Interval preserving map approximation of $3x + 1$ problem

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What is $3x+1$ problem?

Consider a function $f : \mathbf{N} \to \mathbf{N}$;

$$f(x) = \begin{cases} 3x+1, & \text{if } x \text{ is odd,} \\ x/2, & \text{if } x \text{ is even.} \end{cases}$$

Conjecture 1

For any natural number n, the sequence

$$f(n), f^2(n), f^3(n), \ldots$$

eventually reaches to 1.

(Posed by L. Collatz in 1930's)

Still attracting me for a quarter of a century...

Ultimate challenges

A million of people (mathematicians, computer scientists or math-lovers) has been attacking this problem.

Surveys

G. J. Wirsching,
"The dynamical system generated by the $3n+1$ function", Springer, 1998.
J. C. Lagarias,
"The Ultimate Challenge: The $3x+1$ Problem", AMS, 2011.

Verification by computer

Verified up to

$$20 \cdot 2^{58} = 5764607523034234880 > 5.764 \cdot 10^{18}.$$

(by Oliveira e Silva, Jan. 2009.)

Erdös commented: "*Mathematics is not yet ready for such problems.*"

As a Dynamics in $\mathbf{Z}_2$

The dyadic integers $\mathbf{Z}_2$:

$$\mathbf{Z}_2 = \{x = (\cdots x_2 x_1 x_0)_2 \mid x = \sum_{k=0}^{\infty} x_k \cdot 2^k,\ x_k = 0, 1\}$$

equipped with a distance $d_2(x, y)$; for $x, y \in \mathbf{Z}_2$,

$$d_2(x, y) = 2^{-\ell}, \quad \text{where } \ell = \min_k \{x_k \neq y_k\},$$

and carries $c_k(x, y)$; the addition $x + y$ is given by

$$(x + y)_k = x_k + y_k + c_{k-1}(x, y) \mod 2$$
$$c_k(x, y) = \left[\frac{x_k + y_k + c_{k-1}(x, y)}{2}\right].$$

Natural numbers are identified with finite sequences in dyadic numbers:

$$\mathbf{N} = \{x \in \mathbf{Z}_2 \mid \exists \ell\ x_k = 0 \text{ for any } k \geq \ell\} \subset \mathbf{Z}_2.$$

As a Dynamics in $\mathbf{Z}_2$ (a natural idea)

The process $3x+1$ can be interpreted as (for odd x)

$$
\begin{array}{r|ccccc|l}
x & \cdots & x_3 & x_2 & x_1 & 1 & \\
2x & \cdots & x_2 & x_1 & 1 & 0 & \text{shift to upper digits,} \\
+ \quad 1 & \cdots & 0 & 0 & 0 & 1 & \text{odometer.} \\
\hline
3x+1 & * & * & * & x_1 & 0 &
\end{array}
$$

The process $x/2$ can be interpreted as (for even x)

$$
\begin{array}{r|ccccc|l}
x & \cdots & x_3 & x_2 & x_1 & 0 & \\
\hline
x/2 & \cdots & x_4 & x_3 & x_2 & x_1 & \text{shift to lower digits.}
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This kind of approaches often has been done. (cf. Lagarias's book)

...but I'd like to visualize these processes...

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Embedding $\mathbf{Z}_2$ into $[0,1]$

Consider $\beta : \mathbf{Z}_2 \to [0,1]$ given by

$$\beta((\cdots x_2 x_1 x_0)_2) = (0.x_0 x_1 x_2 \cdots)_2 = \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}}.$$

- Carrying to upper digits in $\mathbf{Z}_2$ corresponds to carrying to lower digits in $[0,1]$:

$$\begin{array}{rr|cccc} & 3 & \cdots 0 & 0 & 1 & 1 \\ + & 1 & \cdots 0 & 0 & 0 & 1 \\ \hline & 4 & \cdots 0 & 1 & 0 & 0 \end{array} \iff \begin{array}{rr|ccccc} & \beta(3) & 0. & 1 & 1 & 0 & 0\cdots \\ \beta(+) & \beta(1) & 0. & 1 & 0 & 0 & 0\cdots \\ \hline & \beta(4) & 0. & 0 & 0 & 1 & 0\cdots \end{array}$$

- β maps even numbers to $[0,1/2)$ and odd numbers to $[1/2,1)$.
- $\{\beta(n) \mid n \in \mathbf{N}\}$ and $\{\beta(3n+1) \mid n \in \mathbf{N}\}$ are dense in $[0,1]$ respectively.

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$$\begin{array}{rr|cccc} & 3 & \cdots 0 & 0 & 1 & 1 \\ + & 1 & \cdots 0 & 0 & 0 & 1 \\ \hline & 4 & \cdots 0 & 1 & 0 & 0 \end{array} \quad \Longleftrightarrow \quad \begin{array}{rr|ccccc} & \beta(3) & 0. & 1 & 1 & 0 & 0\cdots \\ \beta(+) & \beta(1) & 0. & 1 & 0 & 0 & 0\cdots \\ \hline & \beta(4) & 0. & 0 & 0 & 1 & 0\cdots \end{array}$$

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Conjugacy of Collatz procedure

Definition 4.1

The conjugacy $F : [0,1] \to [0,1]$ of the Collatz procedure f is defined by

$$F(x) = \begin{cases} 2x, & \text{for } x \in [0, 1/2), \\ \lim_{k\to\infty} \beta(3\beta^{-1}(x|_k) + 1), & \text{for } x \in [1/2, 1], \end{cases}$$

where $x|_k$ stands for the truncation of x at k-th digit in binary expansion:

$$x|_k = (0.x_1 x_2 \cdots x_k)_2 \ \text{ for } \ x = (0.x_1 x_2 \cdots x_k \cdots)_2.$$

Then we have the following commutative diagram.

$$\begin{array}{ccc} \mathbf{N} & \xrightarrow{f} & \mathbf{N} \\ \beta \big\downarrow & & \big\downarrow \beta \\ [0,1] & \xrightarrow{F} & [0,1] \end{array}$$

Conjugacy of Collatz procedure

Proposition 4.2

For any odd number n and $k \in \mathbf{N}$, F gives a right continuous bijection

$$F : [\ \beta(n)\ )_k \to [\ \beta(3n+1)\ )_k .$$

Here $[\ x\ )_k$ stands for an interval $\left[\ x|_k, x|_k + 2^{-k}\ \right)$ (called k-th segment).

- F is not left continuous: e.g.,

$$\lim_{\substack{w \to (0.11)_2 \\ w < (0.11)_2}} F(w) = (0.001)_2 \neq (0.0101)_2 = F((0.11)_2).$$

- This Proposition means that F behaves like an 'interval exchange map' on $[1/2, 1)$.

→ a graph of F

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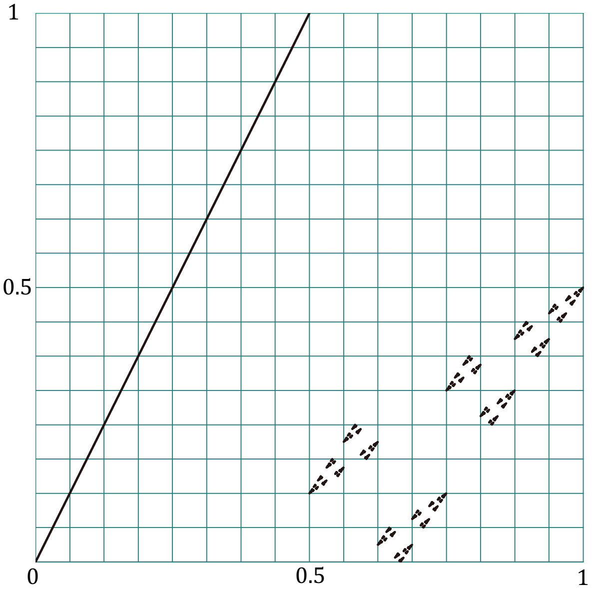
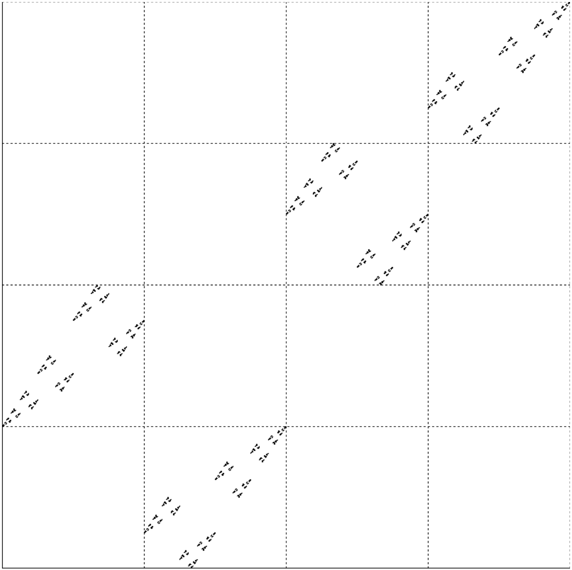
Graph of the conjugacy F



Collatz set $\mathfrak{C}$ – closure of graph F on $[1/2, 1]$



Collatz set $\mathfrak{C}$ – geometry

Theorem 5.1 (Y.H. 1998, 2007.)

$\mathfrak{C}$ is a Cantor space (perfect, compact, totally disconnected and metrizable), isometric to a self-similar set generated by the following iterated functional system on $[0,1]^2$,

$$\begin{aligned}
g_1(x,y) &= \frac{1}{2}(x+1, y+1), && \textit{fixes } (1,1),\\
g_2(x,y) &= \frac{1}{4}(x+1, y), && \textit{fixes } (1/3,0),\\
g_3(x,y) &= \frac{1}{4}(1-x, 2-y), && \textit{fixes } (1/5,2/5),
\end{aligned}$$

which has the Hausdorff dimension 1.

It seems to be difficult to analyze the dynamics on $\mathfrak{C}$...

Piecewise linear approximation of F

As F gives a right continuous bijection (Proposition 4.2),

$$F : [\ \beta(n)\ )_k \to [\ \beta(3n+1)\ )_k\,,$$

we consider a piecewise liner approximant of F:

Definition 6.1

For each $k \in \mathbf{N}$, we define the k-th approximant F_k as

$$\begin{aligned} F_k(x) &= \begin{cases} 2x, & \text{for } x \in [0, 1/2), \\ x - x|_k + F(x|_k)|_k, & \text{for } x \in [1/2, 1], \end{cases} \\ &= \begin{cases} 2x, & \text{for } x \in [0, 1/2), \\ x - x|_k + \beta(3\beta^{-1}(x|_k) + 1)|_k, & \text{for } x \in [1/2, 1]. \end{cases} \end{aligned}$$

cf.

$$F(x) = \lim_{k\to\infty} \beta(3\beta^{-1}(x|_k) + 1), \qquad \text{for } x \in [1/2, 1].$$

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- For $x \in [\ \beta(n)\ )_k$, we see

$$F_k(x) = x - \beta(n)|_k + \beta(3n+1)|_k.$$

- $F_k([\ \beta(n)\ )_k) = [\ \beta(3n+1)\ )_k$ for any odd number n and $k \in \mathbf{N}$.
- Thus the sequence F_k, $k = 1, 2, \ldots$ approximates F uniformly on $[0, 1]$.

Behavior of carries

-To observe the dynamics of F_k

For $n, k \in \mathbf{N}$, we define an integer valued function

$$\tau_k(n) = \left[\frac{3n|^k + 1}{2^k}\right].$$

Here, for a binary expression $n = (\cdots a_k a_{k-1} \cdots a_0)_2$, $n|^k$ denotes an *upper cut off* of n at k-th order;

$$n|^k = (a_{k-1} a_{k-2} \cdots a_0)_2 \equiv n \mod 2^k.$$

The function τ_k describes the number of bits carried in the calculation of $3n + 1$ at k-th bit.

Behavior of carries

Proposition 6.2

Given an odd number n and take $k \in \mathbf{N}$, then we have

- $\tau_k(n) \in \{0, 1, 2\}$.
- $$\tau_{k+1}(n|^k) = \begin{cases} 0, & \textit{if } \tau_k(n) = 0, 1, \\ 1, & \textit{if } \tau_k(n) = 2, \end{cases}$$
  $$\tau_{k+1}(n|^k + 2^k) = \begin{cases} 1, & \textit{if } \tau_k(n) = 0, \\ 2, & \textit{if } \tau_k(n) = 1, 2. \end{cases}$$

Here

$$\tau_k(n) = \left[\frac{3n|^k + 1}{2^k}\right].$$

Behavior of carries

- Note that

$$\left[\ \beta(n|^k)\ \right)_k = \left[\ \beta(n|^k)\ \right)_{k+1} \oplus \left[\ \beta(n|^k + 2^k)\ \right)_{k+1},$$

e.g., $[\ (0.111)_2\ )_3 = [\ (0.1110)_2\ )_4 \oplus [\ (0.1111)_2\ )_4$.

Proposition 6.3

If $\tau_k(n) = 0$ *or* 2,

$$\left[\ F(\beta(n|^k))\ \right)_k = \left[\ F(\beta(n|^k))\ \right)_{k+1} \oplus \left[\ F(\beta(n|^k + 2^k))\ \right)_{k+1}.$$

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Substitution dynamics

For an odd number n and $k \in \mathbf{N}$, we label the segments $[\ \beta(n)\ )_k$ as follows. $\rightarrow$

$\tau_k(n)$	Label of $[\ \beta(n)\ )_k$
0	S
1	E
2	U

- From Proposition 6.2 and 6.3, to increment the approximation order k by 1 causes a division of each segment, and induces a substitution

$$\sigma : S \to SE \qquad E \to SU \qquad U \to EU,$$

which are mapped by F as

$$F(\sigma) : F(S) \to F(S)F(E) \quad F(E) \to F(U)F(S) \quad F(U) \to F(E)F(U).$$

- The original segment $[\ \beta(1)\ )_1 = [\ 1/2, 1\ )$ is labeled as U.

picture of σ

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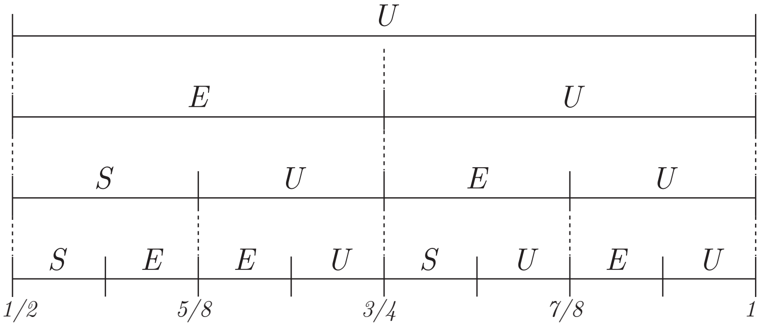


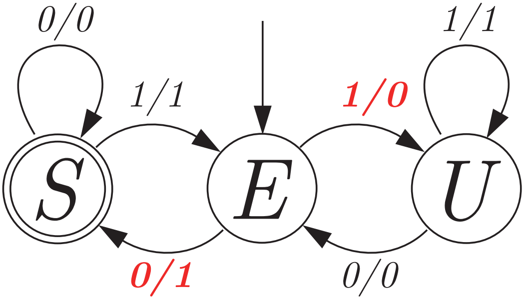
Transducer

The calculation

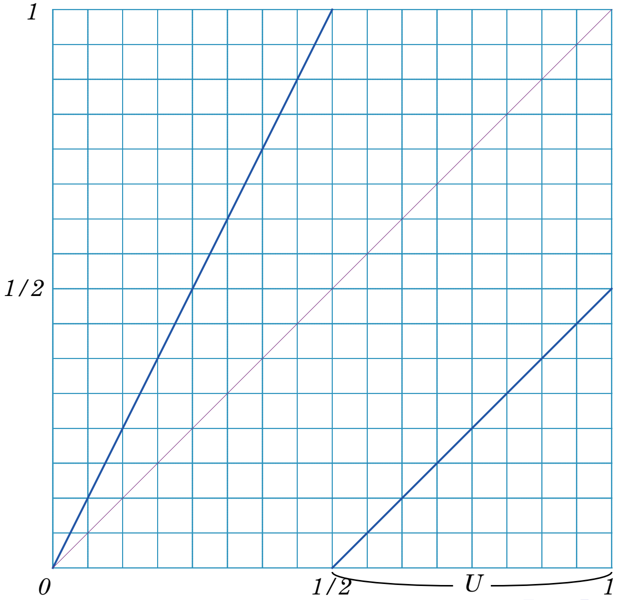
$$f : 3 = (0011)_2 \mapsto 3 \times 3 + 1 = (1010)_2$$

is given by the transducer as follows:

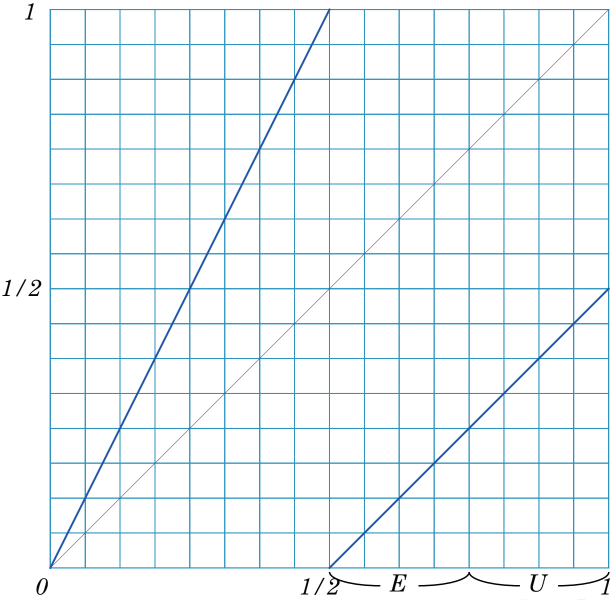
$$\text{state: } E \underset{\substack{\downarrow \\ 0}}{\overset{\substack{1 \\ \downarrow}}{\to}} U \underset{\substack{\downarrow \\ 1}}{\overset{\substack{1 \\ \downarrow}}{\to}} U \underset{\substack{\downarrow \\ 0}}{\overset{\substack{0 \\ \downarrow}}{\to}} E \underset{\substack{\downarrow \\ 1}}{\overset{\substack{0 \\ \downarrow}}{\to}} S.$$



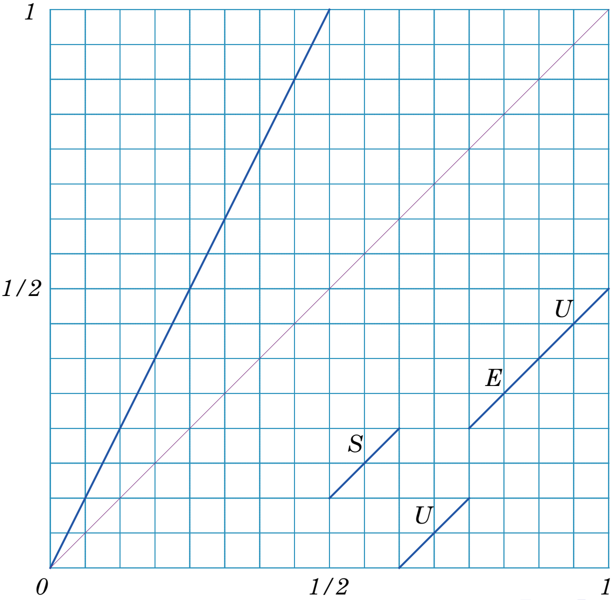
Approximation order $k = 1$



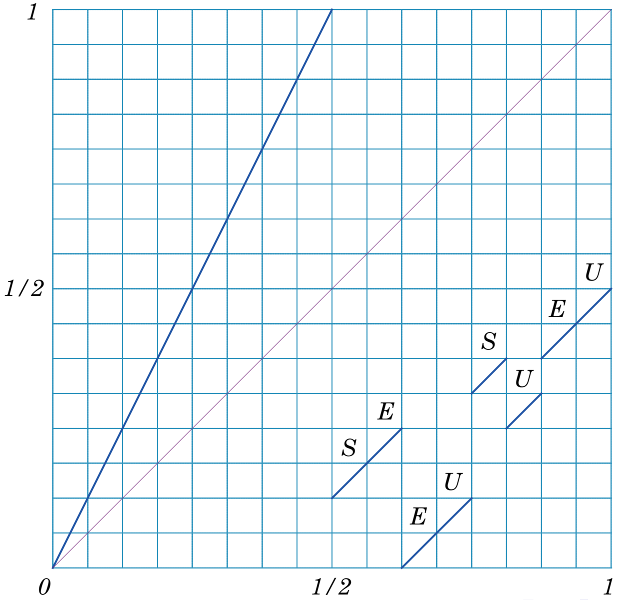
Approximation order $k = 2$



Approximation order $k = 3$



Approximation order $k = 4$



Property and Problem on F_k

F_k just exchanges the segments $[\ x\ )_k$ on $[1/2, 1]$ and expands the segments on $[0, 1/2)$: eg.,

$$F_2 : \beta(13) = (0.\underline{10}11)_2 \xrightarrow{F_2} (0.\underline{00}11)_2 \xrightarrow{2x} (0.011)_2 \xrightarrow{2x} (0.11)_2 = \beta(3).$$

Proposition 6.4

The k-th approximant F_k contracts any finitely long binaries to at most k-bit sequences. That is, for any natural number n, there exists $t \in \mathbf{N}$ such that

$$\beta^{-1}(F_k^t(\beta(n))) \leq 2^k.$$

- Thus the orbit starts from any finitely long binary sequence $\beta(n)$ is attracted to the orbit of some k-bit sequence.

Property and Problem on F_k

- Then we just observe the orbits consist of k-bit sequences to answer the following 'Collatz-like' problem on F_k:

Problem 2 ($3x + 1$ problem on F_k)

Show that for any natural number n, there exists $t \in \mathbf{N}$ such that

$$F_k^t(\beta(n)) = 0 \text{ or } 1/2(= \beta(1)).$$

- But the expanding part $F_k(x) = 2x$ on $[0, 1/2)$ causes some difficulties to analyze the dynamics.

Interval preserving approximation of F

We introduce a modification of F_k.

Definition 7.1

For each $k \in \mathbf{N}$, we define the k-th approximant G_k as

$$
\begin{aligned}
G_k(x) &= x - x|_k + F(x|_k)|_k, && \text{for } x \in [0,1), \\
&= \begin{cases} x + x|_k, & \text{for } x \in [0,1/2), \\ x - x|_k + \beta(3\beta^{-1}(x|_k)+1)|_k, & \text{for } x \in [1/2,1), \end{cases} \\
&= \begin{cases} x + \beta(n)|_k, & \text{for } x \in [\ \beta(n)\ )_k \subset [0,1/2), \\ x - \beta(n)|_k + \beta(3n+1)|_k, & \text{for } x \in [\ \beta(n)\ )_k \subset [1/2,1). \end{cases}
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- G_k is just a translation of each segment $[\ x\ )_k$ to another one:

$$G_k : [\ x\ )_k \to [\ G_k(x)\ )_k\ .$$

- Thus the orbit of any point $x \in [0,1)$ is eventually periodic, described completely by the orbit of the k-bit sequence $x|_k$.

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Interval preserving approximation of F

Problem 3 ($3x + 1$ problem on G_k)

Show that for any $x \in [0, 1)$, there exists $t \in \mathbf{N}$ such that

$$G_k^t(x) \in [\ 0\ )_k \cup [\ \beta(1)\ )_k = \left[\ 0,\ 1/2^k\ \right) \cup \left[\ 1/2,\ 1/2 + 1/2^k\ \right).$$

- Problem 3 reduces Problem 2 to a finite combinatorics.
- However Problem 3 seems to be not trivial.

  Consider the map $5x + 1$ (instead of $3x + 1$).

  $\Rightarrow$ *Lots of periodic orbits appear as increasing the approximation order k.*

$\Rightarrow$ Problem 3 indicates the unique characteristics of the map $3x + 1$.

From G_k to original F

The original $3x+1$ problem can be solved by two processes:

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and

Problem 4

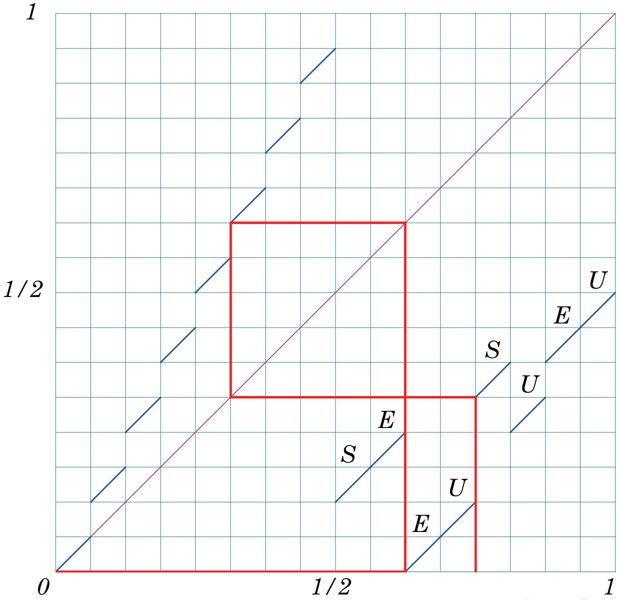
Show that for any $n \in \mathbf{N}$, there exists $k \in \mathbf{N}$ such that for any $t \in \mathbf{N}$

$$F^t(\beta(n)) = G_k^t(\beta(n))$$

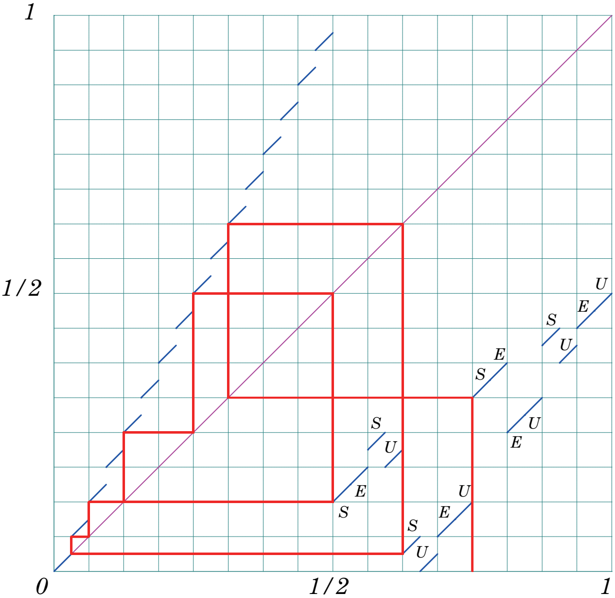
holds.

Maybe hard problem...

Orbit of $3 = (11)_2$ under G_4



Orbit of $3 = (11)_2$ under G_5



A conjecture arisen from a graph symmetric to F

Consider a right continuous map H:

$$H(x) = \begin{cases} \lim_{k\to\infty} \beta(3\beta^{-1}(x|_k) + 1), & x \in [0, 1/2), \\ 2x - 1, & x \in [1/2, 1). \end{cases}$$

- H is symmetrical about $(1/2, 1/2)$ with F on $\beta(\mathbf{N})$.
- H is a 'left continuous' version of F.
- H is the conjugacy of the following arithmetic procedure h:

$$h(n) = \begin{cases} 3n + 1, & \text{if } n \text{ is even}, \\ (n-1)/2, & \text{if } n \text{ is odd}. \end{cases}$$

$\rightarrow$ graph of H

A conjecture arisen from a graph symmetric to F

Consider a right continuous map H:

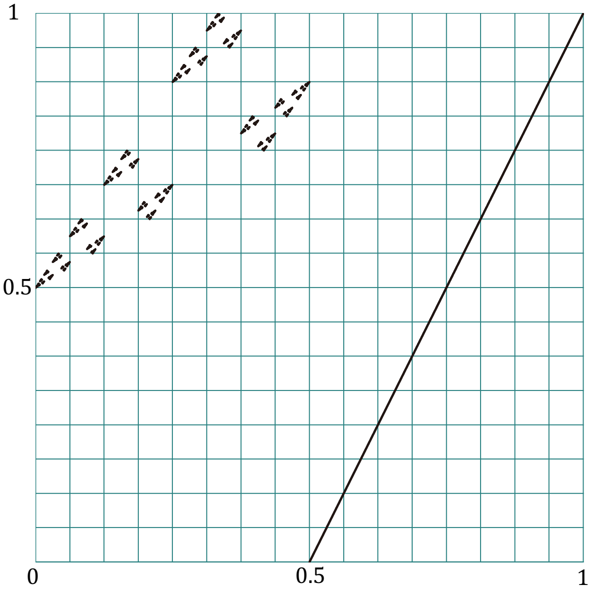
$$H(x) = \begin{cases} \lim_{k\to\infty} \beta(3\beta^{-1}(x|_k) + 1), & x \in [0, 1/2), \\ 2x - 1, & x \in [1/2, 1). \end{cases}$$

- H is symmetrical about $(1/2, 1/2)$ with F on $\beta(\mathbf{N})$.
- H is a 'left continuous' version of F.
- H is the conjugacy of the following arithmetic procedure h:

$$h(n) = \begin{cases} 3n + 1, & \text{if } n \text{ is even,} \\ (n-1)/2, & \text{if } n \text{ is odd.} \end{cases}$$

→ graph of H

A conjecture arisen from a graph symmetric to F



A conjecture arisen from a graph symmetric to F

- H is the conjugacy of the following arithmetic procedure h:

$$h(n) = \begin{cases} 3n+1, & \text{if } n \text{ is even,} \\ (n-1)/2, & \text{if } n \text{ is odd.} \end{cases}$$

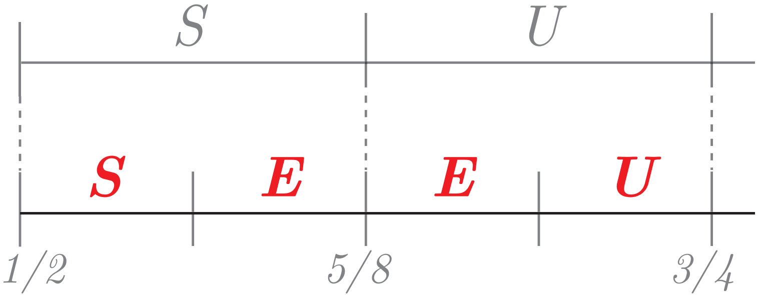
By a computer verification, we pose the following conjecture.

Conjecture 5

For any natural number n, the sequence

$$h(n), h^2(n), h^3(n), \ldots$$

eventually reaches to 1, 4 or 16.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1/2
5/8
3/4