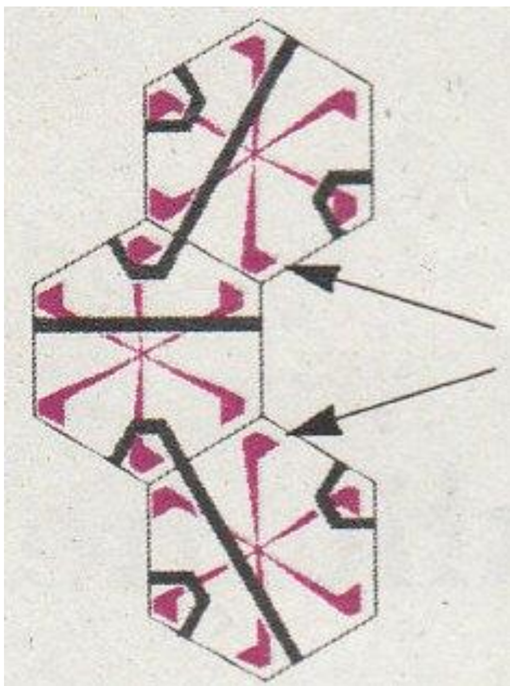
Both tiles of (b) have the same color arrangement except one diagonal.
Let us call this diagonal having different colors "red-blue diagonal".

Next, we prove Lemma 1,2 and five Propositions.
Later, using them, we prove the Main Proposition.

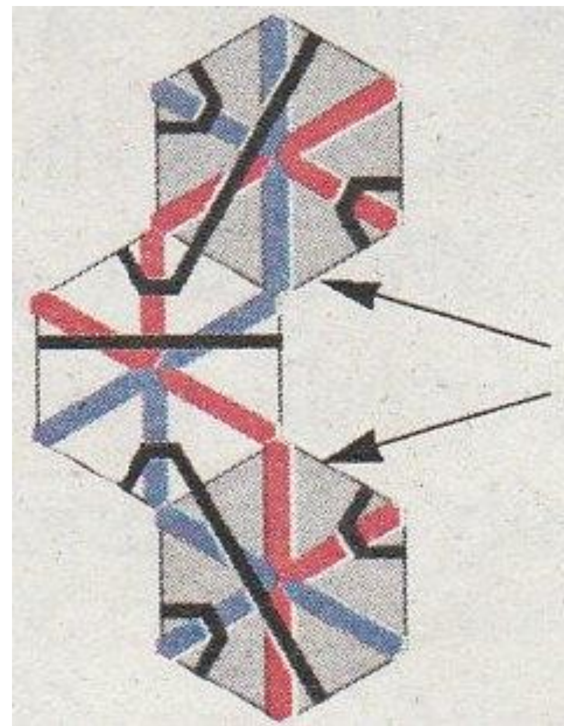
Socolar-Taylor’s tile has two matching rules.

1. Black stripes must be continuous across shared edges.
2. The colored (red or blue) segments that meet a given edge at opposite endpoints and are collinear with that edge must not be the same color (as shown below).

(a)

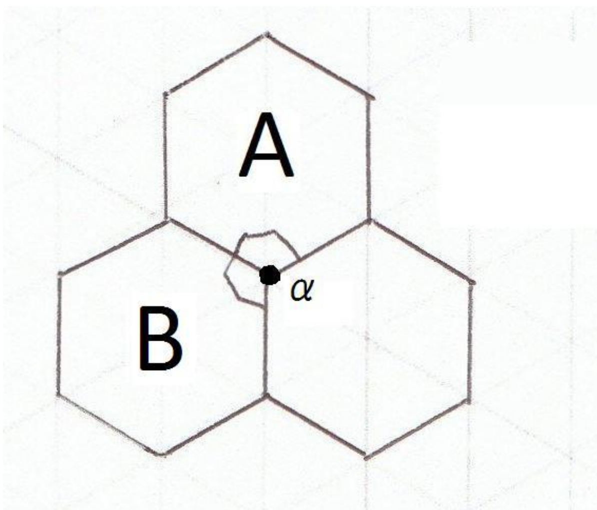


(b)

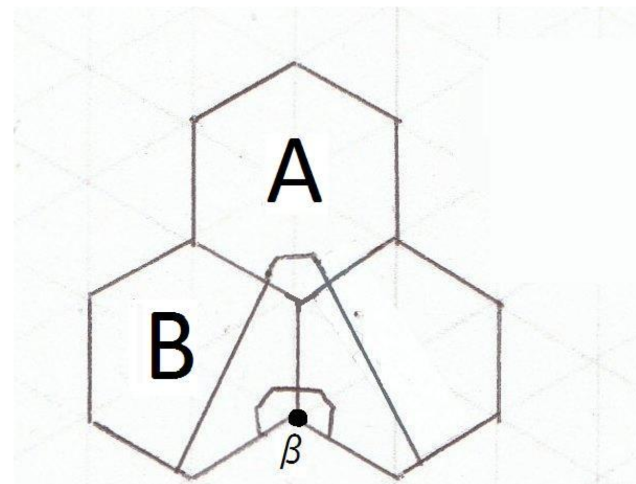


[Lemma 1]
If the tiles cover the plane, then a small black ring like below must be formed. And it is formed at the opposite vertex, as well.

<Proof>
First, we prove that a small ring must be formed.
Assuming that the direction of Tile A is like this, there are two cases.



A small ring is formed at vertex α by matching rule of black stripe.

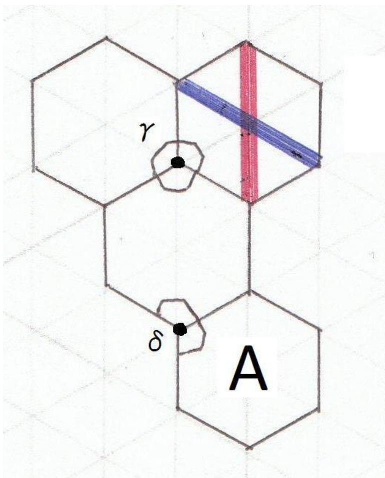


A small ring is formed at vertex β by matching rule of black stripe.

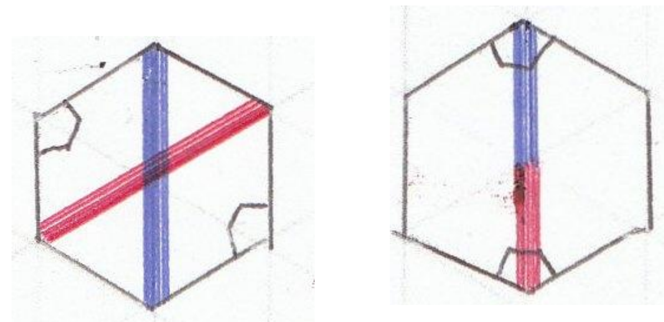
Therefore, a small ring must be formed.

[Lemma 1]
If the tiles cover the plane, then a small black ring like below must be formed. And it is formed at the opposite vertex, as well.

<Proof: continued>
Next, we prove that a small ring is formed at the opposite vertex.



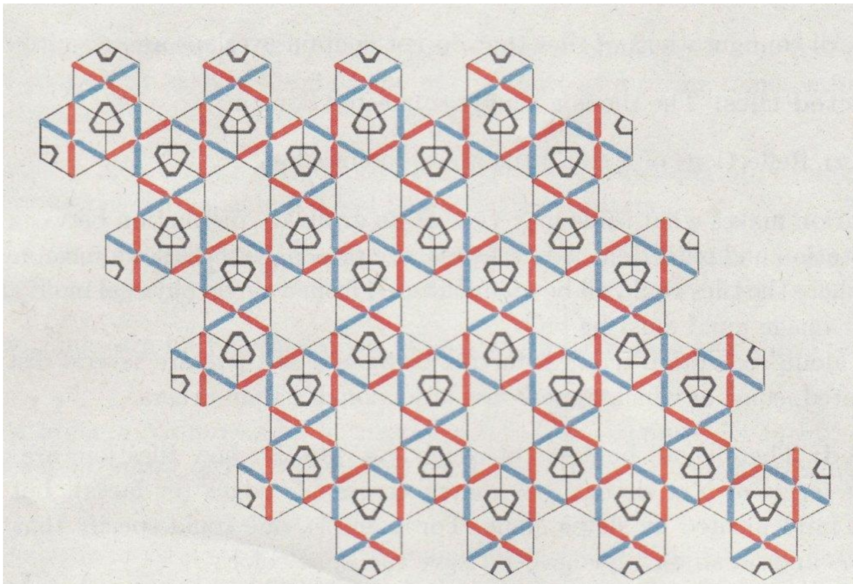
Assume that a small ring is formed at vertex γ.



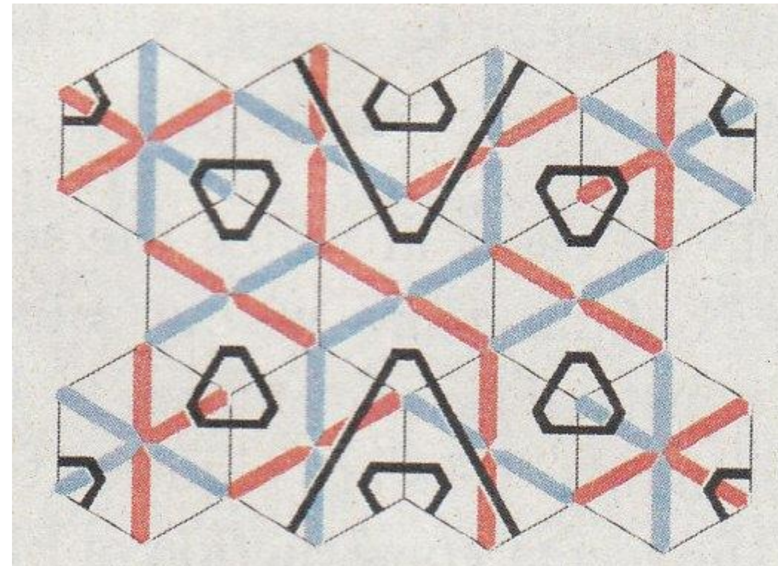
By color matching rule, there are two ways to configure A as in the figure above, but by matching rule of black stripe, the former is correct.

Therefore, vertex δ has a small ring. □

The following lattice of holes is formed by Lemma 1.
There are three ways to fill a hole.



Hexagonal lattice of holes.

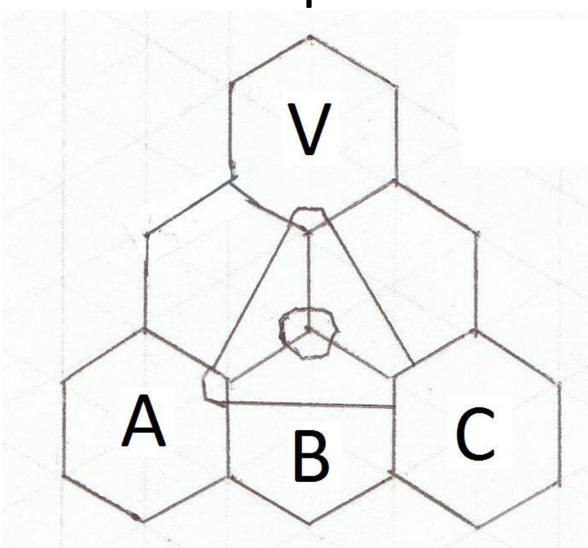


Putting a tile in the hole.

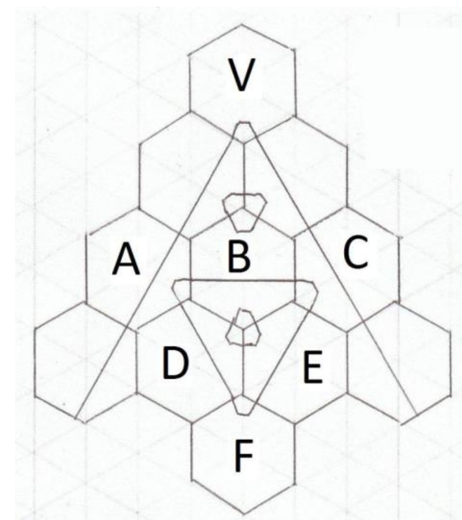
Let us call a tile in the hole in the lattice “vertex tile” and draw “V” in it to fix the orientation.

[Lemma 2]
A black triangle using three tiles for an edge (Level 1 triangle) must be formed. And it is formed at the opposite side, as well.

<Proof>
First, we prove that Level 1 triangle must be formed. As shown below, according to the lattice, there are two directions at A by matching rule of black stripes.



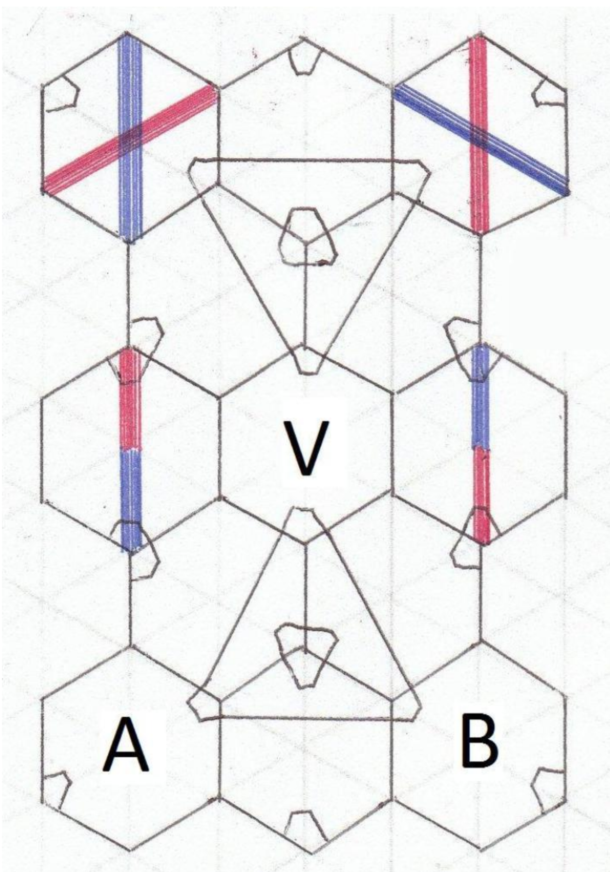
The direction of tile A, B, C is decided, and Level 1 triangle is formed here.



The direction of tile A, B, C, D, E, F is decided, and Level 1 triangle is formed below.

[Lemma 2]
A black triangle using three tiles for an edge (Level 1 triangle) must be formed. And it is formed at the opposite side, as well.

<Proof: continued>
Next, we prove that if Level 1 triangle is formed, it is formed at the opposite side, too.



The directions of tile A and B force the left color configurations, therefore, Level 1 triangle is formed at the opposite side, too.

□

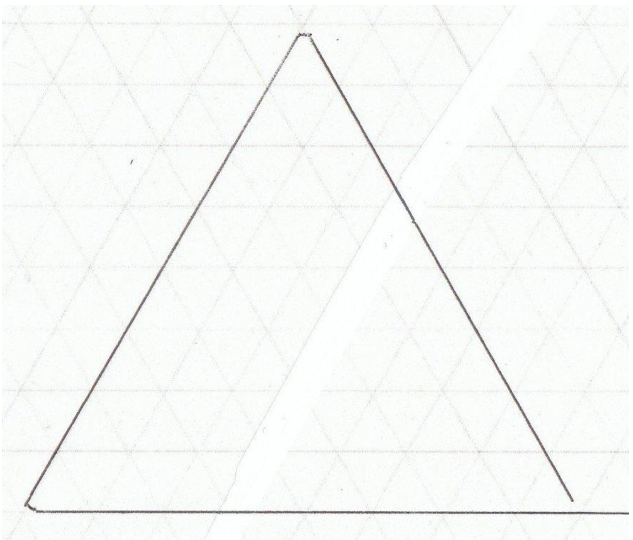
Two triangles at the opposite sides share a "vertex tile".

[Definition]

- If number of tiles of one side is $N_n = 2^n + 1$, then the triangle is called Level n triangle.
  We do not count a small ring as a triangle.
- Tiles whose red-blue diagonals are collinear are called “ tiles in the same column”.
- If consecutive tiles from tile A to B contain a straight line, then denote their length by N_{AB} .

[Proposition 1]
If there is a straight line of m tiles which can't be extended any more, then the line produce a triangle having m tiles at each side.

<Proof>

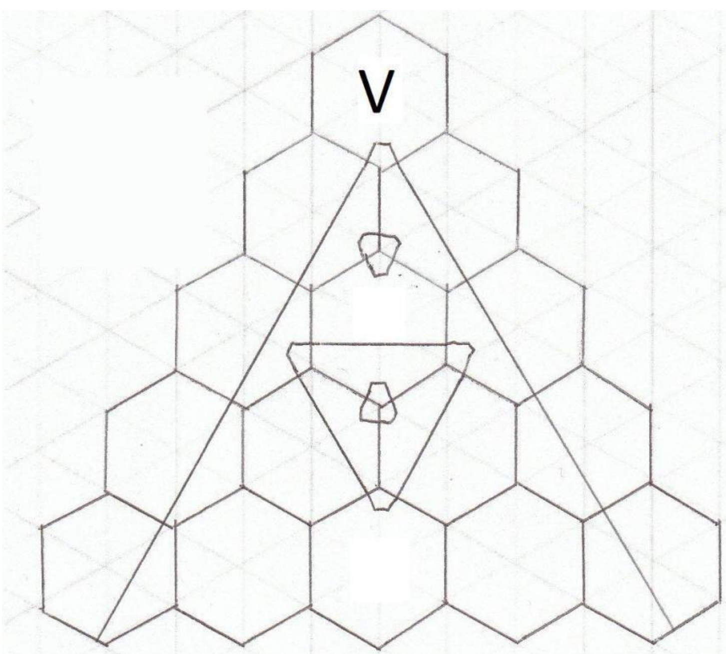


When the line turns at two ends, then the direction of turns must coincide by matching rule of black stripe. Therefore, the other two sides starting from two ends must be of length m as in the figure, by matching rule of black stripe.

□

[Proposition 2]
If there are two straight lines starting from a vertex tile, whose length is longer than $N_n + 1$, then we observe lined up triangles whose vertex tiles are in the same column in the middle of the two straight lines. They appear in the increasing order like, the vertex tile, a small ring, Level 1 triangle and Level 2 triangle ・・・ and form a Sierpinski's triangle shape.

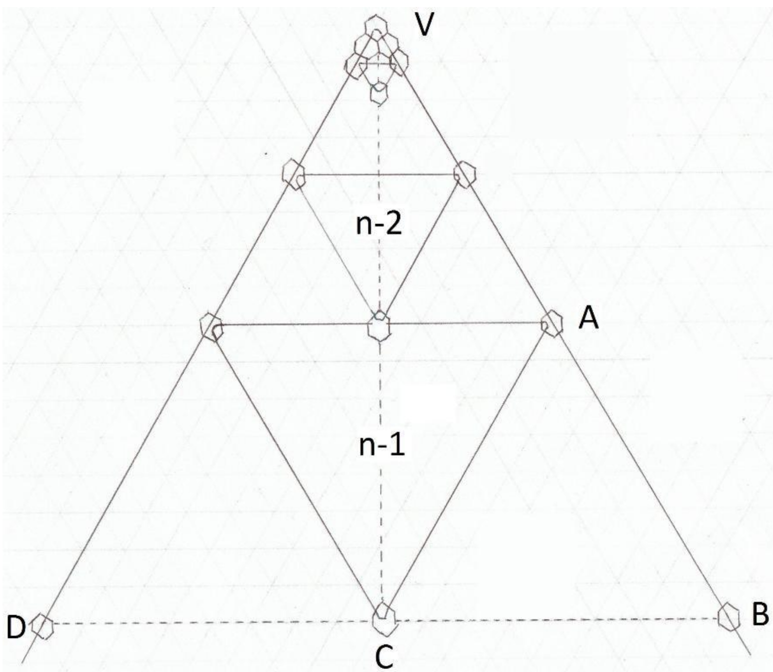
<Proof>
By induction on n.



A small ring and Level 1 triangle are formed, as shown on the left, by matching rule of black stripe. (n=1 is satisfied because of $N_1 = 3 = 2^1 + 1$)

[Proposition 2]
If there are two straight lines starting from a vertex tile, whose length is longer than $N_n + 1$, then we observe lined up triangles whose vertex tiles are in the same column in the middle of the two straight lines. They appear in the increasing order like, the vertex tile, a small ring, Level 1 triangle and Level 2 triangle ・・・ and form a Sierpinski's triangle shape.

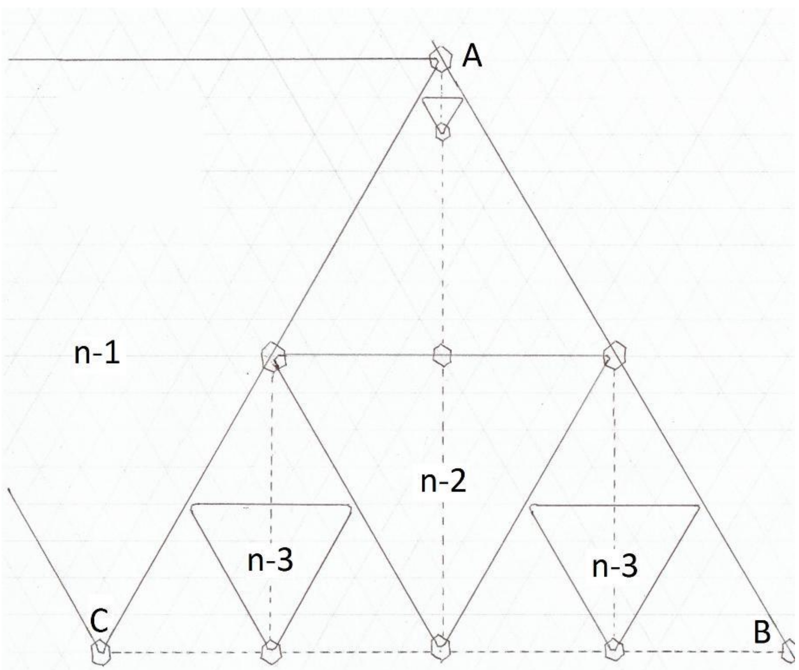
<Proof: continued>



$$N_{VA} = 2 + \sum_{i=1}^{n-3} (N_i - 1) + N_n - 2$$
$$= N_n - 1$$
$$= N_{AB}$$

[Proposition 2]
If there are two straight lines starting from a vertex tile, whose length is longer than N_n+1, then we observe lined up triangles whose vertex tiles are in the same column in the middle of the two straight lines. They appear in the increasing order like, the vertex tile, a small ring, Level 1 triangle and Level 2 triangle ・・・ and form a Sierpinski's triangle shape.

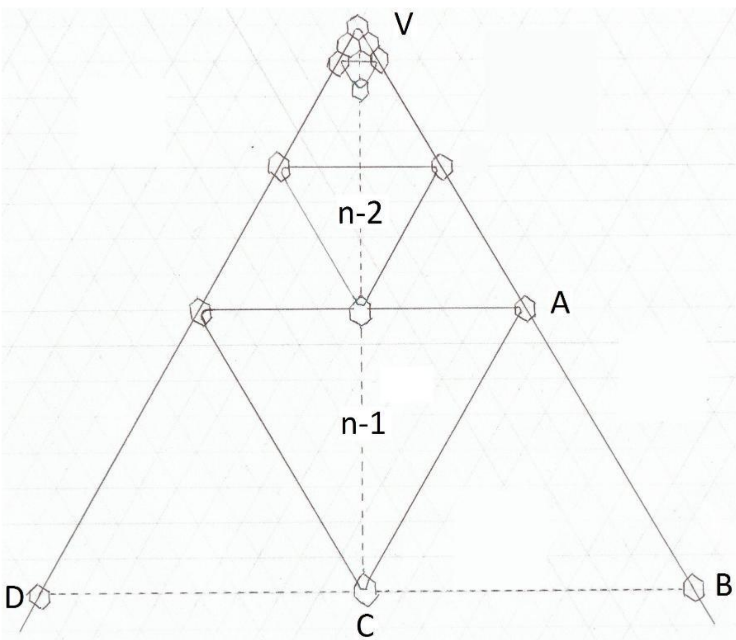
<Proof:continued>



Formation of triangles is repeated, because tile V and A are the same role and from the left figure.

[Proposition 2]
If there are two straight lines starting from a vertex tile, whose length is longer than $N_n + 1$, then we observe lined up triangles whose vertex tiles are in the same column in the middle of the two straight lines. They appear in the increasing order like, the vertex tile, a small ring, Level 1 triangle and Level 2 triangle ・・・ and form a Sierpinski's triangle shape.

<Proof:continued>



From the left figure,

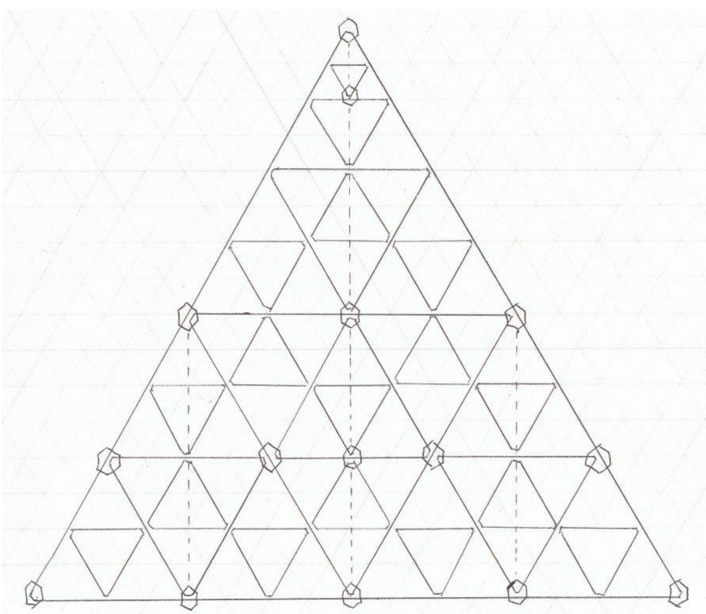
$$N_{DB} = 2N_{n-1} - 1 = 2\left(2^{n-1} + 1\right) - 1$$

$$= 2^n + 1 = N_n$$

Therefore, the straight line which connects tile F to B form Level n triangle by proposition 1.

[Proposition 2]
If there are two straight lines starting from a vertex tile, whose length is longer than N_n+1, then we observe lined up triangles whose vertex tiles are in the same column in the middle of the two straight lines. They appear in the increasing order like, the vertex tile, a small ring, Level 1 triangle and Level 2 triangle ･･･ and form a Sierpinski's triangle shape.

<Proof:end>

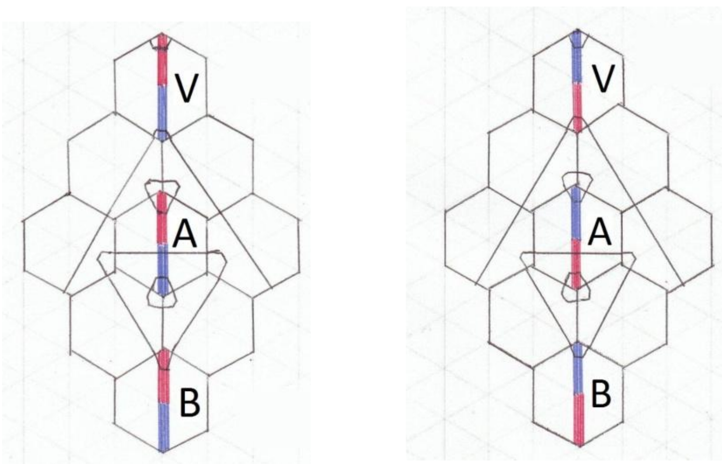


Therefore, as in the figure, triangles form a Sierpinski's triangle shape.

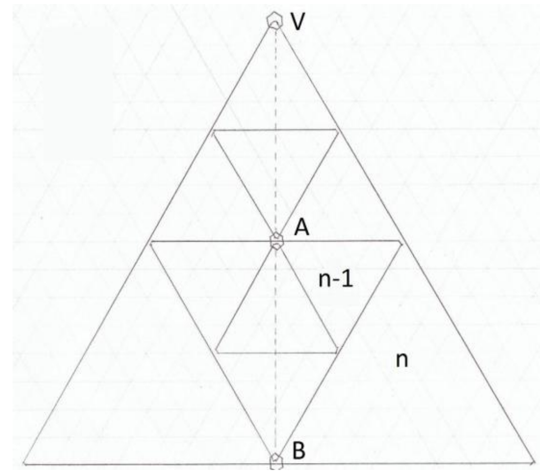
□

[Proposition 3]
The direction of red-blue diagonal of the vertex tile of Level n triangle and that of the middle tile in the base line being opposite of the vertex tile, are the same.

<Proof>
By induction.



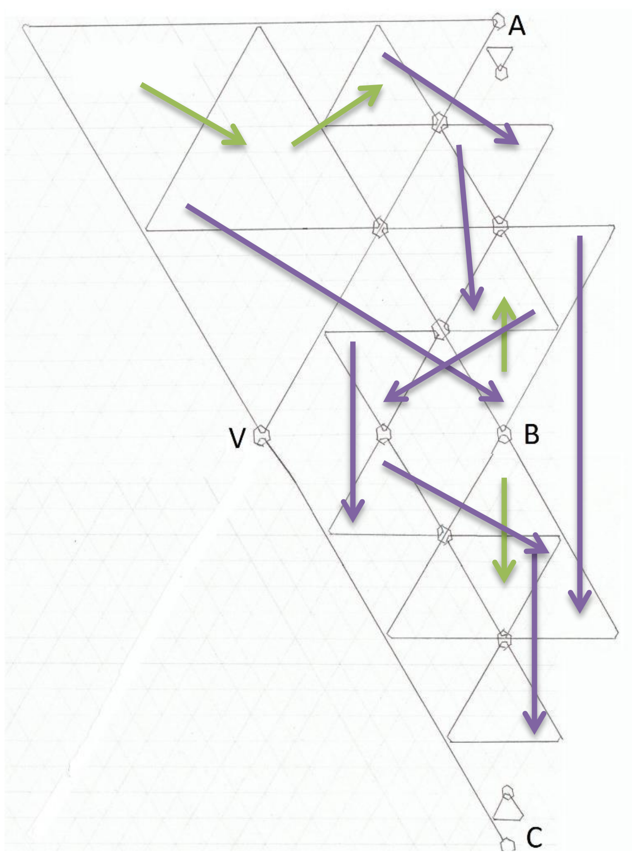
First, the direction of colors of red-blue diagonal of the vertex tile, tile A and B are the same as shown.



From assumption, direction of colors of red-blue diagonal of tile B and A is the same. Iterating the same method, that of tiles from tile V to A is the same. □

[Proposition 4]
If Level n triangle is formed, it must be formed at the opposite, too.

<Proof>
By induction. Assume that for n-1 (Existence of Level 1 triangle was already shown.) the triangle is formed.

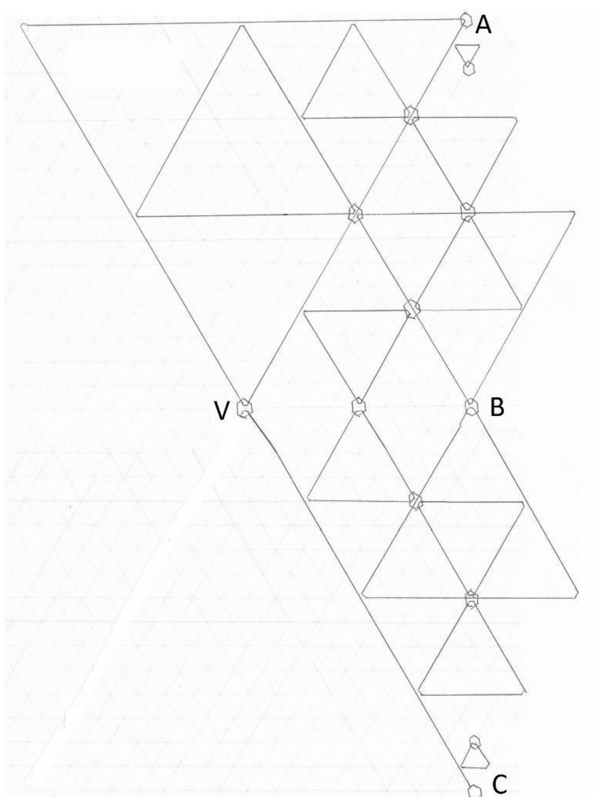


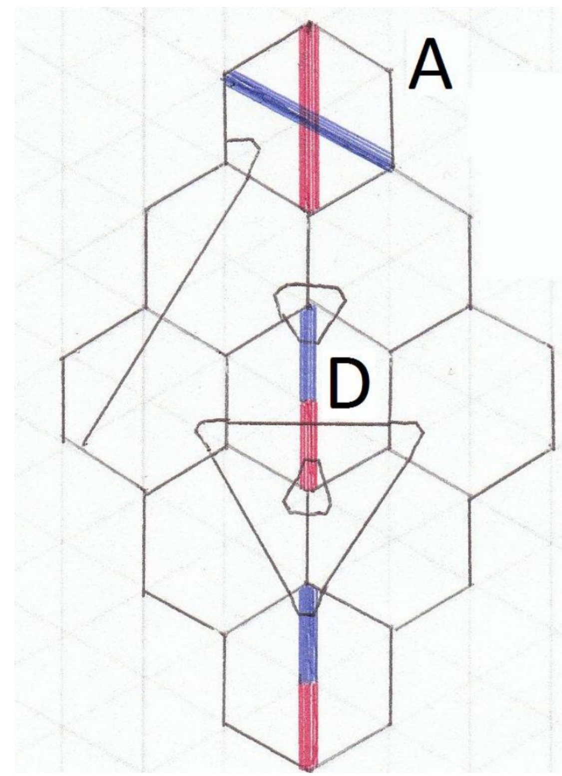
※Tile A, B and C is in the same column.
And clearly $N_{AV} = N_{VC} = N_n$.

From proposition 2
From assumption of induction

[Proposition 4]
If Level n triangle is formed, it must be formed at the opposite, too.

<Proof:continued>

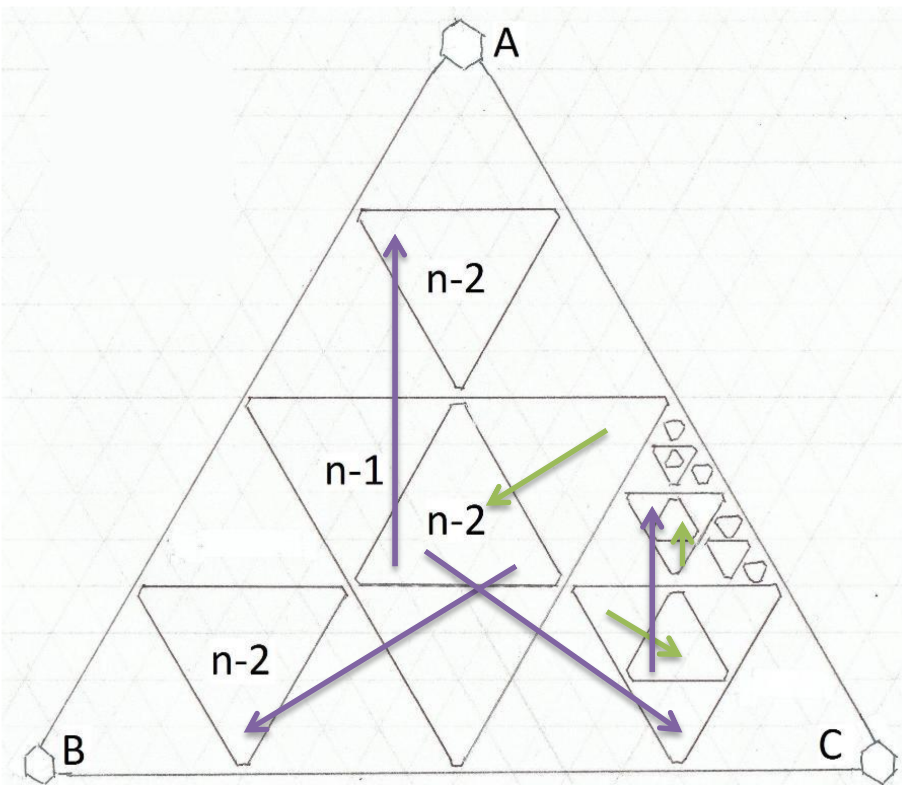




Color information is transmitted as A→D→B→C by proposition 3.
Direction of tile C is determined by matching rule of black stripe.
Therefore, proposition 4 is shown by proposition 1. □

[Proposition 5]
For all n, Level n triangle is formed.

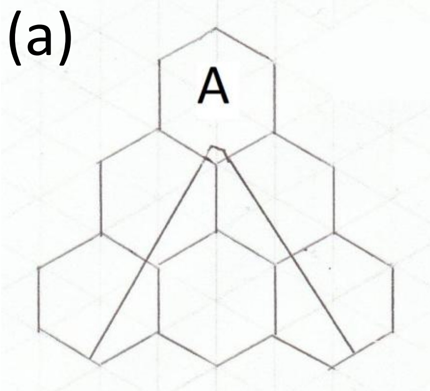
<Proof>

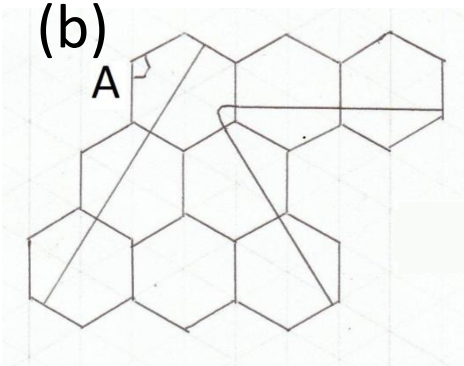


Assuming the existence of a Level n-1 triangle, triangles whose level is less than n-1 are formed. And, consecutive tiles from tile A to B, from tile B to C and from tile C to A contain a straight line.
Then we have,
$N_{AB} = N_{BC} = N_{CA} = N_n$.

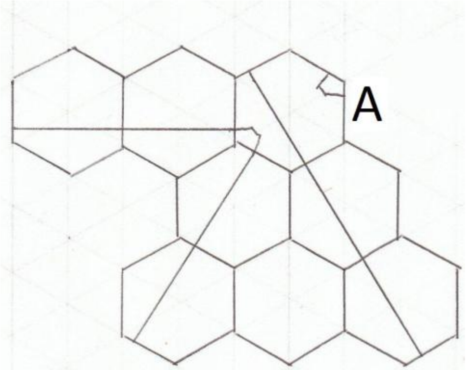
[Proposition 5]
For all n, Level n triangle is formed.

<Proof:continued>
By induction. Assume it for n-1. (Existence of Level 1 triangle was already shown.)

(a)


(b)


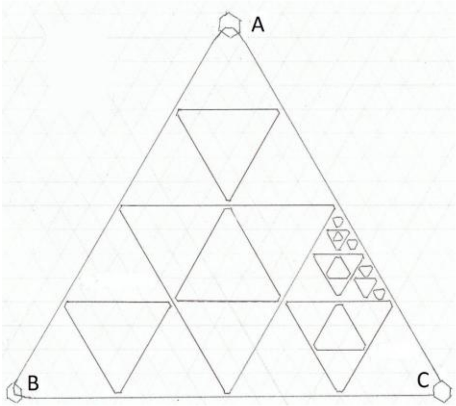
There are three directions of tile A. But I prove only (a). Because rotating figure (b) and (c), they and (a) are same.

(c)


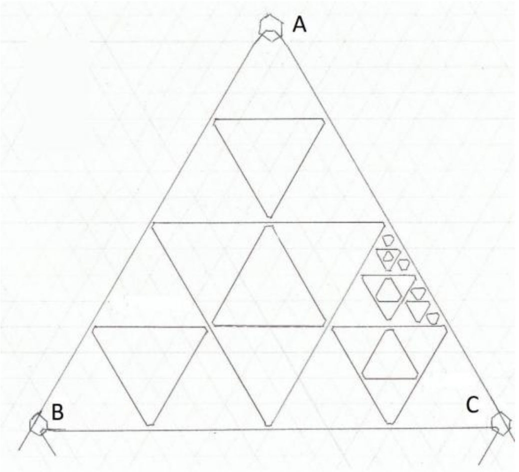
[Proposition 5]
For all n, Level n triangle is formed.

<Proof:end>

Look at the direction of tile B.



1. If tile B doesn't extend a straight line.
   Proposition 5 is shown by proposition 1.



2. If B extends a straight line. Tile B and C must be as in figure. Therefore, proposition 5 is shown by proposition 2.

□

If the tiling has a period p, length of sides of triangles can not exceed p. This is inconsistent with lemma 5.

Therefore, the tiling is not periodic.

$\blacksquare$