Can we construct an easy mono-tile ?

Shigeki Akiyama, Niigata Univ.

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A **tile** is a compact set which is the closure of its interior. Consider a finite set of tiles $\mathcal{A}$ as an **alphabet**. A **tiling** $\mathcal{T}$ of $\mathbb{R}^d$ is a collection of tiles which covers $\mathbb{R}^d$ without interior intersections, and each tile is congruent to an element of $\mathcal{A}$ under rigid motion of $\mathbb{R}^d$. A tiling $\mathcal{T}$ has a **period** $p \in \mathbb{R}^d$ when the tiling exactly matches with its translation by $p \in \mathbb{R}^d$. A tiling $\mathcal{T}$ is **non periodic** if the only period of $\mathcal{T}$ is 0. A set of tiles $\mathcal{A}$ is called **aperiodic** if $\mathcal{A}$ generates a tiling, but all tilings generated by $\mathcal{A}$ are non periodic. A well known example of aperiodic tiles is due to Penrose which consists of two kinds of tiles: kites and darts with matching rules as in Figure 1:

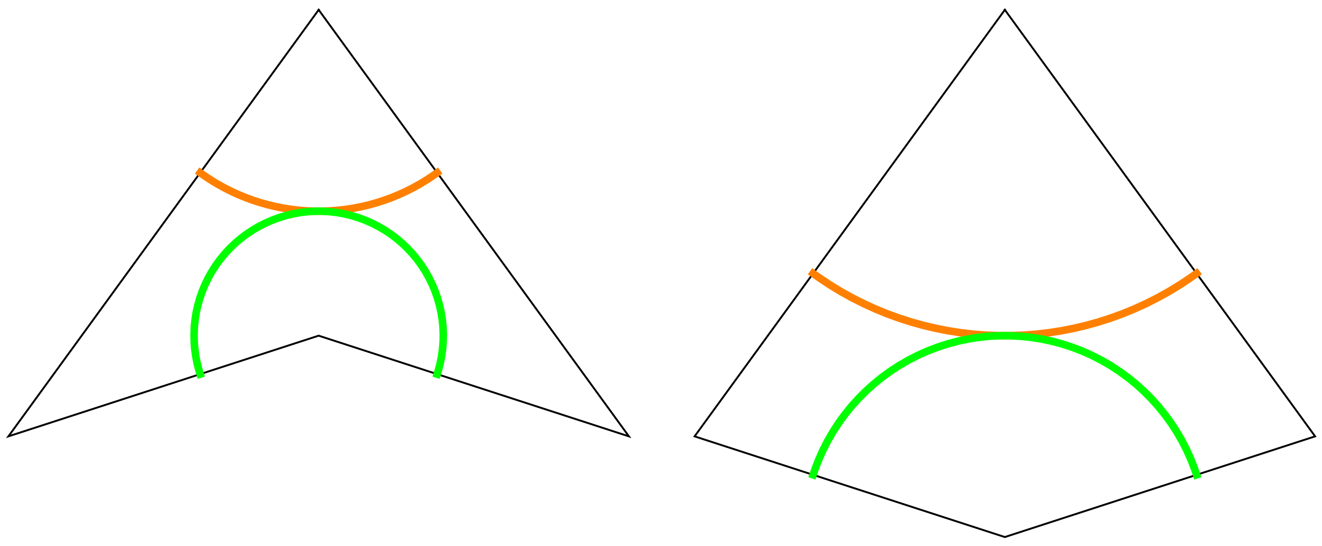


Figure 1: Penrose Tile

that the circular markings must match at the boundary like:

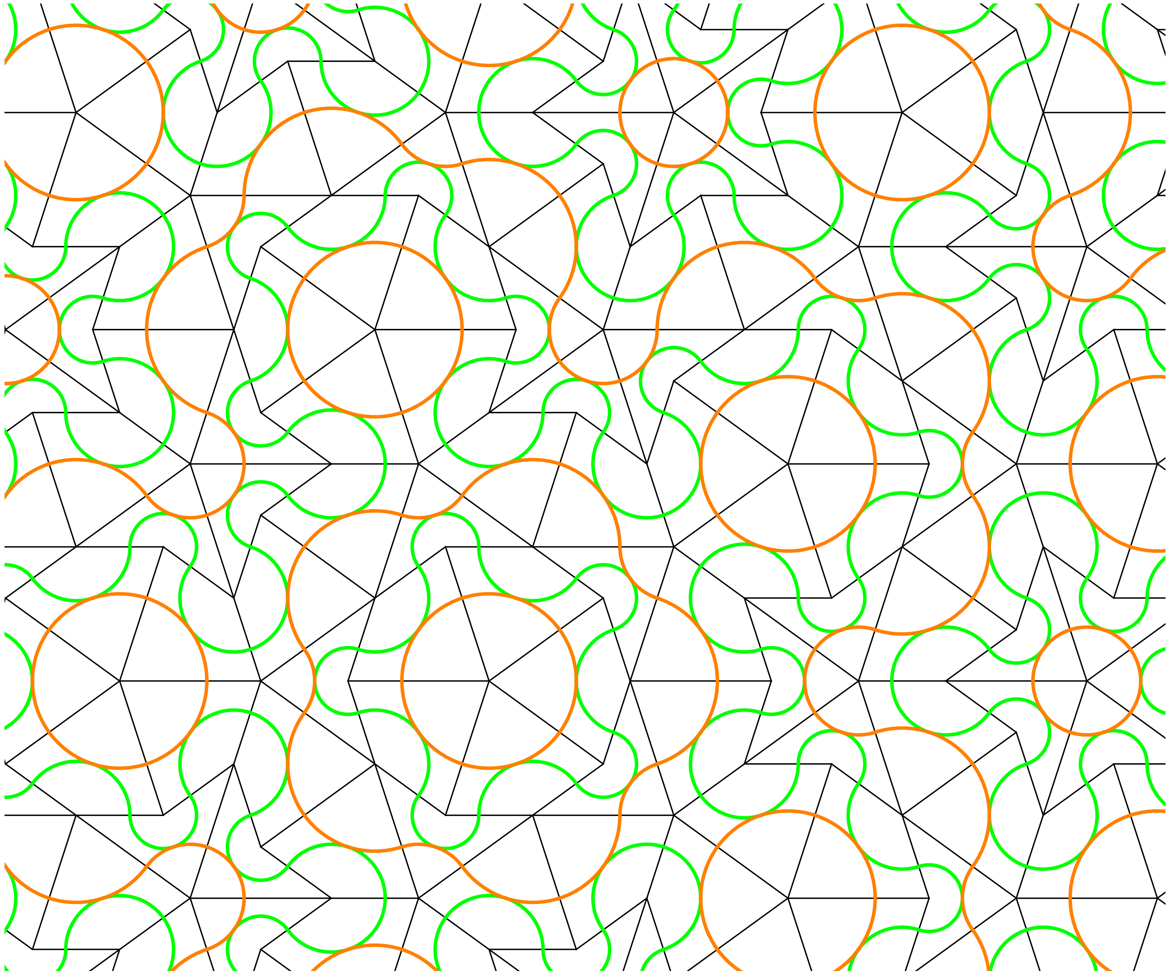


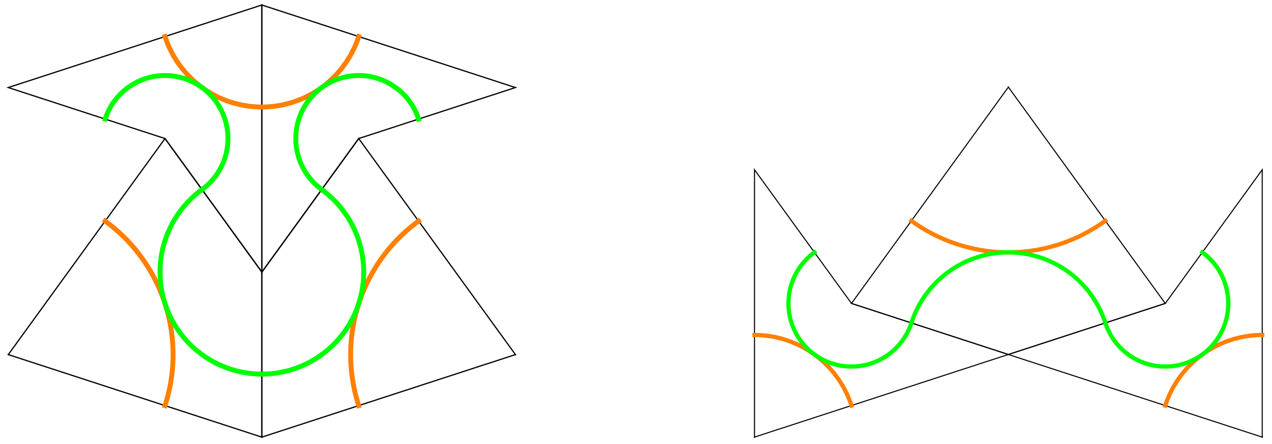
Figure 2: Penrose tiling

To show that Penrose tiles are aperiodic, we have to show two things.

- They admit a tiling.

- Each tiling generated by kite and dart has no period.

Both are shown through **self-similarity**. First we show that there is a substitution rule which make a tile larger and then substitute to a larger patch. Second we prove the **unique composition** property. However both of them are quite non-trivial.



(a) Kite (b) Dart

Figure 3: Substitution rule

Fix a point on the kite. Then the points on the kite of the same orientation forms a **Delone set**, the distance to the nearest point is bounded from below, and there is no big hole without a point. de Bruijn [5] showed that Penrose tiling is really a good model of quasi-crystal by showing that such Delone set is understood by **cut and projection**. More precisely this Delone set is produced by the projection of 5-dim lattice points which lies in some irrational band. The cut and projection from 2-dim to 1-dim is described in Figure 4.

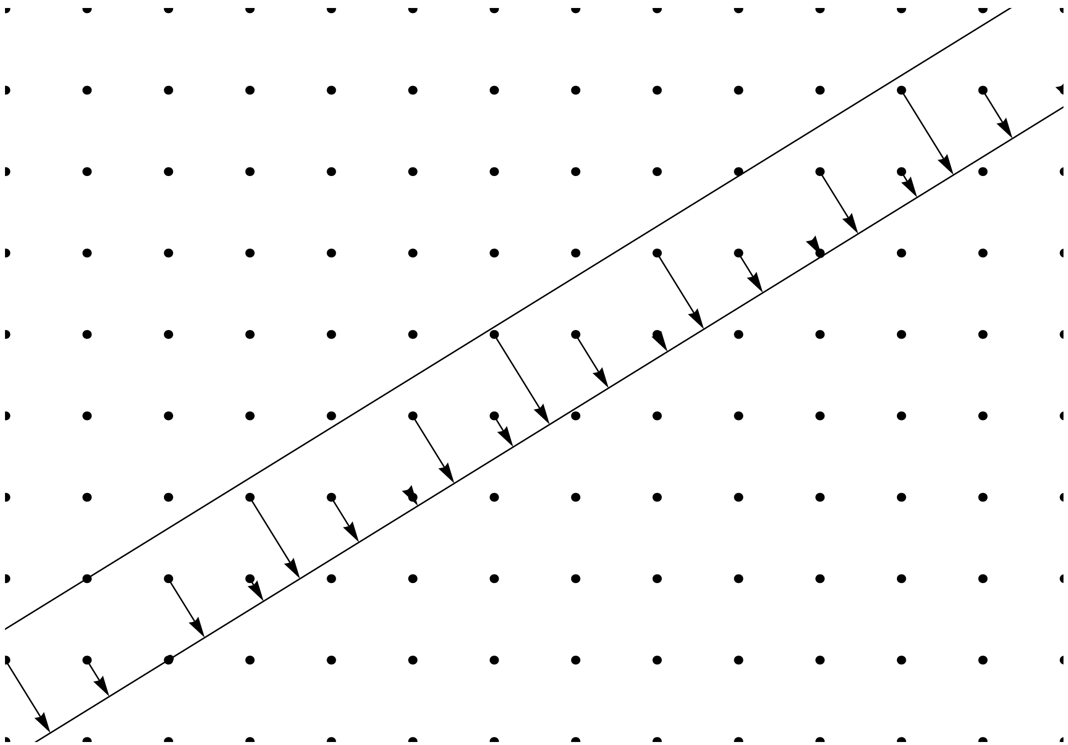


Figure 4: Cut and Projection

Tiling dynamical system and diffraction

Tiling dynamical system is a topological dynamics generated by the orbit closure of translation of a single tiling. This can be viewed as a generalization of symbolic substitution dynamical system.

To study diffraction of such Delone sets, tiling dynamical system is studied in detail. Under certain condition, this dynamics becomes uniquely ergodic and admit a spectral study. It is known that pure pointedness of dynamical spectrum is equal to those of diffraction spectrum.

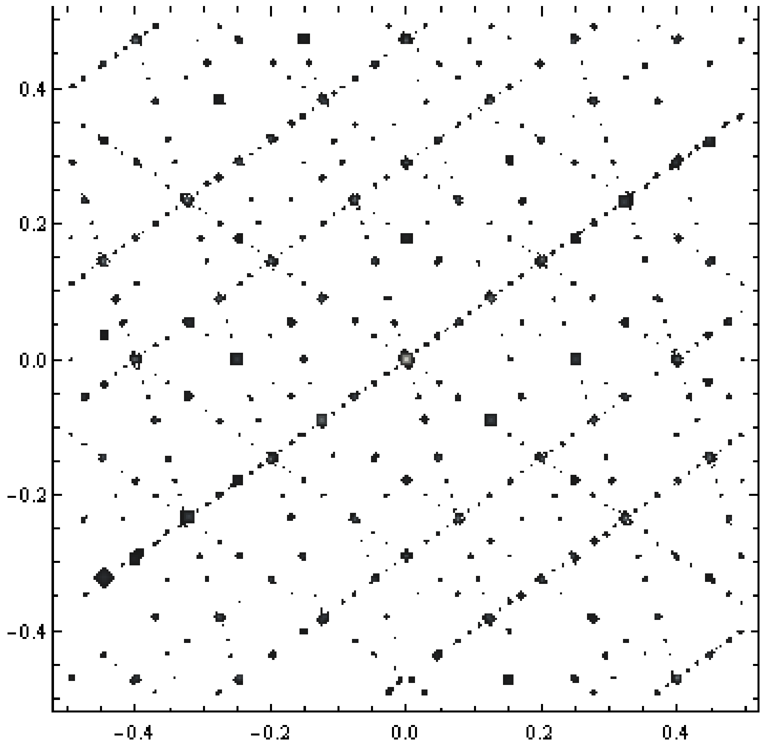


Figure 5: Penrose Diffraction

From this diffraction, we can imagine the 5-dim structure.

R. Ammann, an amateur mathematician working at post office, was strongly motivated by Penrose tiling when he read the article in Scientific American. He found other type of aperiodic tiles (c.f. [7, 3, 11]). Here is one of them with two different markings:

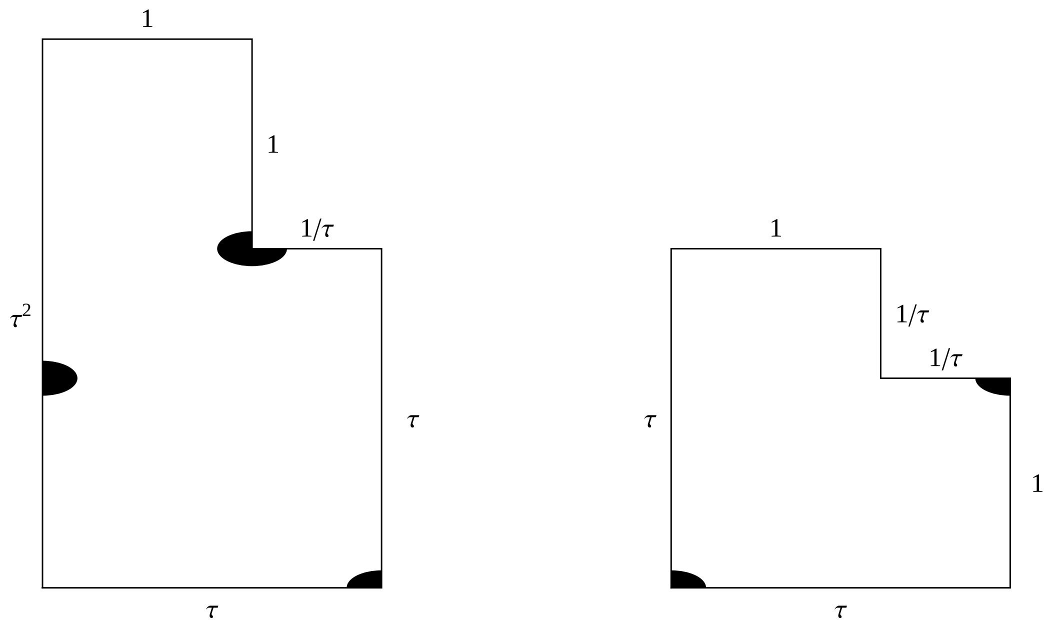


Figure 6: Ammann Tile

The black markings must form complete ovals at the boundary:

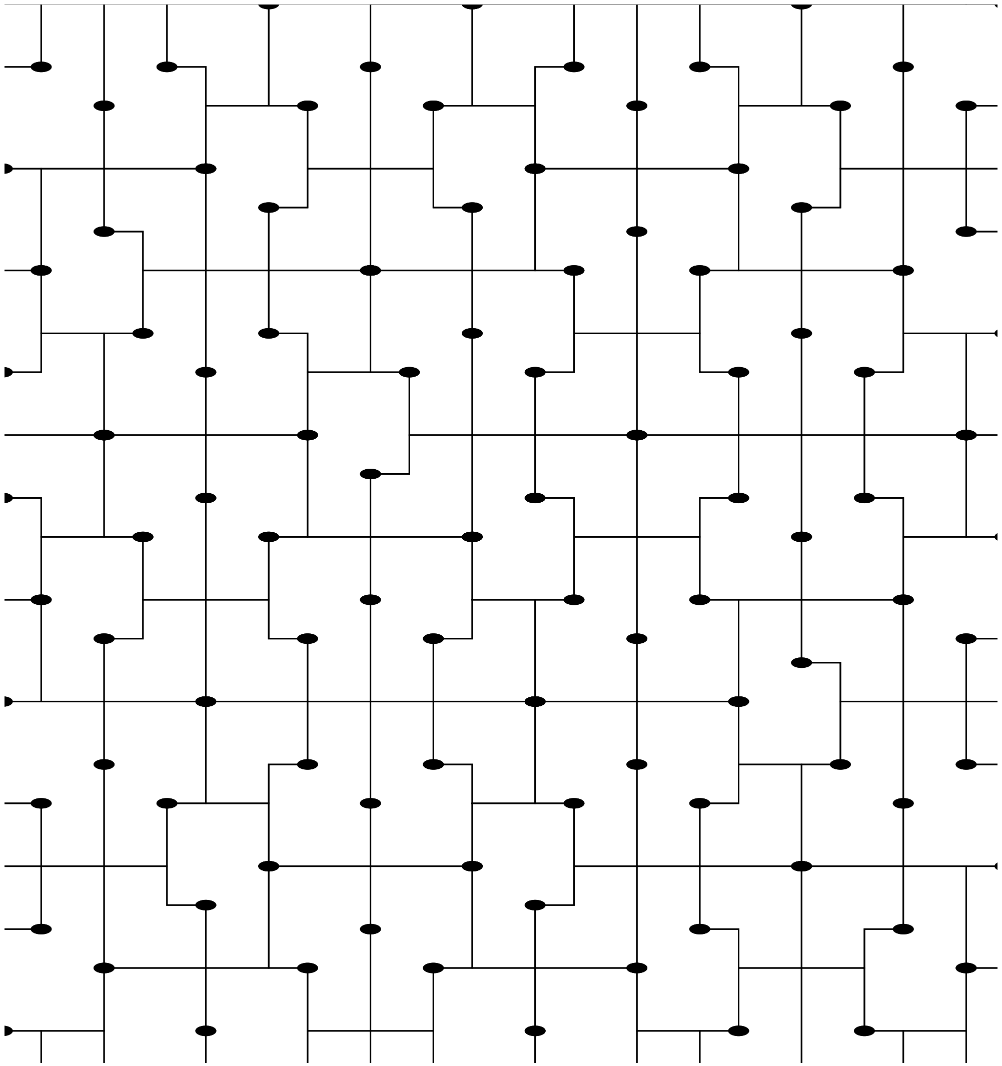


Figure 7: Ammann Tiling

The second striking marking by segments must be continued to a straight line or a dashed line, now called **Ammann Bars**.

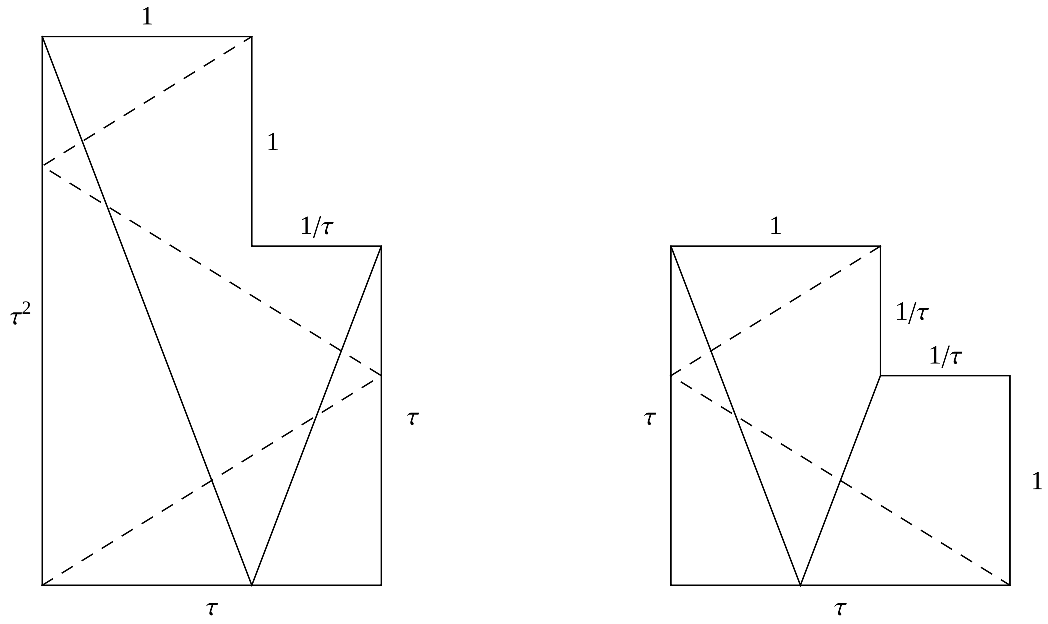


Figure 8: Ammann Tile 2

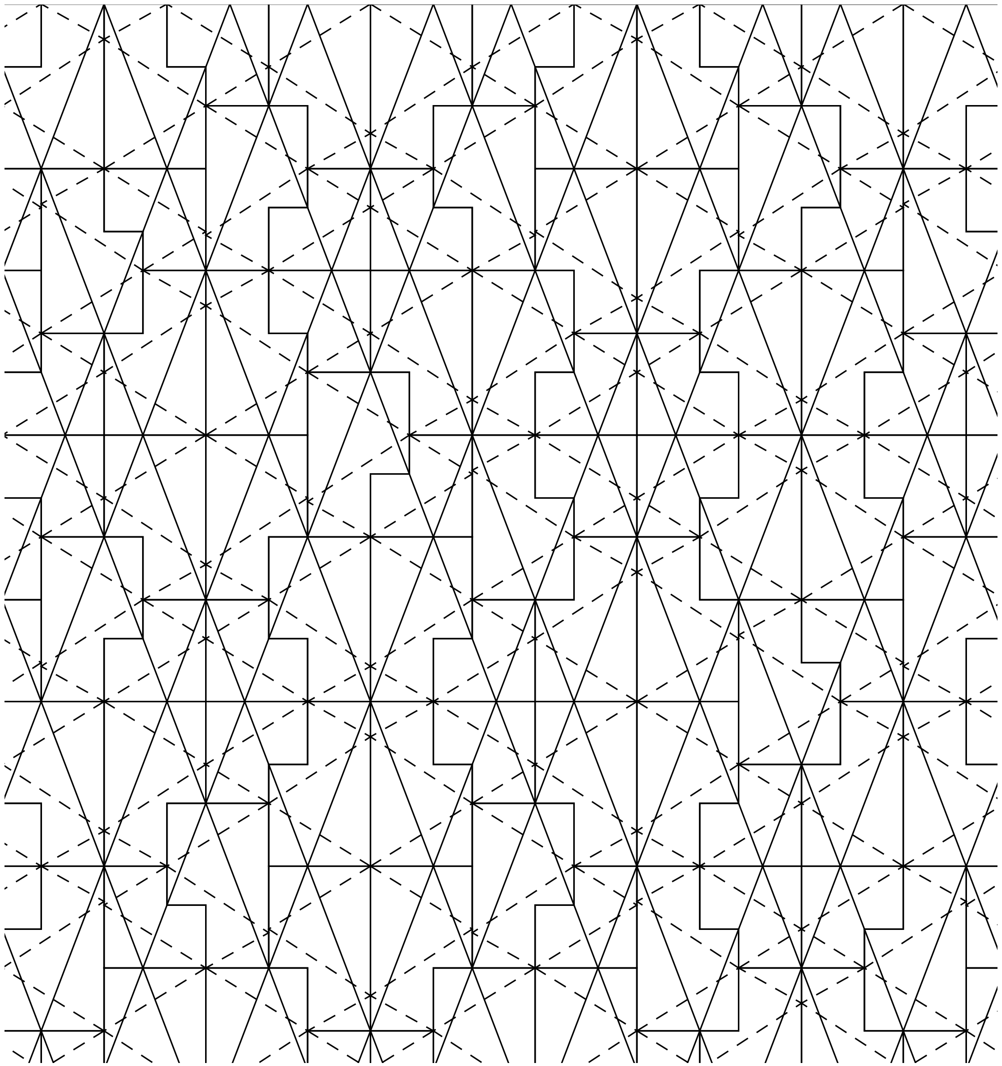


Figure 9: Ammann Bar

We guess that these lines came from higher dimensional lattice structure, which shows Ammann's strong mathematical talent. As far as I know, it is still a mysterious construction to this date. The proof of aperiodicity of Ammann tiles is again by self-similarity in the same course as Penrose tiling. None of them is trivial.

In this talk, we propose a variant of Ammann tiles which provides us the simplest marking, as far as I know:

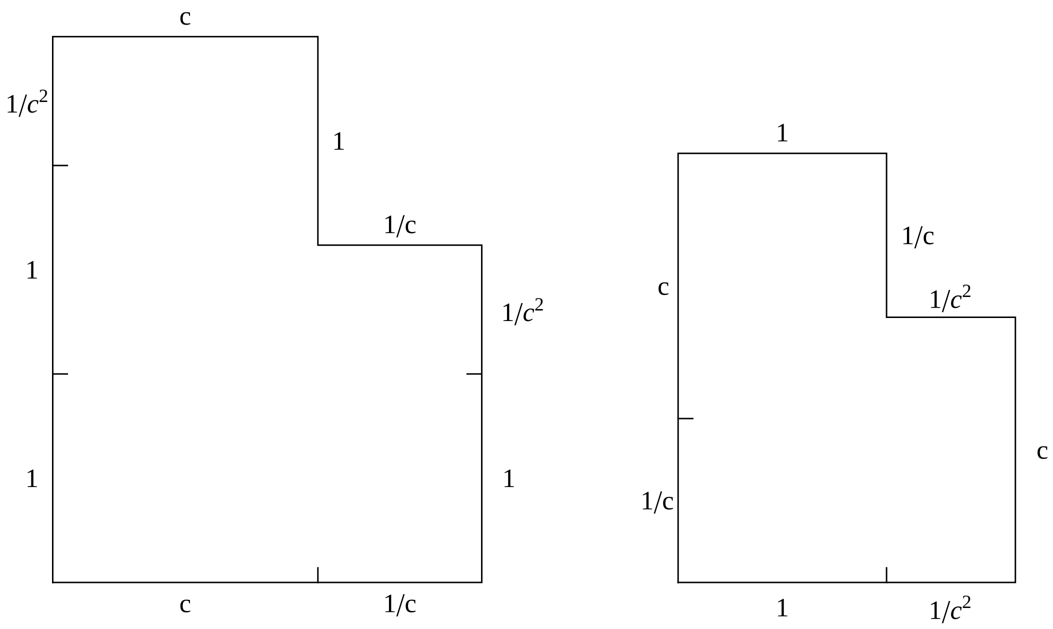


Figure 10: Similar Ammann tiles A and B

having slightly modified shapes on which Grünbaum-Shephard [7] claimed:

'However it does not seem possible to devise a simple matching condition for these tiles involving markings which are such that the similarity between the tiles extends to markings as well.'

and proposed a pretty complicated matching rules on the edges to enforce aperiodicity. Here $c = 1.272\ldots$ is a square root of $(1+\sqrt{5})/2$. The marking on the edges means that edges are cut into smaller segments at those points. Our only matching rule is **edge to edge**: the edges exactly match from one end to the other in the tiling.

In fact, this work is motivated by our former work [2] to find a self-similar dissection of a give sets. The shapes seem to be independently found in his similar dissection puzzle of Scherer [10]. He called it ‘Golden Bee’, which is a polygon composed of two similar copies of itself (see Figure 15).

Tiling property

We show that two tiles in Figure 10 admit a tiling. First we discard the marking and consider only shapes. Our substitution rule makes c-times larger A and B. Then A is divided into A and B as in Figure 15 and B becomes A. The results are depicted in Figure 11.

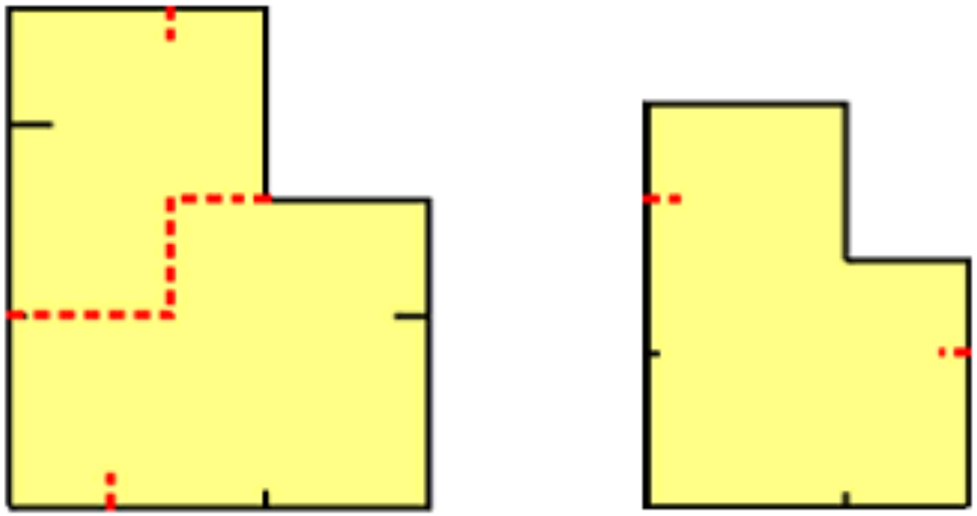


Figure 11: Super-tiles cA and cB: broken lines indicate 4 ghost markings and the former boundary

Starting from the tile A, we successively obtain a larger patch by this substitution rule like Figure 12.

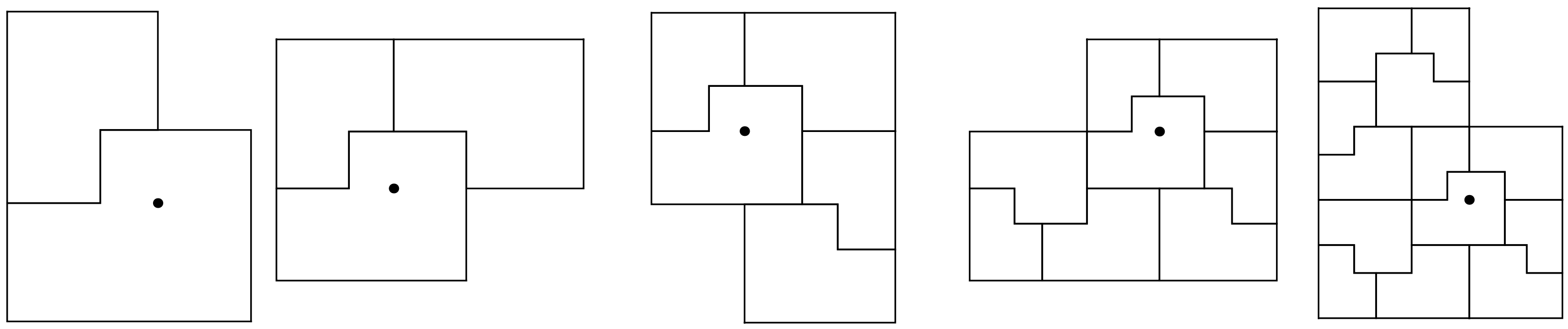


Figure 12: Inflation subdivision

This gives a tiling of the plane by A and B, which is called a fixed point of the substitution rule:

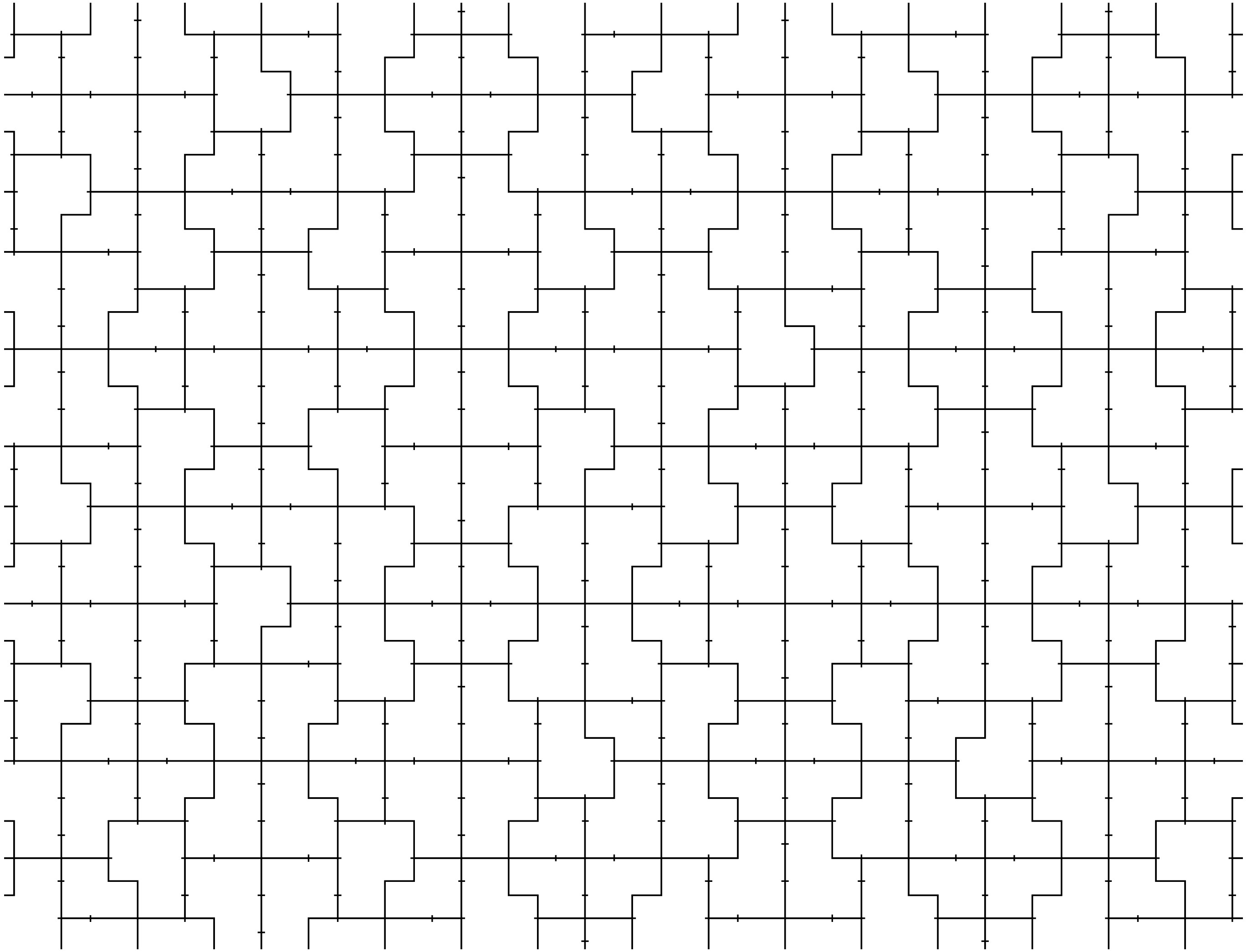


Figure 13: A patch of tiling by A and B

We may view this fixed point as a super-tiling by c^3A and c^3B without markings. Then c^3A is the composition of three A's and two B's, and c^3B is a composition of two A's and one B. Boundaries of A and B give natural markings of super-tiles c^3A and c^3B as black points in Figure 14. Confirm that they are exactly located at the c^3 scaled markings of A and B.

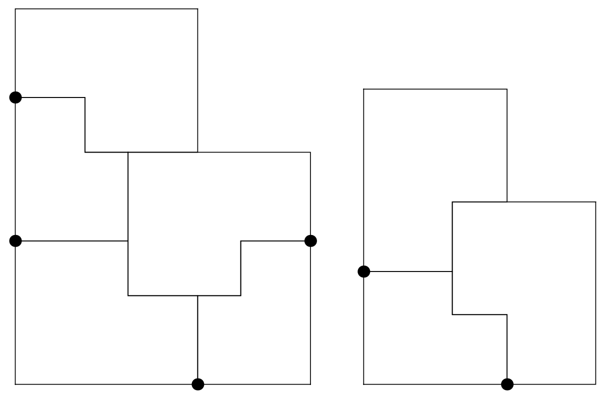


Figure 14: Markings from $c^3 A$ and $c^3 B$

As the tiling by $c^3 A$ and $c^3 B$ surely exists, we know that there exists a tiling by A and B with the markings scaled down by ratio $1/c^3$.

Aperiodicity

Now we show the aperiodicity of two tiles in Figure 10 by proving **unique composition property**, that is, there is a unique super-tiling structure. Firstly, it is plain to see that there are no way to tile $\mathbb{R}^2$ only by A and the dent of B can be filled only by A as in Figure 15.

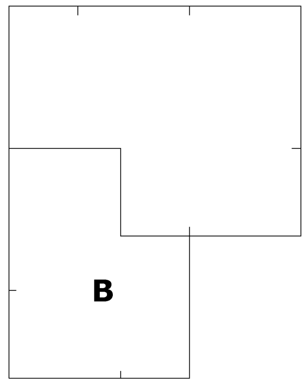


Figure 15: Filling the dent of B

As we already see that this composition of A and B is the inflated image of A by ratio c. We call this super-tile A'. The remaining A which does not fill the dent of B is seen as a super-tile B', the inflated image of B by ratio c. We have to show that additional marking can be removed. In other word, if we could show that the tiles in Figure 11 can be identified with cA and cB **including markings**, we are done.

Now we meet the main difficulty. We have to remove the red dashed markings indicated in Figure 11. However, the effect of removal is involved. So we treat them as **Ghost markings**, meaning that we can think that they exist or not exist as we wish.

So a ghost marking can be used to cut the edge but we can also neglect it. Under this treatise, the set of tilings by two tiles in Figure 11 is surely larger than those without such ghost markings. Then we show that ghost markings do no harm by showing that the patches which essentially use ghost markings, do not grow into a tiling. We have 20 cases to check out.

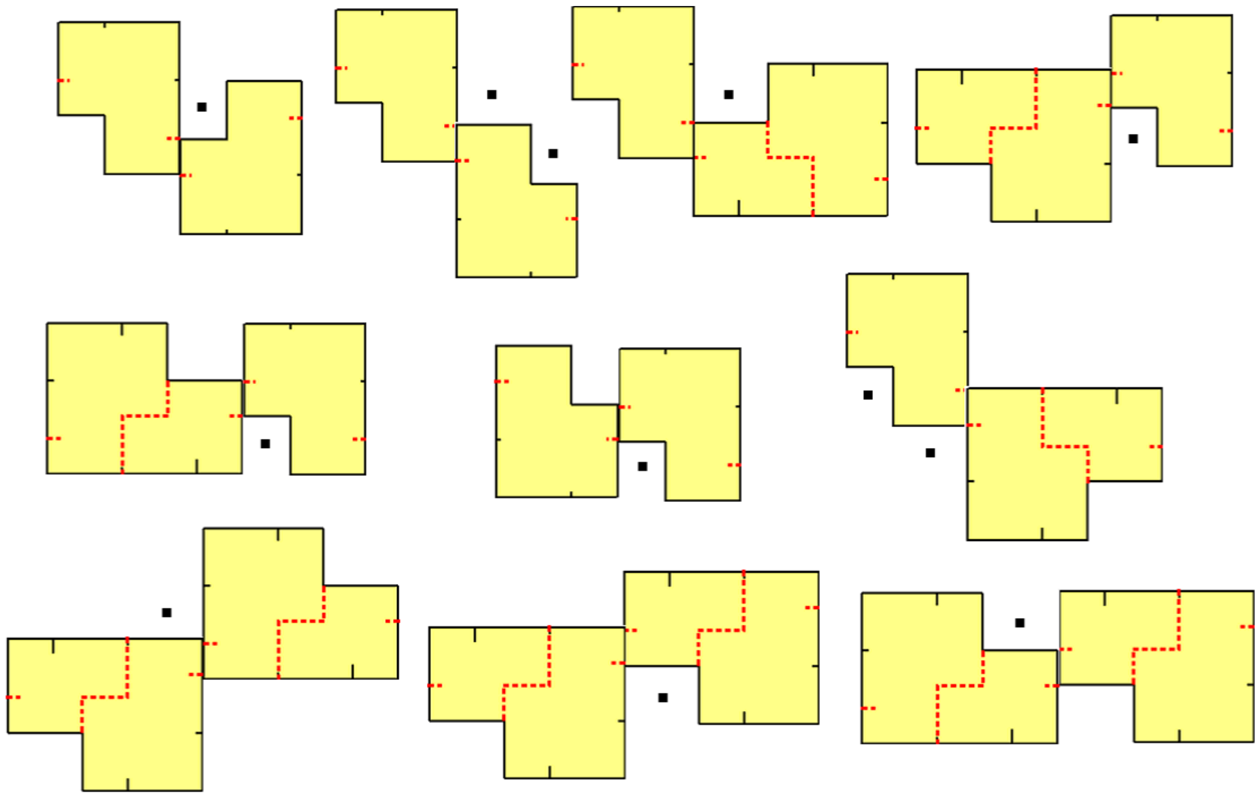


Figure 16: Illegal patches of the 1-st kind

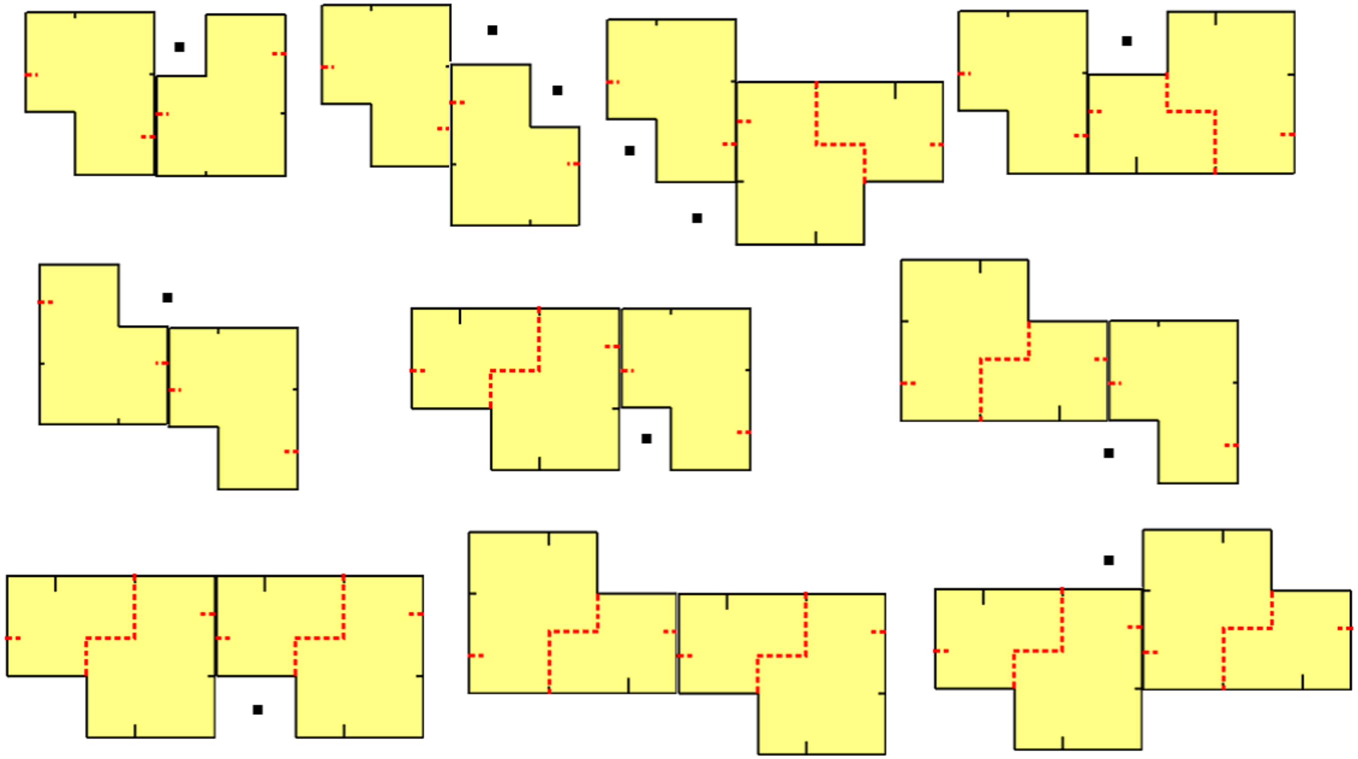


Figure 17: Illegal patches of the 2nd kind

Diffraction property

Similar to Penrose tiling, D. Frettlöh [6] showed that the Delone set of Ammann tiling is given by cut and projection. Thus Ammann tiling gives a pure diffractive point set.

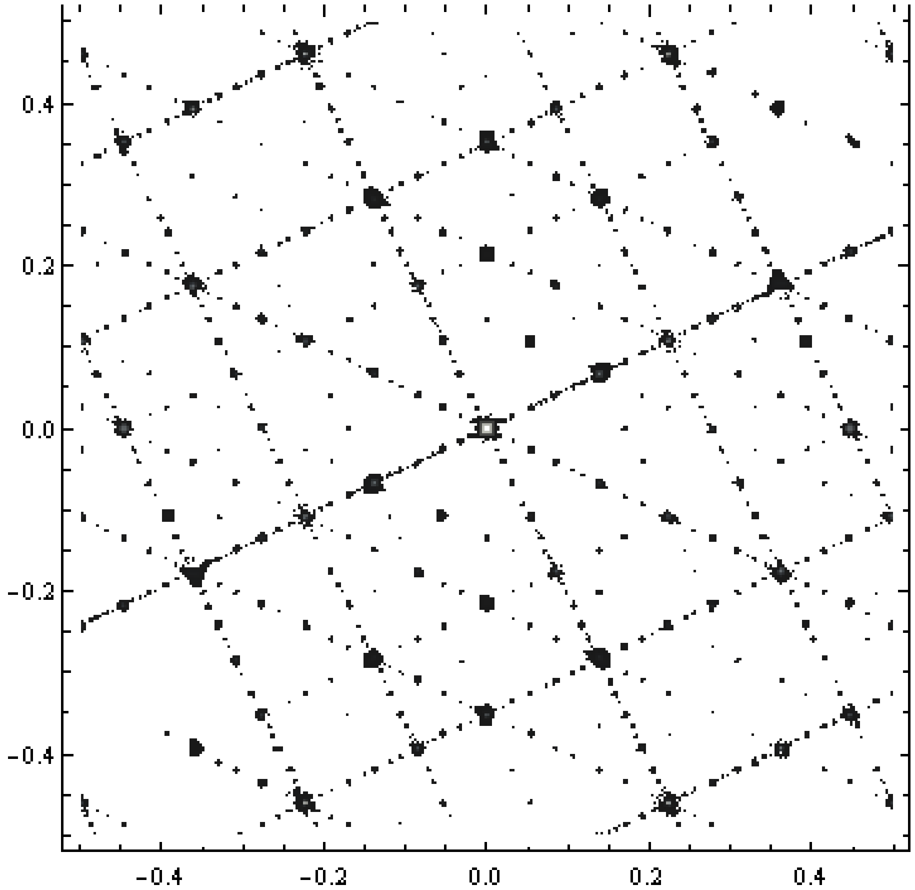


Figure 18: Diffraction pattern of Ammann tiling

Questions

- Is there an aperiodic mono-tile ? **Solved**: Socolar-Taylor [12].

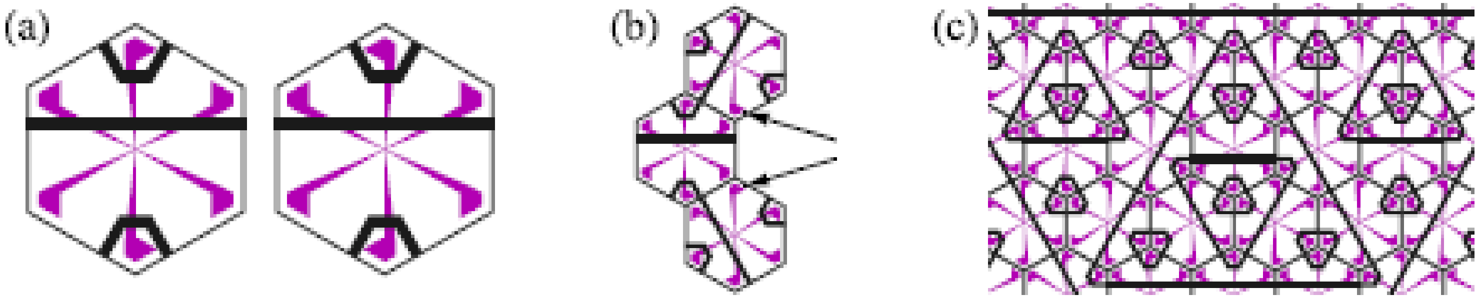


Figure 19: Socolar-Taylor's aperiodic monotile

- Is there an edge to edge aperiodic mono-tile ? Our construction suggests some possibility using rep-tile, that is, a tile consisting of same-sized similar copies of itself.

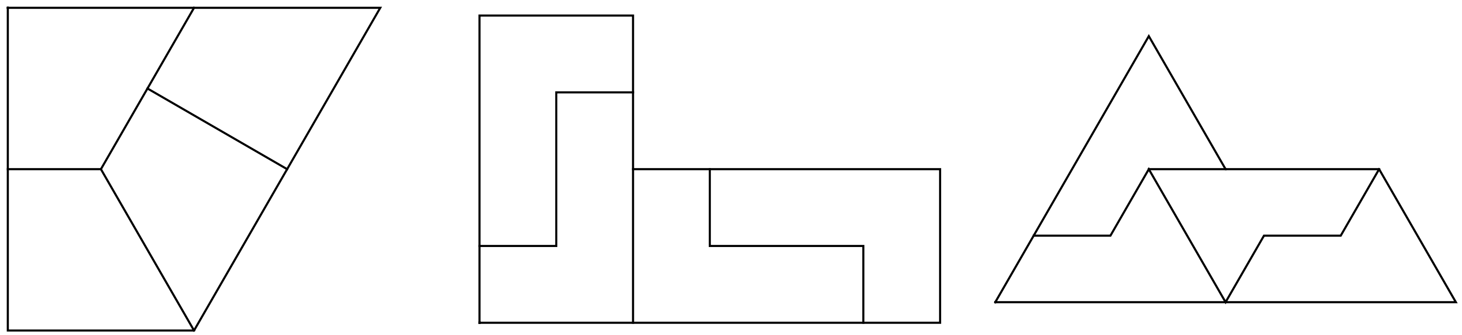


Figure 20: Rep-tiles

- Related question: Must all rep-tile be a fundamental region of a crystallographic group ?

- **Tiling Question** Substitution rule of tiles $\{T_1, \dots, T_m\}$ is written as:
$$QT_j = \bigcup_i T_i + D_{ij}$$
where Q is an expanding matrix and D_{ij} are finite sets of translations. By Lagarias-Wang's duality, the associated Delone set satisfies
$$\Lambda_i = \bigcup_j Q\Lambda_j + D_{ij}.$$
Given expanding Q, which digit set D_{ij} gives rise to a tiling

?

- For which Q and D_{ij}, the associated Delone set dynamics contains pure parts ? **We know a lot.**: E.Bombieri-J.E.Taylor [4], B. Solomyak, J.-Y. Lee [13, 9]

- Characterize pure diffractive substitution tilings. **Difficult**: Overlap coincidence by Solomyak [13], Algorithm by Akiyama-Lee [1]

- Pure diffractive substitution tiling is given by cut and projection ? **Mostly solved**: Lee [8]

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